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CORRIGENDUM TO EXPLICIT ESTIMATES ON SEVERAL SUMMATORY FUNCTIONS INVOLVING THE MOEBIUS FUNCTION

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ABSTRACT. Our earlier paper [Math. Comp. 84 (2015), no. 293, 1359–1387] had a sign mistake (in the definition of h' in Lemma 3.2) that has some consequences. We present the required modifications. The final results are all improved.

1. Overview of the modifications

- In the last parenthesis of the paragraph following Theorem 1.4, the value 0.321 should be replaced by 0.310.
- Theorem 1.5 becomes:

Theorem 1.5. *We have*

$$|\check{m}(x) - 1| \leq \frac{0.00140}{\log x} \quad (x \geq 9\,950).$$

The following simpler bound also holds:

$$|\check{m}(x) - 1| \log x \leq \begin{cases} 0.213 < 3/14 & \textit{when } x \geq 1, \\ 0.0203 < 4/197 & \textit{when } x \geq 16, \\ 1/396 & \textit{when } x \geq 3\,162 \\ 1/714 & \textit{when } x \geq 9\,549. \end{cases}$$

The subsequent corollaries are unchanged.

- Theorem 1.7 becomes:

Theorem 1.7. *We have, when $x \geq 1$:*

$$|\check{\check{m}}(x) - 2\log x + 2\gamma_0| \leq \frac{3/2}{x^2}\int_1^x |M(t)|dt + \frac{4 + 2\gamma_0}{x}$$

The subsequent corollaries are unchanged.

- Theorem 1.8 becomes:

Theorem 1.8. *We have*

$$|\check{\check{m}}(x) - 2\log x + 2\gamma_0| \leq \frac{0.00949\log x - 0.0813}{(\log x)^2} \quad (x \geq 5760)$$

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The following simpler bounds also hold:

$$|\check{\check{m}}(x) - 2\log x + 2\gamma_0|\log x \leq \begin{cases} 0.2062 < 5/24 & \text{when } x > 1, \\ 1/105 & \text{when } x \geq 9. \end{cases}$$

The subsequent corollaries are unchanged.

2. An additional lemma for clarification

Lemma 2.0. *Let φ be continuous and piecewise continuously differentiable over $[1, x]$, and let (a_n) be a sequence of complex numbers. We have*

$$\sum_{n \leq x} a_n \varphi(n/x) = \varphi(1) \sum_{n \leq x} a_n + \int_1^x \sum_{n \leq x/t} a_n \varphi'(t) dt.$$

Assuming φ to be absolutely continuous would be enough with the proper definition of φ'.

3. Modifications inside the paper

- Lemma 3.2 and its proof becomes:

Lemma 3.2. *When $x \geq 1$, we have*

$$\check{m}(x) - 1 = \frac{6 - 8\gamma_0}{3x} - \frac{5 - 4\gamma_0}{x^2} + \frac{6 - 4\gamma_0}{3x^4} + \frac{1}{x}\int_1^x M(x/t) h'(t) dt,$$

where $h'(t) = (\frac{3}{2} - \gamma_0)\varepsilon_1'(t) - g'(t)$ is continuous and differentiable except at integers where it has left and right derivatives. It satisfies

$$0 \leq t^2 |h'(t)| \leq \tfrac{7}{4} - \gamma_0 \tag{3.6}$$

and $t^2 h'(t) = (\frac{7}{4} - \gamma_0)(2\{t\} - 1)^2 - \frac{1}{6} + \mathcal{O}^*(\frac{7}{25}/t)$.

Theorem 1.4 is a straightforward consequence of this lemma.

