ACTA ARITHMETICA
Online First version

An application of counting ideals in ray classes

by

SANOLI GUN (Chennai), OLIVIER RAMARÉ (Marseille) and
JYOTHSNAA SIVARAMAN (Bengaluru)

Abstract. We prove a fully explicit generalized Brun–Titchmarsh theorem for an imaginary quadratic field $\mathbf{K}$. More precisely, for any finite family of linearly independent linear forms with coefficients in $\mathcal{O}_{\mathbf{K}}$, we count the number of integers at which all these linear forms take prime values in $\mathcal{O}_{\mathbf{K}}$.

1. Introduction and statement of the theorem. Throughout this article, $\mathbf{K}$ will denote an imaginary quadratic field with discriminant $d_{\mathbf{K}}$, $h_{\mathbf{K}}$ the class number of $\mathcal{O}_{\mathbf{K}}$ and $|\mu_{\mathbf{K}}|$ the number of roots of unity in $\mathcal{O}_{\mathbf{K}}$. We will denote by $\zeta_{\mathbf{K}}$ the Dedekind zeta-function of $\mathbf{K}$, and its residue at $s = 1$ by $\alpha_{\mathbf{K}}$. Further, we use $\mathcal{P}_{\mathbf{K}}$ to denote the set of prime ideals of $\mathcal{O}_{\mathbf{K}}$, and $\mathcal{Q}$ for the set of all prime elements of $\mathcal{O}_{\mathbf{K}}$. A non-zero element α of $\mathcal{O}_{\mathbf{K}}$ which is not a unit is said to be *prime* if it generates a prime ideal. Two prime elements α and β in $\mathcal{O}_{\mathbf{K}}$ are called *associates* if there exists a unit $u \in \mathcal{O}_{\mathbf{K}}$ such that $\alpha = u\beta$. While associate primes generate the same prime ideal, we will count primes in $\mathcal{O}_{\mathbf{K}}$ along with their associates. For example, when $\mathbf{K} = \mathbb{Q}$, we will count both the associate primes 3 and -3 in $\mathbb{Z}$. We will denote by $\omega_{\mathbf{K}}(\mathfrak{b})$ the number of distinct prime ideals of $\mathcal{O}_{\mathbf{K}}$ which appear in the factorization of the ideal $\mathfrak{b}$ in $\mathcal{O}_{\mathbf{K}}$, by $\mathfrak{N}$ the (absolute) norm and by $\pi_{\mathbf{K}}(x)$ the number of prime ideals of $\mathcal{O}_{\mathbf{K}}$ with norm at most x.

Our aim is to prove a fully explicit generalization of the Brun–Titchmarsh theorem for several linear forms taking values in $\mathcal{Q}$. This is a natural generalization of the problem of finding an upper bound for the number of prime values that can be taken by a set of n linear forms simultaneously. This question has been addressed in considerable detail in the literature.

2020 *Mathematics Subject Classification*: Primary 11R44; Secondary 11N35, 11N36, 11N32.
Key words and phrases: Brun–Titchmarsh theorem, Selberg sieve, primes represented by polynomials.
Received 6 April 2023; revised 28 May 2024.
Published online *.

DOI: 10.4064/aa230406-10-6 © Instytut Matematyczny PAN, ***

The study of such generalizations finds its origin in the twin prime and prime k-tuple conjectures. However, the problem has been placed in the more general context of linear forms by Dickson's conjecture [5], which states the following.

CONJECTURE 1 (Dickson's conjecture [5]). Let $F_1, \dots, F_n \in \mathbb{Z}[x]$ be distinct irreducible linear polynomials with positive leading coefficients. Also suppose that the product $\prod_{i=1}^n F_i(x)$ has no fixed prime divisor. Then the polynomials $F_i(x)$ simultaneously take prime values infinitely often.

