Graph Tower Dichromatic Polynomial

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Abstract

Starting from a non-oriented graph G and an integer s, we define the graph tower $G(s)$. In the linear graph $G = L_r$ case, this results in the classical square on $r \times s$ vertices. The aim of this paper is to describe an effective method to compute this dichromatic polynomial $Z_{G(s)}(q, v)$ and to prove rationality of the series $\Sigma_G(q, v)[X] = \sum_{s\geq 1} Z_{G(s)}(q, v)X^{s-1}$. The functionals created for this purpose are implemented using MuPAD and may be obtained under GPL licence.

1 Dichromatic Polynomial.

Let us begin with the definition of the dichromatic polynomial as can be found in [3, Chapter X] or [2].

Connection-contraction principle: The *dichromatic polynomial* of a (non oriented) graph $G = (V, E)$ is a two variables polynomial, denoted by $Z(G)$ or $Z_G(q, v)$ if we need this specification. It is entirely characterised by

$$\begin{cases} Z(G) = Z(G - e) + vZ(G/e), \\ Z(E_r) = q^r, \end{cases} \tag{1}$$

where

– e is any edge of G

– $G - e$ is the graph obtained from G by canceling e,

– G/e is the graph obtained from G by canceling e *and* by gluing together the two ends of e. This construction is called *connection-contraction of the edge e.*

– E_r is the graph on r vertices without edges.

Directly from definition, we get

$$Z_G(q, 0) = q^{|V|},$$

where $|V|$ denotes the number of vertices of G. The *classical chromatic polynomial* is obtained by specializing $v = -1$ in the dichromatic polynomial.

Determining $Z(G)$ can be done by proceeding edge after edge, following the above definition. One can also write $Z(G)$ as the sum over all possible states where a state is the choice for each edge between cancelation and connection-contraction. In this latter case, we say we are proceeding by "clusters".

Partition function: The dichromatic polynomial can be interpreted in Statistical Mechanics as the partition function P_G of a Potts model with q states on G, through the relation

$$P_G(q, \beta) = e^{-\beta|E|} Z_G(q, v)\,, \tag{2}$$

where $|E|$ denotes the number of edges of G and $v = e^{\beta} - 1$. We do not give more details here, refering the reader to [3, Section X.3]. Within the framework of Statistical Mechanics, one of the most important open problem is the solvability of the partition function for particular graphs like the graph on the infinite regular lattice of dimension 2. The graph towers we introduce below were motivated by this problem, quoted by V.F.R. Jones in [4]. We also may observe that case $q = 2$, called Ising model, has been solved by L. Onsager [5]. An explicit presentation of Potts models on different lattices can be found in [1].

Tutte Polynomial: The Tutte polynomial $T_G(x, y)$ of a graph G is related to its dichromatic polynomial by the formula:

$$T_G(x, y) = (x-1)^{-m}(y-1)^{-|V|} Z_G((x-1)(y-1), (y-1)) \tag{3}$$

where m is the number of connected components of G and $|V|$ the number of its vertices, cf. [3, Section X.2]. Tutte and dichromatic polynomial contain many informations on the nature of the graph G, cf. [3, Section X.4].

Graph Towers: Let s be an integer and $G = (E, V)$ be a non-oriented graph with r vertices. Denote by $x_1, \ldots, x_r$ the vertices of G. We consider s copies of G built on the vertices $x_{1,j}, \ldots, x_{r,j}$ ($j = 1, \ldots, s$) such that the number of edges between $x_{a,j}$ and $x_{b,j}$ is exactly the number of edges between x_a and x_b in G. We add one edge between $x_{a,j}$ and $x_{a,j+1}$ for $j = 1, \ldots, s-1$. The graph we obtain by this process is called a *graph tower* and denoted by $G(s)$. For instance, we have $G(1) = G$, up to the name of its vertices.

The edges of the copies of G (between $x_{a,j}$ and $x_{b,j}$) are called *vertical* while the edges between $x_{a,j}$ and $x_{a,j+1}$are called *horizontal*.

In this paper we describe an effective method to compute the dichromatic polynomial of $G(s)$. We also prove that the series

$$\Sigma_G(q, v)[X] = \sum_{s \geq 1} Z_{G(s)}(q, v) X^{s-1} \tag{4}$$

is rational in q and v and we compute it.

Denote by L_r the "linear graph on r vertices", meaning the graph with $\{1, \ldots, r\}$ as set of vertices and having one and only one edge between a and $a+1$ if $a \in \{1, \ldots, r-1\}$. The graph tower $L_r(s)$ is the squaring $H_{r,s}$ with r lines and s columns built on $r \times s$ points. The dichromatic polynomial $Z(H_{r,s})$ is the main motivation of this work.

Generalised graph tower: We could have chosen for this study a more general framework which we describe here. Let s be an integer, $G = (E, V)$ a non-oriented graph on r vertices and Γ a bipartite graph, on two sets of r elements.