Proof. Indeed, we have already reached

$$\check{m}(x) - 1 = (\tfrac{3}{2} - \gamma_0) m(x) - (\tfrac{3}{2} - \gamma_0)\frac{M(x)}{x} - \frac{2}{x} + \frac{1}{x^2} - \frac{1}{x}\int_1^x M(x/t) g'(t) dt,$$

where the continuous and piecewise differentiable function g is defined by:

$$g(x) = \frac{-1}{12x} + x\varepsilon_{6,0}(x) \quad (g(1) = \tfrac{1}{2} - \gamma_0). \tag{3.7}$$

We further recall (2.1):

$$m(x) = \frac{M(x)}{x} + \frac{4(1 - x^{-1})^2}{x} - \frac{4(1 - x^{-1})^3}{3x} + \frac{1}{x}\int_1^x M(x/t)\varepsilon_1'(t) dt.$$

This leads to

$$x(\check{m}(x) - 1) = \frac{6 - 8\gamma_0}{3} - \frac{5 - 4\gamma_0}{x} + \frac{6 - 4\gamma_0}{3x^3} - (\tfrac{1}{2} - \gamma_0)\frac{M(x)}{x} + \int_1^x M(x/t)((\tfrac{3}{2} - \gamma_0)\varepsilon_1'(t) - g'(t)) dt.$$

With the notation

$$h'(t) = (\tfrac{3}{2} - \gamma_0)\varepsilon_1'(t) - g'(t),$$

we find that (with $u = \{t\}$)

$$(3.1)\quad t^2 h'(t) = u^2 - u + \tfrac{1}{12} + (\tfrac{3}{2} - \gamma_0)(2u-1)^2 - t^2\int_t^\infty \frac{B_2(v)dv}{v^3} + (3 - 2\gamma_0)\frac{(2u-1)(u-u^2)}{t} + (\tfrac{3}{2} - \gamma_0)\frac{(u-u^2)^2}{t^2}.$$

The polynomial in u of the first line reads $(\frac{7}{4} - \gamma_0)(2u-1)^2 - \frac{1}{6}$. What remains is $h_*'(t)$, and by using the relation of the Bernoulli polynomials $B_{n-1} = B_n'/n$, we obtain:

$$(3.2)\quad th_*'(t) = \frac{B_3(t)}{3} - t^3\int_t^\infty \frac{B_3(v)dv}{v^4} + (3 - 2\gamma_0)(2u-1)(u-u^2) + (\tfrac{3}{2} - \gamma_0)\frac{(u-u^2)^2}{t}.$$

Recall that $B_3(t) = u^3 - \frac{3}{2}u^2 + \frac{1}{2}u$ is bounded in absolute value by $\sqrt{3}/36 = 0.04811\cdots$. Hence

$$|th_*'(t)| \le 2\frac{0.0482}{3} + (3 - 2\gamma_0)\times 0.0973 + \frac{\frac{3}{2} - \gamma_0}{16} \le 0.27 \le \tfrac{7}{25}.$$

On the other hand, we recalled after (2.3) that $0 \le \varepsilon_1'(t) \le 1/t^2$. Concerning $g'(t)$, we first notice that $|B_2(v)| \le 1/6$, so that

$$|t^2 g'(t)| \le \left|\{t\}^2 - \{t\} + \tfrac{1}{12}\right| + \frac{1}{6\times 2} \le 1/4.$$

The lemma readily follows. □

- Lemma 4.1 and its proof now becomes:

Lemma 4.1. *With the notation of Lemma 3.2, we have*

$$\int_n^{n+1} |h'(t)|dt \le 0.310 \int_n^{n+1} dt/t^2$$

Proof. We use (3.1) to compute h', though we need to find a better expression for $\int_t^\infty B_2(v)dv/v^3$. When b is an integer and $b \le a < b+1$, we have

$$\begin{aligned}\int_a^{b+1} B_2(v)\frac{dv}{v^3} &= \int_a^{b+1} \frac{(v-b)^2 - (v-b) + \frac{1}{6}}{v^3}dv \\ &= \int_a^{b+1} \frac{v^2 - (2b+1)v + b^2 + b + \frac{1}{6}}{v^3}dv \\ &= \log\frac{b+1}{a} - \frac{2b+1}{a} + \frac{2b+1}{b+1} + \frac{b^2+b+\frac{1}{6}}{2a^2} - \frac{b^2+b+\frac{1}{6}}{2(b+1)^2}.\end{aligned}$$