A quantitative version of Dickson's conjecture was given by Bateman–Horn in 1962 (see [1, 2]). For any polynomial $F \in \mathcal{O}_{\mathbf{K}}[x]$ and prime ideal $\mathfrak{p}$, let us first define

$$\rho_F(\mathfrak{p}) = \#\{m \in \mathcal{O}_{\mathbf{K}}/\mathfrak{p}\mathcal{O}_{\mathbf{K}} : F(m) \equiv 0 \bmod \mathfrak{p}\}.$$

We now state the Bateman–Horn conjecture.

CONJECTURE 2 (Bateman–Horn conjecture [1, 2]). Let $F_1, \dots, F_n \in \mathbb{Z}[x]$ be distinct irreducible linear polynomials with positive leading coefficients. Also suppose that the product $\prod_{i=1}^n F_i(x)$ has no fixed prime divisor. Then

$$\sum_{\substack{1 \le k \le x \\ F_i(k) \text{ is prime } \forall i}} 1 = \prod_p \left\{ \left(1 - \frac{1}{p}\right)^{-n} \left(1 - \frac{\rho_F(p)}{p}\right) \right\} \cdot \int_2^x \frac{dt}{\log^n t} \, (1 + o(1))$$

as $x \to \infty$.

The only case in which these conjectures have been resolved is that of a single linear polynomial, which is same as the prime number theorem for primes in arithmetic progressions. For other well known cases, finding even a lower bound in place of the asymptotic is notoriously difficult. For instance, the case of the polynomials

$$F_1(x) = x \quad \text{and} \quad F_2(x) = x + 2$$

is nothing other than the twin prime conjecture. More generally, for an appropriate choice of a k-tuple $(h_1, \dots, h_k)$, the polynomials

$$F_1(x) = x + h_1, \ \dots, \ F_k(x) = x + h_k$$

give the Hardy–Littlewood k-tuple conjecture. The case of the polynomials

$$F_1(x) = x \quad \text{and} \quad F_2(x) = 2x + 1$$

amounts to finding Sophie Germain primes.

However, upper bounds close to the one suggested by the asymptotic are known using Selberg sieve techniques. For instance, one may find the following theorem in [8, pp. 157–159].

THEOREM 1. *Given distinct irreducible linear polynomials $F_1, \ldots, F_n$ in $\mathbb{Z}[x]$ with positive leading coefficients, let $F(x) = \prod_{i=1}^n F_i(x)$. If $\rho_F(p) < p$ for all primes p, then*

$$\sum_{\substack{1\le k\le x\\ F_i(k)\ \text{is prime}\ \forall i}} 1 \le \prod_p \left\{ \left(1-\frac{1}{p}\right)^{-n} \left(1-\frac{\rho_F(p)}{p}\right) \right\} \frac{2^n n! x}{\log^n x}\left(1+\mathrm{O}_F\left(\frac{\log\log 3x}{\log x}\right)\right).$$

In this article, we show that an analogous bound can be obtained if we consider prime elements in an imaginary quadratic field instead of the rationals. Further, our bounds are fully explicit. An application of such a bound is presented in [9]. On the other hand, the current paper demonstrates an application of the main theorems of [7].

THEOREM 2. *Let u be a positive real number, $n > 1$ be an integer and $a_i \in \mathcal{O}_{\mathbf{K}} \setminus \{0\}$ for $1 \le i \le n$ be distinct. Assume that $(a_i\mathcal{O}_{\mathbf{K}}, b_i\mathcal{O}_{\mathbf{K}}) = \mathcal{O}_{\mathbf{K}}$ for $1 \le i \le n$, $(a_i\mathcal{O}_{\mathbf{K}} : 1 \le i \le n) = \mathcal{O}_{\mathbf{K}}$ and*

$$E = \prod_{i=1}^n a_i \prod_{1\le i<j\le n} (a_ib_j - a_jb_i) \neq 0.$$

Further, let $F = \prod_{i=1}^n (a_ix + b_i)$ and let $\mathcal{Q}$ denote the set of prime elements of $\mathcal{O}_{\mathbf{K}}$. Then for $u \ge [U(\mathbf{K}, a_1b_1)]^4$, we have