We define $G \otimes \Gamma^s$ as follows:

– we consider s copies of G (the "columns" whose edges are "vertical"),

– we link these columns with the graph Γ adding "horizontal edges" by this process.

The graph tower $G(s)$ is the particular case $G \otimes \Gamma^s$ with

$$\Gamma = \{(i, i), 1 \leq i \leq r\}. \tag{5}$$

In the rest of this paper we restrict our attention to this case.

2 Presentation of the library.

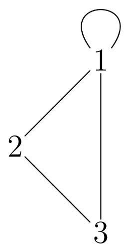
The script of this library may be obtained (under a GPL licence) at

`http://www-gat.univ-lille1.fr/~ramare/ServeurPerso/`

under the names `Dichromate.mu` et `Dichromate.help`.
An elementary example: In our program, a graph is a list whose first element is the set of vertices and the second one the list of edges. Each edge is the list of its two adjacent vertices. For instance

$$G := [\{1,2,3\}, [[1,1],[1,2],[2,3],[3,1]]] \tag{6}$$

is the complete graph on three vertices with a loop around vertex 1:



Functionals and variables : The vertices are expressions whose equality is checked with "=". We first have a functional Dichromate :: DiChromatic(G) which uses the connection-contraction principle. For the previous example, we get:

$$\text{Dichromate} :: \text{DiChromatic}(G) = q^3 + 3qv^2 + 3q^2v + 4qv^3 + q^3v + qv^4 + 3q^2v^2.$$

Since variables q and v could be reserved for another use, we use DiChromateq and DiChromatev internally and substitute q and v only for the output.

As for the usual chromatic polynomial, if we wish MuPAD to work with one variable polynomials, we are simply to specify:

$$\text{DiChromatev} := -1. \tag{7}$$

Such an explicit specification can also be done for the variable DiChromateq. Output variables q and v are defined in DiChromateqvxyDefault whose default value is

$$\text{DiChromateqvxyDefault} := [\text{hold}(q), \text{hold}(v), \text{hold}(x), \text{hold}(y)]. \tag{8}$$

Variables x and y are DiChromatex and DiChromatey internally and are treated like q and v. They appear in the Tutte polynomial.

Dichromatic polynomial of a graph tower: Let G be the following graph

encoded by $G := [\{1,2\},[[1,1],[1,2]]]$. The polynomial $Z_{G(2)}(q,v)$ can be determined using

$$\mathrm{Dichromate} :: \mathrm{RectDiChromatic}([\{1,2\},[[1,1],[1,2]]],2) \tag{9}$$

which gives

$$\begin{aligned}24qv^4 + q^4v + 68qv^5 + 71qv^6 + 35qv^7 + 9qv^8 + qv^9 + 21q^2v^3 + 8q^3v^2 + 51q^2v^4 \\ + 18q^3v^3 + 2q^4v^2 + 40q^2v^5 + 12q^3v^4 + q^4v^3 + 11q^2v^6 + 2q^3v^5 + q^2v^7\end{aligned} \tag{10}$$

The series $\Sigma_G(q,v)[X]$ (cf. 4) can be computed by

$$\mathrm{Dichromate} :: \mathrm{RectDiChromatic}([\{1,2\},[[1,1],[1,2]]],\mathrm{hold}(All)) \tag{11}$$

which gives

$$\frac{qv(1+v) + q^2(1+v) - Xq^2v^2(1+v)^3}{1 - X(1+v)(v^2(4+v) + 3qv + q^2) + X^2v^2(1+v)^3(v^2 + 2qv + q^2)} \tag{12}$$

We can specialise q and/or v for this computation but not X which stays $\mathrm{hold}(X)$ all along the evaluation.

Moreover since the linear graph on r vertices L_r is of special interest to us, we can denote it by r in Dichromate :: RectDiChromatic (but not in Dichromate :: DiChromatic) in such a way that

$$\begin{aligned}&\mathrm{Dichromate} :: \mathrm{RectDiChromatic}(2,3) \\ &\qquad = \mathrm{Dichromate} :: \mathrm{RectDiChromatic}([\{1,2\},[[1,2]]],3).\end{aligned} \tag{13}$$

This function owns a third optional argument whose special value is TRUE. Indeed, we have two ways to compute

$$\mathrm{Dichromate} :: \mathrm{RectDiChromatic}(r,s).$$

We can iterate s times the functional ψ_G described below, or we can compute the corresponding power series and take its s-coefficient. As it is, Dichromate :: RectDiChromatic(r, s) chooses the first procedure if $s \leq r^2$ and the second one otherwise. Still, the user can prefer iteration, in which case he should use Dichromate :: RectDiChromatic(r, s, TRUE). This is because if the computation of $Z_{L_4(1000)}(q, v)$ is clearly faster using the series, the case $Z_{L_4(15)}(q, v)$ is unclear. Observe that

$$\text{Dichromate} :: \text{RectDiChromatic}(G, 1) = \text{Dichromate} :: \text{DiChromatic}(G)$$

though the two procedures are distinct: Dichromate :: DiChromatic uses the connection-contraction principle, while Dichromate :: RectDiChromatic works with cluster expansion in case $s = 1$.

