Explicit estimates on the summatory functions of the Moebius function with coprimality restrictions

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Abstract

We prove that $|\sum_{\substack{d\le x,\\(d,q)=1}} \mu(d)/d| \le 2.4\,(q/\varphi(q))/\log(x/q)$ for every $x > q \ge 1$ and similar estimates for the Liouville functions. We give also better constants when x/q is larger.

1 Introduction

In explicit analytic number theory, one needs very often to evaluate the average of a multiplicative function, say f. The usual strategy is to compare this function to a more usual model f_0, as in [12, Lemma 3.1]. This process is also well detailed in [2]. When the model is $f_0 = 1$, the situation is readily cleared out; it is also well studied when this model is the divisor function in [1, Corollary 2.2]. We signal here that the case of the characteristic function of the squarefree numbers is specifically handled in [4]. The next problem is to use the Moebius function as a model. In this case the necessary material can be found in [13], though of course the saving is much less and may be insufficient: when the model is 1 or the divisor function, or the characteristic

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function of the squarefree integers, the saving is a power of the size of the variable, while now it is only a logarithm (or the square of one according to whether one says that the trivial estimate for $\sum_{d\le D}\mu(d)/d$ is 1 or $\log D$). One of the consequences is that one has to be more careful, and thrifty, when it comes to small variations. The variations we consider here is the addition of a coprimality condition $(d,q)=1$, for some fixed q, on the ranging variable d. Our first aim is thus to show how to get explicit estimates for the family of functions

$$m_q(x)=\sum_{\substack{n\le x,\\ (n,q)=1}}\mu(n)/n,\quad m(x)=m_1(x) \tag{1.1}$$

from explicit estimates concerning solely $m(x)$. The definition of the Liouville function $\lambda(n)$ is recalled below in (1.3), while the auxiliary function ℓ_q is defined in (1.4).

Theorem 1.1. *We have, when $1\le q<x$, where q is an integer and x a real number,*

$$\Biggl|\sum_{\substack{n\le x,\\ (n,q)=1}}\frac{\mu(n)}{n}\Biggr|\le\frac{q}{\varphi(q)}\frac{2.4}{\log(x/q)},\quad \Biggl|\sum_{\substack{n\le x,\\ (n,q)=1}}\frac{\lambda(n)}{n}\Biggr|\le\frac{q}{\varphi(q)}\frac{0.79}{\log(x/q)}.$$

Moreover $\log(x/q)|\ell_q(x)|\le 0.155\,\frac{q}{\varphi(q)}$ *and* $\log(x/q)|m_q(x)|\le\frac{3}{2}\frac{q}{\varphi(q)}$ *when* $x/q\ge 3310$. *We also have* $\log(x/q)|m_q(x)|\le\frac{7}{8}\frac{q}{\varphi(q)}$ *when* $x/q\ge 9960$.

The sole previous estimate on $m_q(x)$ seems to be [7, Lemma 10.2] which bounds $|m_q(x)|$ uniformly by 1. The estimate for $m(x)$ that will provide the initial step comes from [13]

$$|m(x)|\le 0.03/\log x\quad(x\ge X_0=11\,815). \tag{1.2}$$

Let us first note that the simplest treatment of this condition via the Moebius function, i.e. writing

$$\mathbb{1}_{(d,q)=1}=\sum_{\substack{\delta|q,\\ \delta|d}}\mu(\delta),$$

does not work here. Indeed we get:

$$\sum_{\substack{d\le D,\\ (d,q)=1}}\frac{\mu(d)}{d}=\sum_{\delta|q}\mu(\delta)\sum_{\delta|d\le D}\frac{\mu(d)}{d}=\sum_{\delta|q}\frac{\mu(\delta)^2}{\delta}\sum_{\substack{d\le D/\delta,\\ (d,\delta)=1}}\frac{\mu(d)}{d}$$

and we are back to the initial problem with different parameters. The classical workaround (used for instance in [10, near (7)] but already known by Landau) runs as follows: we determine a function g_q such that $\mathbb{1}_{(n,q)=1}\mu(n) = g_q \star \mu(n)$, where $\star$ denotes the arithmetic convolution product. The drawback of this method is that the support of g is not bounded (determining g_q by comparing the Dirichlet series is a simple exercise). So if we write

$$\sum_{\substack{d\le D,\\ (d,q)=1}} \mu(d)/d = \sum_{\delta\le D} \frac{g_q(\delta)}{\delta} \sum_{d\le D/\delta} \frac{\mu(d)}{d},$$

we are forced to two things:

1. using estimates for $\sum_{d\le D/\delta} \mu(d)/d$ when D/δ can be small,
2. completing the sum over δ to get a decent result.