We use the above with $a = t$ and $b = [t]$ and then with $a = b = n$ for integral n. We thus find that

$$\int_t^\infty \frac{B_2(v)dv}{v^3} = \log\frac{[t]+1}{t} - \frac{2[t]+1}{t} + \frac{2[t]+1}{[t]+1} + \frac{[t]^2+[t]+\frac{1}{6}}{2t^2} - \frac{[t]^2+[t]+\frac{1}{6}}{2([t]+1)^2} + \sum_{n\geq[t]+1}\left(\log\frac{n+1}{n} - \frac{(2n+1)(6n^2+6n-1)}{12n^2(n+1)^2}\right).$$

Each summand of the sum over n is negative. By using this expression and the numerical integration of GP-pari with 70 digits of `realprecision`, we check that

$$\forall n \in \{1,\cdots,400\}, \qquad \int_n^{n+1} |h'(t)|dt \leq 0.30922\cdots\int_n^{n+1} dt/t^2.$$

The worst constant is reached when $n = 1$. We specify that the successive constants, when n varies, are *not* decreasing. When n is larger we use

$$\int_n^{n+1} |h'(t)|dt \leq \int_n^{n+1} \frac{|(\frac{7}{4}-\gamma_0)(2(t-n)-1)^2 - 1/6|}{t^2}dt + \frac{7(2n+1)}{50n^2(n+1)^2}.$$

Concerning the remaining integral, we set $\epsilon = 1/\sqrt{42-24\gamma_0}$ and first note that

$$\begin{aligned} n^2\int_n^{n+1} &\frac{|(\frac{7}{4}-\gamma_0)(2(t-n)-1)^2-1/6|}{t^2}dt \\ &\leq \int_0^{\frac{1}{2}-\epsilon}\big((\tfrac{7}{4}-\gamma_0)(2u-1)^2-1/6\big)du \\ &\quad - \int_{\frac{1}{2}-\epsilon}^{\frac{1}{2}+\epsilon}\big((\tfrac{7}{4}-\gamma_0)(2u-1)^2-1/6\big)du \\ &\quad + \int_{\frac{1}{2}+\epsilon}^{1}\big((\tfrac{7}{4}-\gamma_0)(2u-1)^2-1/6\big)du, \end{aligned}$$

which is not more that 0.3081. Hence

$$n(n+1)\int_n^{n+1}|h'(t)|dt \leq 0.3081\frac{n+1}{n} + \frac{7(2n+1)}{50n(n+1)},$$

which is indeed not more than 0.310 when $n \geq 310$. □

- The end of section 4 contains a proof of Theorem 1.5 and Corollary 1.6 and should be modified as follows:

Thus, when $x \geq D_0$ and on recalling Lemma 2.3:

$$\begin{aligned} \int_1^x |M(x/t)h'(t)|dt &\leq \int_1^{x/D_0}|M(x/t)h'(t)|dt + (\tfrac{7}{4}-\gamma_0)\frac{15\,512\,101}{x} \\ &\leq 0.310\cdot x\int_1^{x/D_0}\frac{0.0146\log(x/(t+1))-0.1098}{(\log(x/(t+1)))^2}\frac{dt}{t^2(t+1)} + \frac{18\,192\,350}{x} \\ &\leq 0.310\cdot x\int_{xD_0/(x+D_0)}^{x/2}\frac{0.0146\log u - 0.1098}{(\log u)^2}\frac{udu}{(x-u)^2} + \frac{18\,192\,350}{x} \\ &\leq 0.00139x/\log x \end{aligned}$$

by Lemma 4.2. We then use Lemma 3.2. (notice that $\frac{1}{2} - \gamma_0 < 0$). This yields

$$|\check{m}(x) - 1| \le \frac{2}{x} + \frac{0.00139}{\log x} \le \frac{0.00140}{\log x}$$

when $x \ge D_0$.

We then appeal to Lemma 10.2 to extend the range to $x \ge 18\,100$ and check numerically its extension to the range $9\,550 \le x \le 18\,100$. Theorem 1.5 follows. The GP-script is called `AsymptoticBoundsFor_checkm.gp` and the main function is `getboundsbisaux`.

The proof of Corollary 1.6 is immediate.