The Tutte polynomial can be obtained by Dichromate :: RectTutte with the same parameters than Dichromate :: RectDiChromatic.

By calling

$$\text{Dichromate} :: \text{RectDiChromatic}(G, \text{hold}(All))$$

with $G = [\{1, 2, 3\}, [[1, 1], [1, 2], [2, 3], [3, 1]]]$ already considered in (6), the reader will discover how intricate our results can be.

3 Graph tower dichromatic polynomial: the problematic.

Let s be an integer and $G = (E, V)$ a non-oriented graph on r vertices. Recall that we are interested in computing $Z(G(s))$ where $G(s)$ is the graph tower built on G.

We describe now an algorithm working column by column starting from the right hand. There are three steps: an initialisation process, a repetitive step (iteration of the operator ψ_G) and a way-out. This algorithm uses partitions on the set V of vertices of G. We view these partitions as a peculiar kind of graph and reciproquely will treat this kind of graphs like partitions whenever needed. Here is the correspondence:

Partition of V and associated graph: Start with a non-oriented graph $G = (V, E)$ endowed with a partition $C = (C_1, \ldots, C_k)$ of its set V of vertices. We construct a new graph $G[C]$ obtained from the following completion of the graph G:

(H) *For each element C_i of the partition, we create a point ℓ_i and we connect it by an edge to each vertex belonging to C_i.*

Reciprocally, consider a graph H of the following shape: H is obtained from G by adding new vertices, connected to one or more vertices of G by one and only one edge and such that each vertex of G is reached by such edge. Then such a graph H gives a partition of V.

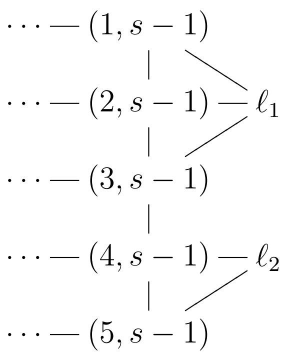
Later on, we shall use regularly this correspondence.

Initialisation: Consider the vertical edges of the first column (on the right). For each edge, we decide first if we delete it or if we contract it by using the connection-contraction principle. This is equivalent to choosing a subset $W \subseteq E$ of edges which will be contracted. We have now to determine $Z(H)$ for a collection of graphs H constituted of $G(s-1)$ and of a certain number of points (located on the right) and connected to some vertices of the right column of $G(s-1)$. This gives us a partition, $\mathrm{Part}(W)$ of the set V of vertices of the column $s-1$ by using the procedure described above. This can be done for instance by identifying the two vertices of each edge in W. Here ends the initialisation procedure. Let us describe an example.

Initialisation of a squarring: Consider the following squarring:

$$\begin{array}{ccccc}
\cdots — & (1, s-1) & — & (1, s) & \\
 & | & & |_1 & \\
\cdots — & (2, s-1) & — & (2, s) & \\
 & | & & |_2 & \\
\cdots — & (3, s-1) & — & (3, s) & \\
 & | & & |_3 & \\
\cdots — & (4, s-1) & — & (4, s) & \\
 & | & & |_4 & \\
\cdots — & (5, s-1) & — & (5, s) &
\end{array}$$

The choice $W = \{1, 2, 4\}$ gives the new graph



with the partition $\{\{(1,s-1),(2,s-1),(3,s-1)\},\{(4,s-1),(5,s-1)\}$ on column $s-1$.
Repetitive Step: From the initialisation process, we get a collection of graphs, H, formed of the tower $G(s-1)$ and of some new vertices ℓ_i connected at the right column of $G(s-1)$. Recall that such a data is equivalent to $G(s-1)$ together with a partition of V. At this moment, we will consider first edges adjacent to ℓ_i and later the vertical edges of the last right hand side column of $G(s-1)$. At the end of this procedure, we will get situation to the initial one, where tower $G(s-2)$ replaces tower $G(s-1)$, namely, $G(s-2)$ and a partition of V.

Denote by $\mathcal{P}(r)$ the set of all partitions on a set of r elements corresponding to the set of vertices of the graph G or of one of its copies. The iterative step can be describe as a linear operator $\psi_G\colon \mathbb{Z}[\mathcal{P}(r)] \to \mathbb{Z}[\mathcal{P}(r)]$. The determination of ψ_G is the main part of this work, see Theorem 1.

Way-out: The previous procedure ends after $(s-1)$ iterations of ψ_G. At this moment, we get a set of vertices without edges. Define the function σ on the set of graphs by $\sigma(G)=q^n$, where n is the number of connected components of G. The last step is the application of σ.