Both steps introduce quite a loss when q is not specified. We propose here a different approach by introducing the Liouville function as an intermediary. This function $\lambda(n)$ is the completely multiplicative function that is 1 on integers that have an even number of prime factors – counted with multiplicity – and -1 otherwise. It satisfies

$$\sum_{n\ge 1} \frac{\lambda(n)}{n^s} = \frac{\zeta(2s)}{\zeta(s)}. \tag{1.3}$$

We introduce the family of auxiliary functions

$$\ell_q(x) = \sum_{\substack{n\le x,\\ (n,q)=1}} \lambda(n)/n, \quad \ell(x) = \ell_1(x). \tag{1.4}$$

Our process runs as follows: we derive bounds for $\ell(x)$ from bounds on $m(x)$ and some computations, derive bounds on $\ell_q(x)$ from bounds on $\ell(x)$, and finally derive bounds on $\mu_q(x)$ from bounds on $\ell_q(x)$. The theoretical steps are contained in the three Lemmas 2.3, 2.5 and 3.2.

We complete this introduction by signalling that [14] contains explicit estimates with a large range of uniformity for sums of the shape

$$\sum_{\substack{d\le x,\\ (d,r)=1}} \frac{\mu(d)}{d^{1+\varepsilon}}$$

and for a similar sum but with the summand $\mu(d)\log(x/d)/d^{1+\varepsilon}$. The path we followed there is essentially elementary and the saving is less.

I thank Harald Helfgott for interesting discussions that pushed me into pulling this note out of its drawer. Special thanks are also due to the referee for his/her very careful reading: several numerical errors have been corrected in the process, and the arguments are also now better exposed.

2 From the Moebius function to the Liouville function

Lemma 2.1. *For* $2 \leq x \leq 906\,000\,000$*, we have* $|\ell(x)| \leq 1.347/\sqrt{x}$*.*
For $2 \leq x \leq 1.1 \cdot 10^{10}$*, we have* $|\ell(x)| \leq 1.41/\sqrt{x}$*.*
For $1 \leq x \leq 1.1 \cdot 10^{10}$*, we have* $|\ell(x)| \leq \sqrt{2/x}$*.*

The computations have been run with PARI/GP (see [11]), speeded by using gp2c as described for instance in [1]. We mention here that [6] proposes an algorithm to compute isolated values of $M(x)$. This can most probably be adapted to compute isolated values of $\ell(x)$, but does not seem to offer any improvement for bounding $|\ell(x)|$ on a large range. In [3], the authors show that

$$\ell(x) \geq 0, \quad (x < 72\,185\,376\,951\,205)$$

and that

$$\ell(x) \geq -2.0757642 \times 10^{-9}, \quad (x \leq 75\,000\,000\,000\,000)$$

This takes care of the lower bound for $\ell(x)$. The computations we ran are much less demanding in time and algorithm, but however rely on a large enough sieve-kind of table to compute values of $\lambda(n)$ on some very large range. Harald Helfgott (indirectly) pointed out to me that the RAM-memory can be very large nowadays, allowing to precompute large quantities to which one has an almost immediate access. Here is a simplified version of the main loop:

```
{getbounds(zmin:small, valini:real, zmax:small) =
   my(maxi:real, valuesliouville:vecsmall, gotit:vecsmall,
      valuel:real, bound:small, pa:small);

   /* Precomputing lambda(n): */
   valuesliouville = vectorsmall(zmax-zmin+1, m, 1);
   gotit = vectorsmall(zmax-zmin+1, m, 1);
   forprime (p:small = 2, floor(sqrt(zmax+0.0)),
```

```
            bound = floor(log(zmax+0.0)/log(p+0.0));
            pa = 1;
            for(a:small = 1, bound,
               pa *= p;
               for(k:small = 1, floor((zmax+0.0)/pa),
                  if(k*pa >= zmin,
                     valuesliouville[k*pa-zmin+1] *= -1;
                     gotit[k*pa-zmin+1] *= p,))));

   /* Correction in case of a large prime factor: */
   for(n:small = zmin, zmax,
      if(gotit[n-zmin+1] < n,
         valuesliouville[n-zmin+1] *= -1,));

   valuel = (valini + 0.0) + valuesliouville[1]/zmin;
   maxi = max( valini*sqrt(zmin+0.0), abs(valuesl*sqrt(zmin+1.0)));

   /* Main loop: */
   for(n:small = zmin+1, zmax,
      valuel += valuesliouville[n-zmin+1]/n;
      maxi = max(maxi, abs(valuel)*sqrt(n+1.0)));

   return([maxi, valuel]);
}
```