$$\sum_{\substack{\mathfrak{N}((\alpha))\le u\\ \forall i,\, b_i+a_i\alpha\in\mathcal{Q}}} 1 \le \frac{5n!\,|\mu_{\mathbf{K}}|}{2\alpha_{\mathbf{K}}^{n-1}h_{\mathbf{K}}}\cdot\mathcal{S}\cdot\frac{u}{(\log Cu^{1/4})^n},$$

where

$$U(\mathbf{K}, a_1b_1) = \frac{\exp(18(n+1)L_{\mathbf{K}})}{C},$$

$$C = \frac{n!\pi}{3^{23n^3} n^{17n}\mathfrak{N}((a_1b_1))\alpha_{\mathbf{K}}^n\sqrt{|d_{\mathbf{K}}|}},$$

$$L_{\mathbf{K}} = n\omega_{\mathbf{K}}((E)) + n\omega_{\mathbf{K}}\Big(\prod_{\mathfrak{N}(\mathfrak{p})\le n}\mathfrak{p}\Big) + 10n^3 + n\frac{e^{28}|d_{\mathbf{K}}|^{1/3}\log|d_{\mathbf{K}}|}{\alpha_{\mathbf{K}}},$$

$$\mathcal{S} = \Bigg(\prod_{\mathfrak{N}(\mathfrak{p})\le n}\frac{\mathfrak{N}(\mathfrak{p})}{\mathfrak{N}(\mathfrak{p})-1}\Bigg)^n \prod_{\mathfrak{N}(\mathfrak{p})>n}\left(1-\frac{\rho_F(\mathfrak{p})}{\mathfrak{N}(\mathfrak{p})}\right)\left(1-\frac{1}{\mathfrak{N}(\mathfrak{p})}\right)^{-n}.$$

The paper is organized as follows. In Section 2, we will state some notations and preliminaries required for the proof of our main theorem. In the same section, we will also recall the results used from [7]. In Section 3, we will prove some auxiliary lemmas, and finally we will use them in Section 4 to prove our theorem.

2. Notation and preliminaries. Let $\mathbf{K}$ be an imaginary quadratic field and $\mathcal{O}_{\mathbf{K}}$ be its ring of integers. For an ideal $\mathfrak{q} \in \mathcal{O}_{\mathbf{K}}$, let $H_{\mathfrak{q}}(\mathbf{K})$ denote the ray class group modulo $\mathfrak{q}$, and $h_{\mathbf{K},\mathfrak{q}}$ denote its cardinality. When $\mathfrak{q} = \mathcal{O}_{\mathbf{K}}$, the ray class group modulo $\mathcal{O}_{\mathbf{K}}$ is $\mathrm{Cl}_{\mathbf{K}}$. In this case, we denote $h_{\mathbf{K},\mathcal{O}_{\mathbf{K}}}$ by $h_{\mathbf{K}}$. Throughout the article, $\mathfrak{p}$ will denote a prime ideal in $\mathcal{O}_{\mathbf{K}}$ and p will denote a rational prime number. Further, we use $\varphi(\mathfrak{q})$ to denote the Euler-phi function:

$$\varphi(\mathfrak{q}) = \mathfrak{N}(\mathfrak{q}) \prod_{\mathfrak{p}|\mathfrak{q}} \left(1 - \frac{1}{\mathfrak{N}(\mathfrak{p})}\right). \tag{1}$$

Throughout this article, given an arithmetic function f and a positive arithmetic function g, $f(z) = \mathrm{O}^*(g(z))$ means that $|f(z)| \leq g(z)$. For any embedding σ of $\mathbf{K}$, the *Minkowski embedding* ψ of $\mathbf{K}$ to $\mathbb{R}^2$ maps x to $(\Re(\sigma(x)), \Im(\sigma(x)))$.

Let us begin with a counting theorem proved in [7].