Finally, we have:

$$Z(G(s)) = \sum_{W\subseteq V} \sigma(\psi_G^{s-1}(\mathrm{Part}(W)))v^{|W|}. \tag{14}$$

where $\mathrm{Part}(W)$ is described in the initialisation procedure. As for the series defined in (4), we get:

$$\Sigma_G(q,v)[X] = \sum_{W\subseteq V} v^{|W|}\sigma\left(\frac{1}{\mathrm{Id} - X\psi_G}(\mathrm{Part}(W))\right) \tag{15}$$

$$= \sigma\left[\frac{1}{\mathrm{Id} - X\psi_G}\left(\sum_{W\subseteq V} v^{|W|}\,\mathrm{Part}(W)\right)\right]. \tag{16}$$

For instance, in the case of the linear graph with r vertices, we find

$$\Sigma_{L_r}(q,0) = \sum_{s\geq 1} q^{r(s-1)+r}X^{s-1} = \frac{q^r}{1-q^rX}. \tag{17}$$

4 Partitions and Hats.

The linear operator $\psi_G \colon \mathbb{Z}[\mathcal{P}(r)] \to \mathbb{Z}[\mathcal{P}(r)]$ allows the determination of the dichromatic polynomial of any graph G. In case of the linear graph $G = L_r$ (more generally when $G \otimes \Gamma^s$ is a planar graph), we can restrict the set of partitions $\mathcal{P}(r)$ to a smaller set denoted by $\mathcal{C}(r)$. The elements of $\mathcal{C}(r)$, called *hats,* are the partitions C such that the associated graph $G[C]$ is planar. We keep the same notation for this operator $\psi_G \colon \mathbb{Z}[\mathcal{C}(r)] \to \mathbb{Z}[\mathcal{C}(r)]$.

Recall that the set of partitions $\mathcal{P}(r)$ is of cardinality the *Bell number*, $p(r)$ whose exponential generating series is

$$\sum_{n\geq 0} p(n)\frac{x^n}{n!} = e^{e^x-1}\,. \tag{18}$$

We readily to see that cardinal $c(r)$ of $\mathcal{C}(r)$ is *Catalan number* characterised by

$$c(r) = \sum_{i=0}^{r-1} c(i)\,c(r-i-1)\,,\ c(0)=1\,, \tag{19}$$

whose values are 5 if $r=3$, 14 if $r=4$, 42 if $r=5$, 132 si $r=6$, ... A closed formula is given by

$$c(r) = \binom{2r+1}{r}/(2r+1). \tag{20}$$

Using hats instead of partitions reduces the dimension of the involved spaces but these remain rather large. For instance, $p(12) = 4\,213\,597$ while $c(12) = 58\,786$.

An ordering: Let C and C' be two partitions. We denote by $C' \leq C$ the usual ordering: for any component C'_j of C', there exists a component C_i of C such that $C'_j \subseteq C_i$.

To keep more compact expressions in some formulae, we need a second ordering. If C and C' are two partitions of $\{1\ldots r\}$ where r is a fixed integer, we set $C' \preccurlyeq C$ when the two following conditions are satisfied:

(a) $C' \leq C$.

(b) By (a), each component C_i of C splits into components of C', $C_i \cap C' = \cup_j M_{i,j}$. For every C_i, we require that at most one of the components $M_{i,j}$ contains more than a point.

5 Möbius function "à la Rota".

In this section, $\mathcal{A}$ represents $\mathcal{P}(r)$, or $\mathcal{C}(r)$. We are interested in functions f on $\mathcal{A} \times \mathcal{A}$ such that $f(x,y) = 0$ if $x \not\leq y$. We define a convolution product of two such functions is by

$$(f \star g)(x,y) = \sum_{z \in [x,y]} f(x,z) g(z,y) \tag{21}$$

where $[x,y]$ is the interval connecting x and y. This product is associative, has a neutral element Δ (whose value is 1 on the diagonal and 0 otherwise) and the inversible elements are exactly the functions f such that $f(x,x) \neq 0$ for every x.

Möbius function : Define the function $\mathbb{1}$ by

$$\mathbb{1}(x,y) = \begin{cases} 1 & \text{si} \quad x \leq y \\ 0 & \text{sinon.} \end{cases} \tag{22}$$

Its inverse is called Möbius function $\mu(x,y)$. The matrix of $\mu(x,y)$ is the inverse of the matrix of $\mathbb{1}(x,y)$. At this point, we recall a classical result saying that there exists an ordering on $\mathcal{A}$, $C_1, \ldots, C_{|\mathcal{A}|}$, such that $C_i \leq C_j \implies i \leq j$. With this ordering, the matrix of $\mathbb{1}$ is triangular with 1 on the diagonal. Observe this proves the existence of μ and give a way to compute it if r is small.