We used this loop to compute our maximum on intervals of length $2\cdot 10^7$. The main function aggregates these results by making the interval vary. The computations took about half a day on a 64 bits fast desktop equipped of 8G of RAM. In the actual script, we also checked that the computed value of $\ell(x)$ is non-negative in this range. Going farther would improve on the final constants, but only when x/q is large. We compared $|\ell(x)|$ with $1/\sqrt{x}$, and this seems correct for small values, but the works [9] and [8] suggest that the maximal order is larger than that.

Lemma 2.2. *The function*

$$T(y) : y \mapsto \frac{\log y}{y}\int_{\sqrt{X_0}}^{y}\frac{dv}{\log v}$$

satisfies $T(y) \leq 1.119$ *for* $y \geq 10^5$.

Proof. A repeated integration by parts shows that

$$\begin{aligned} T(y) &= \frac{\log y}{y}\Big(\frac{y}{\log y} - \frac{\sqrt{X_0}}{\log\sqrt{X_0}} + \frac{y}{(\log y)^2} - \frac{\sqrt{X_0}}{(\log\sqrt{X_0})^2} + 2\int_{\sqrt{X_0}}^{y}\frac{dv}{(\log v)^3}\Big) \\ &\le \frac{\log y}{y}\Big(\frac{y}{\log y} - \frac{\sqrt{X_0}}{\log\sqrt{X_0}} + \frac{y}{(\log y)^2} - \frac{\sqrt{X_0}}{(\log\sqrt{X_0})^2}\Big) + \frac{2T(y)}{(\log\sqrt{X_0})^2} \end{aligned}$$

from which we deduce that

$$T(y) \le 1.1001\cdot\Big(1+\frac{1}{\log y}\Big).$$

This shows that $T(y) \le 1.113$ when $y \ge 10^{40}$. We then check *numerically* that the function T is increasing and then decreasing, with a maximum around 12478.8 with value $1.118\,598 + \mathcal{O}^*(10^{-6})$. But this is only an *observation*, since a computer computes only a sample of values. Since the derivative of T can easily be bounded, we obtain the claimed upper bound. The reader may also consult [5] where a similar process is fully detailed. □

The following lemma is a simple exercise:

Lemma 2.3. *We have*

$$\ell_q(x) = \sum_{\substack{u^2\le x,\\ (u,q)=1}} m_q\big(x/u^2\big)/u^2. \tag{2.1}$$

We shall use it only when $q = 1$, but it is equally easy to state it in general.

Lemma 2.4. *For $x > 1$, we have $|\ell(x)| \le 0.79/\log x$.*
For $x \ge 3310$, we have $|\ell(x)| \le 0.155/\log x$.
For $x \ge 8918$, we have $|\ell(x)| \le 0.099/\log x$.

Proof. We appeal to Lemma 2.3 (with $q = 1$) and separate the sum according to $u \le U$ or $u > U$ where $x/U^2 \ge X_0$. When $u \le U$ we apply (1.2), in the other case we use that $|m(x)| \le 1$

$$|\ell(x)| \;\le\; 0.03\sum_{u\le U}\frac{1}{u^2\log(x/u^2)} + \frac{1+U^{-1}}{U}$$

With $f(t) = 1/(t^2\log(x/t^2))$, we check that

$$f'(t) = -\frac{2}{t^3\log(x/t^2)} + \frac{2}{t^3\log^2(x/t^2)}.$$

This quantity is negative for $1 \le t \le U$, since then $x/t^2 \ge x/U^2 \ge X_0 > e$. We thus have

$$\sum_{u \le U} \frac{1}{u^2 \log(x/u^2)} \le f(1) + \int_1^U f(t)dt = \frac{1}{\log x} + \int_1^U \frac{dt}{t^2 \log(x/t^2)}.$$