- Lemma 5.1 and the beginning of its proof becomes:

Lemma 5.1. *When $x \ge 1$, we have*

$$\check{\check{m}}(x) - 2\log x + 2\gamma_0 = \frac{1}{x}\int_1^x M(x/t)k_2'(t)dt + \frac{K_2(1/x)}{x},$$

where

$$K_2(v) = 1 + 4\gamma_0 + (3 - 2\gamma_0)v - (8\gamma_0^2 - 12\gamma_0 + 2\gamma_1 + 2)\frac{(1-v)(2+v)}{3} \tag{3.3}$$

satisfies $4.5 \le K_2(v) \le 4 + 2\gamma_0$ *when* $0 \le v \le 1$ *and*

$$k_2'(t) = (2\gamma_0^2 - 3\gamma_0 + 2\gamma_1 + \tfrac{1}{2})\varepsilon_1(t) + 2(\gamma_0 - \log t)t\varepsilon_{6,0}(t) + 2t\varepsilon_{6,1}(t) - \frac{2\gamma_0 - 1}{12t} \tag{3.4}$$

is continuous and differentiable except at integers where it has left and right derivatives. It satisfies

$$t^2|k_2'(t)| \le 1.46. \tag{3.5}$$

Theorem 1.7 is a straightforward consequence of this lemma. Let us specify that h' is typically not continuous at integer points.

Proof. Indeed, we have already reached

$$\tfrac{1}{2}\check{\check{m}}(x) + (\gamma_0 - \tfrac{1}{2})(\check{m}(x) - 1) - (\tfrac{1}{2}\gamma_0 + \gamma_1 + \tfrac{1}{4})(m(x) - M(x)x^{-1}) + \frac{1}{x}\int_1^x M(x/t)q'(t)dt = \log x - \gamma_0 + \frac{1}{x^2} - \frac{1}{2x},$$

with the notation

$$q(t) = t(\log t - \tfrac{1}{2})\varepsilon_{6,0}(t) - t\varepsilon_{6,1}(t) \qquad (q(1) = -\tfrac{1}{2}\gamma_0 + \gamma_1). \tag{3.6}$$

During the proof of Lemma 3.2, we have noticed that

$$\check{m}(x) - 1 = (\tfrac{3}{2} - \gamma_0)(m(x) - M(x)x^{-1}) - \frac{2}{x} + \frac{1}{x^2} - \frac{1}{x}\int_1^x M(x/t)g'(t)dt,$$

where the function g is defined by

$$g(t) = \frac{-1}{12t} + t\varepsilon_{6,0}(t). \tag{3.7}$$

Combining both identities leads to

$$\tfrac{1}{2}\check{\check{m}}(x) + (-\gamma_0^2 + \tfrac{3}{2}\gamma_0 - \gamma_1 - \tfrac{1}{4})(m(x) - M(x)x^{-1}) + \frac{1}{x}\int_1^x M(x/t)(q'(t) - (\gamma_0 - \tfrac{1}{2})g'(t))dt = \log x - \gamma_0 + \frac{1+4\gamma_0}{2x} + \frac{3-2\gamma_0}{2x^2}. \tag{3.8}$$

We further recall (2.1):

$$m(x) - \frac{M(x)}{x} = \frac{4(1-x^{-1})^2}{x} - \frac{4(1-x^{-1})^3}{3x} + \frac{1}{x}\int_1^x M(x/t)\varepsilon_1'(t)dt$$

and thus

$$\check{\check{m}}(x) - 2\log x + 2\gamma_0 = \frac{1}{x}\int_1^x M(x/t)k_2'(t)dt + \frac{K_2(1/x)}{x},$$

where

$$k_2'(t) = -2(-\gamma_0^2 + \tfrac{3}{2}\gamma_0 - \gamma_1 - \tfrac{1}{4})\varepsilon_1'(t) - 2q'(t) + (2\gamma_0 - 1)g'(t)$$

and K_2 is defined in (5.1).

The sequel of the proof is unchanged. □

Reference

[1] O. Ramaré, *Explicit estimates on several summatory functions involving the Moebius function*, Math. Comp. **84** (2015), no. 293, 1359–1387, DOI 10.1090/S0025-5718-2014-02914-1. MR3315512

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