Let us also the existence of a closed formula for μ in the case of partitions. In the case of hats, we observe that the product of μ and of $\mathbb{1}$ is equal to the unit Δ. This gives us the relation

$$\sum_{C \leq B \leq C'} \mu(B, C') = 0 \qquad \text{if} \quad C \neq C' \tag{23}$$

and if $C \leq C'$ (otherwise the sum is empty). This relation allows a computation of μ step by step. A hat C such that $C \leq C'$ can be decomposed into components of C', it is therefore sufficient to consider the case where C' is the maximal hat C_{max}.

In the case of the series Σ_G, we shall show later how we can avoid completely the determination of μ.

We end this section with some remarks concerning the ordering. If C is a hat, we can find all the hats which are less than C (for the relation

$\leq$) easily: for any component C_i of C, simply partition it in hats in every possible fashion. This process gives us the number of divisors of C as the product $\prod_i c(|C_i|)$ where the C_i are the components of C.

As for the description of the multiples of a hat, it is more delicate not only because we have to construct the hats on the components but because these components are not naturally ordered.

6 The operator ψ_G.

Theorem 1 *Let $G = (V, E)$ be a non-oriented graph on r vertices and $C = (C_1, \ldots, C_k) \in \mathcal{P}(r)$ be a partition of V. The operator $\psi_G \colon \mathbb{Z}[\mathcal{P}(r)] \to \mathbb{Z}[\mathcal{P}(r)]$, introduced in the repetitive step of Section 3, can be written as*

$$\psi_G(C) = \sum_{C'' \in \mathcal{P}(r)} (C : C'')\; C''$$

where $(C : C'')$ is determined by

$$(C : C'') = \sum_{C' \leq C''} \mu(C', C'')\, \gamma^\flat_{v,q}(C, C')\, (v+1)^{\kappa_G(C')}.$$

In this formula,

– $\kappa_G(C')$ is the number of edges of G that connect two points belonging to the same component of C'

– and

$$\gamma^\flat_{v,q}(C, C') = \prod_{i=1}^{k} \left\{ q + \sum_j \left[(v+1)^{|M_{i,j}|} - 1 \right] \right\}$$

where the $M_{i,j}$ are the components of C' cut out along C_i, i.e. $C_i \cap C' = \cup_j M_{i,j}$.

Proof: Let $C = (C_1, \ldots, C_k) \in \mathcal{P}(r)$ be a partition of the set of vertices V. Denote by $G[C]$ the graph associated to the partition C, as described in Section 3. Recall that, for each component C_i (even for the singletons) of the partition C, we have a point ℓ_i, outside G and connected by an edge (q_{ij}, ℓ_i) at any point q_{ij} of C_i. These edges are called *horizontal*.

Destruction of the horizontal edges: In the set of horizontal edges (q_{ij}, ℓ_i) of $G[C]$, we choose a set $\mathrm{Hor}_{\mathrm{Supp}}$ of edges to cancel and a set $\mathrm{Hor}_{\mathrm{Contr}}$ of edges to connect. We denote by $\alpha(\mathrm{Hor}_{\mathrm{Supp}})$ (resp. $\alpha(\mathrm{Hor}_{\mathrm{Contr}})$) the set of vertices $q_{ij} \in V$ of the edges $(q_{ij}, \ell_i) \in \mathrm{Hor}_{\mathrm{Supp}}$ (resp. $(q_{ij}, \ell_i) \in \mathrm{Hor}_{\mathrm{Contr}}$).

We execute now the operations corresponding to our choices, including the suppression of the vertex ℓ_i if $\mathrm{Hor}_{\mathrm{Contr}} \cap \mathrm{C_i} = \emptyset$. A new partition $C' \in \mathcal{P}(r)$ arises :

– the edges $(q_{ij}, \ell_i) \in \mathrm{Hor}_{\mathrm{Supp}}$ bring a component reduced to a singleton $\{q_{ij}\}$;

– the edges $(q_{ij}, \ell_i) \in \mathrm{Hor}_{\mathrm{Contr}}$ bring a new component

$$M_{ij} = \{q_{ij} \mid (q_{ij}, \ell_i) \in \mathrm{Hor}_{\mathrm{Contr}}\}\,.$$

Denote by $C_i = \cup M_{ij}$ the decomposition of C_i along elements of C' and observe that at most one component of M_{ij} has more than one point, i.e. $C' \preccurlyeq C$. Let us fix C' and determine the choices which give this partition C'. We argue according to the components M_{ij}:

– the configuration where M_{ij} is reduced to a singleton *for any* j can be obtained by following two ways:

— either we have cancelled all edges of C_i (i.e. $\alpha(\mathrm{Hor}_{\mathrm{Contr}}) \cap \mathrm{C_i} = \emptyset$). There is only one way for that and the contribution is q, corresponding to the suppression of the point ℓ_i;

— or we have cancelled all edges of C_i but one which has been connected (i.e. $\alpha(\mathrm{Hor}_{\mathrm{Contr}}) \cap \mathrm{C_i}$ is a singleton). There are $|C_i|$ possibilities for the choice of this edge and each has a contribution in v.