Changing variables we get

$$\sum_{u \le U} \frac{1}{u^2 \log(x/u^2)} \le \frac{1}{\log x} + \frac{1}{\sqrt{x}} \int_{\sqrt{x/U^2}}^{\sqrt{x}} \frac{dv}{2 \log v}.$$

It follows that

$$|\ell(x)| \le \frac{0.03}{\log x} + \frac{0.03}{\sqrt{x}} \int_{\sqrt{X_0}}^{\sqrt{x}} \frac{dv}{2 \log v} + \frac{1 + \sqrt{X_0/x}}{\sqrt{x/X_0}}.$$

We apply Lemma 2.2 at this level. Hence, when $x \ge 10^{10}$,

$$\begin{aligned} |\ell(x)| &\le \frac{0.03}{\log x} + \frac{0.03 \cdot 1.119}{\log x} + \frac{1 + \sqrt{X_0/x}}{\sqrt{x/X_0}} \\ &\le \frac{0.06357}{\log x} + \frac{(1 + \sqrt{X_0/x}) \log x}{\sqrt{x/X_0}} \frac{1}{\log x} \\ &\le \frac{0.089}{\log x} \le \frac{0.099}{\log x}. \end{aligned}$$

We extend it to $x \ge 17\,715$ via Lemma 2.1, part one and two, and to $x \ge 8918$ by direct inspection. This inequality extends to $x \ge 1$ by weakening the constant 0.099 to 0.79. It is straighforward to use some mild computations to check the validity of the bound 0.155 when $x \ge 3310$. □

Adding coprimality conditions

Our tool is provided by the simple elementary lemma.

Lemma 2.5. *We have*

$$\ell_q(x) = \sum_{d|q} \frac{\mu^2(d)}{d} \ell\big(x/d\big).$$

The second part of Theorem 1.1 follows immediately by combining Lemma 2.5 together with Lemma 2.4. Actually, what comes out is the bound

$$|\ell_q(x)| \le \frac{0.79}{\log(x/q)} \sum_{d|q} \frac{\mu^2(d)}{d} = \frac{0.79}{\log(x/q)} \prod_{p|q} \frac{p+1}{p}.$$

As the function $q/\varphi(q)$ is easier to remember and $\prod_{p|q}\frac{p+1}{p} \le q/\varphi(q)$, we simplify the above into

$$|\ell_q(x)| \le \frac{0.79}{\log(x/q)}\frac{q}{\varphi(q)}.$$

When $x/q \ge 3310$, one can replace 0.79 by 0.155, and when $x/q \ge 8918$, by 1/10.

3 Back to the Moebius function with coprimality coditions

Let us start with a wide ranging estimate:

Lemma 3.1. *We have, for every integer $q \ge 1$ and every real number $x \ge 1$, $|\ell_q(x)| \le \pi^2/6$.*

Proof. This a direct consequence of Lemma 2.3 and [7, Lemma 10.2].[1] □

The following lemma is again a simple exercise.

Lemma 3.2. *We have*

$$m_q(x) = \sum_{\substack{u^2 \le x,\\ (u,q)=1}} \frac{\mu(u)}{u^2}\ell_q\big(x/u^2\big).$$

Proof of Theorem 1.1. We have to prove several estimates of type

$$\varphi(q)\log(x/q)|m_q(x)| \le c, \quad x/q \ge N.$$

We put $x^* = x/q$ and $y = \log x^* = \log(x/q)$ and separate the proof in two parts. First we consider the case $1 \le y \le 8$, and later the case $y > 8$.
Case (A) : $1 \le y \le 8$. We appeal to Lemma 3.2. We have for a real parameter U such that $U^2 \le x^*$

$$\begin{aligned} |m_q(x)| &\le \sum_{u^2 \le x} \frac{\mu^2(u)}{u^2}\big|\ell_q\big(x/u^2\big)\big| \qquad (3.1)\\ &\le \sum_{u \le U} \frac{q}{\varphi(q)}\frac{0.79\mu^2(u)}{u^2\log(x/(u^2q))} + \frac{\pi^2}{6}\sum_{u>U}\frac{\mu^2(u)}{u^2}\\ &\le \frac{q/\varphi(q)}{\log(x/q)}\Bigg(\sum_{u\le U}\frac{0.79\mu^2(u)}{u^2(1-\frac{2\log u}{\log(x/q)})} + \frac{\pi^2}{6}\sum_{u>U}\frac{\mu^2(u)}{u^2}\log(x/q)\Bigg). \end{aligned}$$

[1]If we were to adapt the proof presented in [7] to the case of λ instead of μ, we would reach the bound 2 and not $\pi^2/6$.