– if C' contains a component M_{ij} that is not reduced to a singleton, there is only one manner to reach this situation: $\alpha(\mathrm{Hor}_{\mathrm{Contr}}) \cap \mathrm{C_i} = \mathrm{M_{ij}}$. Its contribution is $v^{|M_{ij}|}$. Recall that the other components are reduced to a point corresponding to the suppression of an edge.

If we denote by $\psi^1_G \colon \mathbb{Z}[\mathcal{P}(r)] \to \mathbb{Z}[\mathcal{P}(r)]$ the operation "suppression of the horizontal edges", we have proved

$$\psi^1_G(C) = \sum_{C' \preccurlyeq C} \widetilde{\gamma}^{\#}_{v,q}(C, C')\; C'$$

where $\widetilde{\gamma}^{\#}_{v,q}(C, C')$ is determined by

$$\widetilde{\gamma}^{\#}_{v,q}(C, C') = \prod_i \begin{cases} v^{|\widetilde{M}_{ij}|} & \text{si } |\widetilde{M}_{ij}| \geq 2 \\ \\ q + |C_i| v & \text{si } |\widetilde{M}_{ij}| = 1, \end{cases} \tag{24}$$

where $C_i \cap C' = \cup_j M_{ij}$ and the $\widetilde{M_{ij}}$ are the components containing the greatest number of elements if it exists or $|\widetilde{M_{ij}}| = 1$.

Destruction of the vertical edges: Among the edges of G, let us fix a choice, $\mathrm{Vert}_{\mathrm{Supp}}$ and $W = \mathrm{Vert}_{\mathrm{Contr}}$, of edges which must be, respectively, cancelled or connected. We execute now the operations corresponding to theses choices, once ψ_G^1 has been applied. We denote by C" the new partition and we are interested in the different ways of reaching C''.

Recall that $\mathrm{Part}(W)$ is the partition of V obtained from the identification of the two vertices of any edge in W. The partition C" satisfies $C' \vee \mathrm{Part}(W) = C''$ and therefore $\psi_G(C) = \sum_{C'' \in \mathcal{P}(r)} (C : C'')\, C''$ with

$$(C : C'') = \sum_{\substack{C' \preccurlyeq C \\ W \subseteq V \\ C' \vee \mathrm{Part}(W) = C''}} v^{|W|}\, \widetilde{\gamma}^{\#}_{v,q}(C, C'). \tag{25}$$

The inversion formula of Möbius will lead to a separation of the variables C' and W. We have

$$(C : C\text{"}) = \sum_{C' \leq C\text{"}} \mu(C', C\text{"})[C : C'] \tag{26}$$

with

$$[C : C'] = \sum_{\substack{C_1 \preccurlyeq C \\ C_1 \leq C' \\ W \subseteq V \\ \mathrm{Part}(W) \leq C'}} v^{|W|}\, \widetilde{\gamma}^{\#}_{v,q}(C, C_1). \tag{27}$$

This last expression is determined by:

$$[C : C'] = \sum_{\substack{C_1 \preccurlyeq C \\ C_1 \leq C' \\ W \subset V \\ \mathrm{Part}(W) \leq C'}} v^{|W|} \widetilde{\gamma}^{\#}_{v,q}(C, C_1) = \sum_{\substack{C_1 \preccurlyeq C \\ C_1 \leq C'}} \widetilde{\gamma}^{\#}_{v,q}(C, C_1) \sum_{\substack{W \subset V \\ \mathrm{Part}(W) \leq C'}} v^{|W|}. \tag{28}$$

The last term is easy to compute if we introduce $\kappa_G(C')$ the number of edges of V which are connecting two points belonging to the same component of C'. We get

$$[C : C'] = (1+v)^{\kappa_G(C')} \sum_{\substack{C_1 \preccurlyeq C \\ C_1 \leq C'}} \widetilde{\gamma}^{\#}_{v,q}(C, C_1) = (1+v)^{\kappa_G(C')} \gamma^{\flat}_{v,q}(C, C') \tag{29}$$

(with the convention $0^0 = 1$). It now is sufficient now to compute this last sum component by component. Recall that $C_i \cap C' = \cup_j M_{ij}$ where the M_{ij} are distinct. We get

$$\gamma^\flat_{v,q}(C, C') = \prod_i \left\{ q + \sum_j \left[(v+1)^{|M_{ij}|} - 1 \right] \right\} \tag{30}$$

where the set inducing the product is the set of components C_i of C. Finally, we have

$$(C : C'') = \sum_{C' \leq C''} \mu(C', C'')\, \gamma^\flat_{v,q}(C, C')\, (v+1)^{\kappa_G(C')}. \tag{31}$$

This formula is remarkable in that it allows a perfect determination of the dependence in V. By gluing (30) and (31), we have a clear expression of ψ_G.

Warning!!! This is not a product of convolution because $\gamma^\flat_{v,q}(C, C')$ is not 0 if $C \not\leq C'$.

Remark: The formula (25) gives us

$$\psi_G(C_0) = (q+v)^r \sum_{W \subset V} v^{|W|}\, \mathrm{Part}(W) \tag{32}$$

where C_0 is the hat with r components.

7 Some results.

We use the equation (15) in the case of the linear graph with r vertices, L_r. This equation can be simplified if we observe that the expression of ψ_G takes the shape $\phi_G \times M$ with $M(C'') = \sum_{C'} \mu(C', C'')\, C'$. To avoid computing M, we observe that

$$(\mathrm{Id} - X\phi_G M)^{-1} = \mathbb{1} \cdot (\mathbb{1} - X\phi_G)^{-1} \tag{33}$$

where $\mathbb{1}$ is the matrix $\mathbb{1}_{C \leq C''}$. This has also another advantage: the coefficients of the matrix we have to inverse are simpler. We do not even have to inverse this matrix but to determine a pre-image by it. This can be done by classical Gauss procedures. The result of these remarks is ... remarkable! In case $r = 2$, the series can be computed in a tenth of second. For $r = 3$, we need about ten seconds. With the previous algorithm ignoring (33), we had stopped the computation after 14 hours... The case $r = 4$ is more problematic because of the size of the data. The particular case $r = 4$, $v = -1$ takes 15 seconds.

Let us detail the cases $r = 2$, 3 *and* 4 *:* First, we compute $\Sigma_2(q,v)$:

$$\frac{qv + q^2 + X\left(-q^2v^2 - q^2v^3\right)}{1 - X\left(3qv + q^2 + 4v^2 + v^3\right) + X^2\left(v^4 + v^5 + 2qv^3 + 2qv^4 + q^2v^2 + q^2v^3\right)} \tag{34}$$

For $v = 0, -1$ we have:

$$\Sigma_2(q,0) = \frac{q^2}{1 - Xq^2} \quad , \quad \Sigma_2(q,-1) = \frac{q^2 - q}{1 - (3 - 3q + q^2)X} \tag{35}$$

The corresponding Taylor development is:

$$\begin{aligned}
&\Sigma_2(q,v) = qv + q^2 + X(q^4 + 4qv^3 + 4q^3v + qv^4 + 6q^2v^2) \\
&+X^2(q^6+15qv^5+7q^5v+7qv^6+qv^7+33q^2v^4+35q^3v^3+21q^4v^2+6q^2v^5+2q^3v^4)+\mathcal{O}(X^3)
\end{aligned}$$

We get $\Sigma_3(q,v) = P_3(q,v)/Q_3(q,v)$ with

$$\begin{aligned}
P_3(q,v) =& q^3 + qv^2 + 2q^2v \\
&+ X(qv^6 - 9q^2v^4 - 9q^3v^3 - 2q^4v^2 - 7q^2v^5 - 6q^3v^4 - q^4v^3 - 2q^2v^6 - q^3v^5) \\
&+ X^2(-qv^8 - qv^9 + 3q^2v^7 + 11q^3v^6 + 8q^4v^5 + q^5v^4 + 5q^2v^8 + 17q^3v^7 \\
&\quad + 12q^4v^6 + q^5v^5 + 2q^2v^9 + 7q^3v^8 + 5q^4v^7 + q^3v^9 + q^4v^8) \\
&+ X^3(-q^3v^9 - 2q^4v^8 - q^5v^7 - 3q^3v^{10} - 6q^4v^9 - 3q^5v^8 - 3q^3v^{11} - 6q^4v^{10} \\
&\quad - 3q^5v^9 - q^3v^{12} - 2q^4v^{11} - q^5v^{10})
\end{aligned}$$

where the factor of X is effectively $(qv^6 - 9q^2v^4 \dots)$ and not $(-qv^6 - 9q^2v^4 \dots)$. A similar remark can be done for the factor of X^2. Then