This is our starting inequality. We define

$$\rho(U,y) = 0.79 \sum_{u \le U} \frac{\mu^2(u)}{u^2(1-\frac{2\log u}{y})} + \frac{\pi^2}{6} \sum_{u>U} \frac{\mu^2(u)}{u^2} y. \tag{3.2}$$

Note that $\rho(U,y) = \rho([U],y)$ where $[U]$ is the integer part of U. For each y we need to select one U such that $\rho(U,y) \le 2.4$. We choose $U = 1$ for $y \in [1, a_1]$; $U = 2$ for $y \in [a_1, a_2]$; $U = 3$ for $y \in [a_2, a_3]$; and $U = 7$ for $y \in [a_3, 8]$. Where $a_1 = 1.8665\cdots$ is a solution of $\rho(1,y) = \rho(2,y)$; $a_2 = 2.6774\cdots$ is a solution of $\rho(2,y) = \rho(3,y)$; $a_3 = 4.1237\cdots$ is a solution of $\rho(3,y) = \rho(7,y)$. Each of these three functions is a sum of a linear term ay and terms of type $\frac{Ay}{(y-2\log n)}$ with $A > 0$. These are convex for $y > 2\log n$. In this way it is very easy to show that $\rho(1,y)$ is convex in $[1, a_1]$, $\rho(2,y)$ is convex in $[a_1, a_2]$, $\rho(3,y)$ is convex in $[a_2, a_3]$ and finally that $\rho(7,y)$ is convex in $[a_3, 8]$. So, for example, to show the inequality $\rho(3,y) \le 2.4$ in the interval $[a_2, a_3]$ we only have to show that $\rho(3, a_2)$, and $\rho(3, a_3) \le 2.4$. This presents no difficulty. The maximum value obtained is $\rho(2, a_2) = 2.38790\cdots$ with

$$a_2 = \frac{237 + 100\pi^2 \log 3}{50\pi^2},$$

$$\rho(2,a_2) = \frac{237}{20\pi^2} + \pi^2\Big(\frac{79\log 2}{948 + 400\pi^2\log(3/2)} - \frac{5\log 3}{12}\Big) + \log 243.$$

Case (B) : $y > 8$.

We start from Lemma 3.2, from which we deduce the simpler bound:

$$|m_q(x)| \le \sum_{u^2 \le x} \big|\ell_q(x/u^2)\big|/u^2$$

which we then exploit in the same way as what is done in the proof of Lemma 2.4, replacing the bound $|m(x)| \le 1$ by Lemma 3.1. With $x = eU^2 q$ and $x^* = x/q$, we thus get

$$\begin{aligned} |m_q(x)| &\le \frac{q}{\varphi(q)} \frac{0.79}{\log x^*} + \frac{0.79\,q}{\varphi(q)} \int_1^{\sqrt{x^*/e}} \frac{du}{u^2 \log(x^*/u^2)} + \frac{\pi^2\sqrt{e}}{6} \frac{1+\sqrt{e/x^*}}{\sqrt{x^*}} \\ &\le \frac{q}{\varphi(q)} \frac{0.79}{\log x^*} + \frac{0.79\,q}{\varphi(q)\sqrt{x^*}} \int_{\sqrt{e}}^{\sqrt{x^*}} \frac{dv}{2\log v} + \frac{\pi^2\sqrt{e}}{6} \frac{1+\sqrt{e/x^*}}{\sqrt{x^*}} \\ &\le c(x^*) \frac{q}{\varphi(q)\log x^*} \end{aligned}$$

with

$$c(x^*) = 0.79 + 0.79 \frac{\log x^*}{\sqrt{x^*}} \int_{\sqrt{e}}^{\sqrt{x^*}} \frac{dv}{2\log v} + \frac{\pi^2\sqrt{e}}{6} \frac{1+\sqrt{e/x^*}}{\sqrt{x^*}} \log x^*.$$

Some numerical work shows that $c(x^*) \leq 2.4$ when $x^* \geq 1\,862$, so our inequality is proved for $y > \log 1862 = 7.52941\cdots$. This with part (A) proves that $\varphi(q)\log(x/q)|m_q(x)| \leq 2.4$ for $1 \leq q < x$.