$$\begin{aligned}
Q_3(q,v) =& 1 \\
&+ X(-q^3 - 15v^3 - 6v^4 - v^5 - 12qv^2 - 5q^2v - qv^3) \\
&+ X^2(32v^6 + 30v^7 + 10v^8 + v^9 + 62qv^5 + 51qv^6 + 14qv^7 + qv^8 \\
&\quad + 43q^2v^4 + 15q^3v^3 + 2q^4v^2 + 29q^2v^5 + 9q^3v^4 + q^4v^3 + 5q^2v^6 + q^3v^5) \\
&+ X^3(-15v^9 - 25v^{10} - 12v^{11} - 2v^{12} - 49qv^8 - 79qv^9 - 36qv^{10} - 6qv^{11} \\
&\quad - 62q^2v^7 - 38q^3v^6 - 11q^4v^5 - q^5v^4 - 96q^2v^8 - 56q^3v^7 - 15q^4v^6 \\
&\quad - q^5v^5 - 41q^2v^9 - 22q^3v^8 - 5q^4v^7 - 7q^2v^{10} - 4q^3v^9 - q^4v^8) \\
&+ X^4(v^{12} + 3v^{13} + 3v^{14} + v^{15} + 5qv^{11} + 15qv^{12} + 15qv^{13} + 5qv^{14} \\
&\quad + 10q^2v^{10} + 10q^3v^9 + 5q^4v^8 + q^5v^7 + 30q^2v^{11} + 30q^3v^{10} + 15q^4v^9 \\
&\quad + 3q^5v^8 + 30q^2v^{12} + 30q^3v^{11} + 15q^4v^{10} + 3q^5v^9 + 10q^2v^{13} \\
&\quad + 10q^3v^{12} + 5q^4v^{11} + q^5v^{10})
\end{aligned}$$

We observe that Q is of degree 4 (and not 5 as the dimension of the set of hats) and

$$\Sigma_3(q,-1) = \frac{q - 2q^2 + q^3 + X(q - 4q^2 + 4q^3 - q^4)}{1 + X(10 - 11q + 5q^2 - q^3) + X^2(11 - 24q + 19q^2 - 7q^3 + q^4)}.$$

This example shows that the polynomials in (q, v) which appear (except the coefficient of 1 in the denominator) are not all divisible by $q + v$. Then

$$\Sigma_4(q,-1) = \frac{\begin{array}{c} q - 3q^2 + 3q^3 - q^4 + X(-6q + 32q^2 - 56q^3 + 43q^4 - 15q^5 + 2q^6) \\ + X^2(7q - 53q^2 + 145q^3 - 196q^4 + 144q^5 - 58q^6 + 12q^7 - q^8) \end{array}}{\begin{array}{l} -1 + X(33 - 41q + 23q^2 - 7q^3 + q^4) \\ \quad + X^2(-207 + 517q - 553q^2 + 329q^3 - 116q^4 + 23q^5 - 2q^6) \\ \quad + X^3(279 - 1084q + 1858q^2 - 1829q^3 + 1130q^4 - 449q^5 + 112q^6 \\ \qquad - 16q^7 + q^8) \end{array}}$$

is still a case of degeneracy because the surrounding set is of dimension 14. With the same procedure, 41 seconds give us the dichromatic polynomial of a 4x4 squaring:

$$\begin{aligned} Z_{4,4} = & q^{16} + 100352qv^{15} + 24q^{15}v + 175264qv^{16} + 151160qv^{17} + 83956qv^{18} \\ & + 32888qv^{19} + 9358qv^{20} + 1920qv^{21} + 272qv^{22} + 24qv^{23} + qv^{24} \\ & + 438352q^2v^{14} + 994000q^3v^{13} + 1528336q^4v^{12} + 1760208q^5v^{11} \\ & + 1593044q^6v^{10} + 1161496q^7v^9 + 690436q^8v^8 + 335652q^9v^7 + 132874q^{10}v^6 \\ & + 42324q^{11}v^5 + 10617q^{12}v^4 + 2024q^{13}v^3 + 276q^{14}v^2 + 591072q^2v^{15} \\ & + 1030872q^3v^{14} + 1206968q^4v^{13} + 1041372q^5v^{12} + 689740q^6v^{11} \\ & + 356536q^7v^{10} + 143948q^8v^9 + 44807q^9v^8 + 10440q^{10}v^7 + 1722q^{11}v^6 \\ & + 180q^{12}v^5 + 9q^{13}v^4 + 388261q^2v^{16} + 512408q^3v^{15} + 447544q^4v^{14} \\ & + 281632q^5v^{13} + 131628q^6v^{12} + 45832q^7v^{11} + 11654q^8v^{10} + 2060q^9v^9 \\ & + 228q^{10}v^8 + 12q^{11}v^7 + 161336q^2v^{17} + 157560q^3v^{16} + 99556q^4v^{15} \\ & + 43736q^5v^{14} + 13464q^6v^{13} + 2816q^7v^{12} + 364q^8v^{11} + 22q^9v^{10} \\ & + 46164q^2v^{18} + 32368q^3v^{17} + 14177q^4v^{16} + 4100q^5v^{15} + 752q^6v^{14} + 80q^7v^{13} \\ & + 4q^8v^{12} + 9248q^2v^{19} + 4424q^3v^{18} + 1236q^4v^{17} + 209q^5v^{16} + 16q^6v^{15} \\ & + 1254q^2v^{20} + 368q^3v^{19} + 52q^4v^{18} + 4q^5v^{17} + 104q^2v^{21} + 14q^3v^{20} + 4q^2v^{22} \end{aligned}$$

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