When $x^* \geq 3310$, we can single out the term $u = 1$ in (3.1) and modify the coefficient of the bound on this term from 0.79 into 0.155, then we treat the rest of the sum in the same way as before. We get a similar bound with $c(x^*)$ substituted by:

$$c_1(x^*) = 0.155 + 0.79\frac{\log x^*}{4\log(x^*/4)} + 0.79\frac{\log x^*}{\sqrt{x^*}}\int_{\sqrt{e}}^{\sqrt{x^*/4}} \frac{dv}{2\log v} + \frac{\pi^2\sqrt{e}}{6}\frac{1+\sqrt{e/x^*}}{\sqrt{x^*}}\log x^*.$$

This yields a maximum not more than $1.466 < 3/2$. When $x^* \geq 3 \times 3310$, we single out the terms of index 1, 2, and 3 similarly. This means substituting $c_2(x^*)$ to $c_1(x^*)$ where the $c_2(x^*)$ is defined by

$$c_2(x^*) = 0.155 + 0.155\frac{\log x^*}{4\log(x^*/4)} + 0.155\frac{\log x^*}{9\log(x^*/9)} + 0.79\frac{\log x^*}{25\log(x^*/25)} + 0.79\frac{\log x^*}{\sqrt{x^*}}\int_{\sqrt{e}}^{\sqrt{x^*/25}} \frac{dv}{2\log v} + \frac{\pi^2\sqrt{e}}{6}\frac{1+\sqrt{e}x^{*-1/2}}{\sqrt{x^*}}\log x^*.$$

This yields a maximum not more than $0.871 < 7/8$. The proof of Theorem 1.1 is complete. □

References

[1] D. Berkane, O. Bordellès, and O. Ramaré. Explicit upper bounds for the remainder term in the divisor problem. *Math. of Comp.*, 81(278):1025–1051, 2012.

[2] P. Berment and O. Ramaré. Ordre moyen d'une fonction arithmétique par la méthode de convolution. *Revue de Mathématique Spéciale*, 212(1):1–15, 2012.

[3] P. Borwein, R. Ferguson, and M.J. Mossinghoff. Sign changes in sums of the Liouville function. *Math. Comp.*, 77(263):1681–1694, 2008.

[4] H. Cohen, F. Dress, and M. El Marraki. Explicit estimates for summatory functions linked to the Möbius μ-function. *Funct. Approx. Comment. Math.*, 37(part 1):51–63, 2007.

[5] J. Arias de Reyna and J. van de Lune. A proof of a trigonometric inequality. A glimpse inside the mathematical kitchen. *J. Math. Inequal.*, 5(3):341–353, 2011.

[6] M. Deléglise and J. Rivat. Computing the summation of the Möbius function. *Exp. Math.*, 5(4):291–295, 1996.

[7] A. Granville and O. Ramaré. Explicit bounds on exponential sums and the scarcity of squarefree binomial coefficients. *Mathematika*, 43(1):73–107, 1996.

[8] T. Kotnik and J. van de Lune. On the order of the Mertens function. *Experiment. Math.*, 13(4):473–481, 2004.

[9] H.L. Montgomery. Zeros of approximations to the zeta function. In *Studies in pure mathematics*, pages 497–506. Birkhäuser, Basel, 1983.

[10] Y. Motohashi. Primes in arithmetic progressions. *Invent. Math.*, 44(2):163–178, 1978.

[11] The PARI Group, Bordeaux. *PARI/GP, version* `2.5.2`, 2011. `http://pari.math.u-bordeaux.fr/`.

[12] O. Ramaré. On Snirel'man's constant. *Ann. Scu. Norm. Pisa*, 21:645–706, 1995. `http://math.univ-lille1.fr/~ramare/Maths/Article.pdf`.

[13] O. Ramaré. From explicit estimates for the primes to explicit estimates for the Moebius function. *Acta Arith.*, 157(4):365–379, 2013.

[14] O. Ramaré. Some elementary explicit bounds for two mollifications of the Moebius function. *Functiones et Approximatio*, 12p, 2012.