THEOREM 3 (Gun, Ramaré and Sivaraman). *Let $\mathfrak{a}, \mathfrak{q}$ be coprime ideals of $\mathcal{O}_{\mathbf{K}}$, $\mathfrak{C}$ be the ideal class of $\mathfrak{a}\mathfrak{q}$ in the class group of $\mathcal{O}_{\mathbf{K}}$, and $\Lambda(\mathfrak{a}\mathfrak{q})$ be the lattice $\psi(\mathfrak{a}\mathfrak{q})$ in $\mathbb{R}^2$, where ψ is defined above. Also let*

$$S_\beta(\mathfrak{a}, \mathfrak{q}, t^2) = \{\alpha \in \mathfrak{a} : |\psi(\alpha)|^2 \leq t^2,\ \alpha \equiv \beta \bmod \mathfrak{q}\}$$

for some fixed $\beta \in \mathcal{O}_{\mathbf{K}}$. Then for any real number $t \geq 1$, we have

$$|S_\beta(\mathfrak{a}, \mathfrak{q}, t^2)| = \frac{(2\pi)}{\sqrt{|d_{\mathbf{K}}|}\, \mathfrak{N}(\mathfrak{a}\mathfrak{q})} t^2 + \mathrm{O}^*\left(\frac{10^{13.66}\mathfrak{N}(\mathfrak{C}^{-1})}{|\mathfrak{N}(\mathfrak{a}\mathfrak{q})|^{1/2}} t + 1\right), \tag{2}$$

where

$$\mathfrak{N}(\mathfrak{C}^{-1}) = \max_{\mathfrak{b} \in \mathfrak{C}^{-1}} \frac{1}{|\mathfrak{N}(\mathfrak{b})|^{1/2}}.$$

One can ignore 1 in the error term when $\mathfrak{q} = \mathcal{O}_{\mathbf{K}}$.

The Dedekind zeta-function. For $\Re s = \sigma > 1$, the *Dedekind zeta-function* is defined by

$$\zeta_{\mathbf{K}}(s) = \sum_{\mathfrak{a} \subseteq \mathcal{O}_{\mathbf{K}}} \frac{1}{\mathfrak{N}(\mathfrak{a})^s},$$

where $\mathfrak{a}$ ranges over the integral ideals of $\mathcal{O}_{\mathbf{K}}$. It has only a simple pole at $s = 1$ of residue $\alpha_{\mathbf{K}}$, say. When $\mathbf{K}$ is an imaginary quadratic field, we know from the analytic class number formula that

$$\alpha_{\mathbf{K}} = \frac{2\pi h_{\mathbf{K}}}{|\mu_{\mathbf{K}}|\sqrt{|d_{\mathbf{K}}|}}, \tag{3}$$

where $h_{\mathbf{K}}$, $d_{\mathbf{K}}$ and $|\mu_{\mathbf{K}}|$ are as before.

The next lemma is used to estimate the error term in Theorem 3.

LEMMA 4 (Debaene [3]). *Let $\mathfrak{b}_1, \mathfrak{b}_2, \dots$ be integral ideals of $\mathcal{O}_{\mathbf{K}}$, ordered so that $\mathfrak{N}(\mathfrak{b}_1) \le \mathfrak{N}(\mathfrak{b}_2) \le \cdots$. Then for any real number $y \ge 2$,*

$$\sum_{i=1}^{y} \mathfrak{N}(\mathfrak{b}_i)^{-1/2} \le 12 y^{1/2} (\log y)^{1/2}.$$

Finally, we recall two estimates which will be used in the course of our proof.

LEMMA 5 (Debaene [3]). *For any real number $y \ge 16$, we have*

$$\sum_{p \le y} \frac{1}{p} \le 0.666 + \log\log y.$$

LEMMA 6 (Rosser and Schoenfeld [12]). *For any real number $y \ge 1$, we have*

$$\sum_{p \le y} \frac{1}{p} \ge \log\log y.$$

3. Some intermediate lemmas. Selberg sieve. Let $n \ge 2$ be an integer, and $a_i x + b_i$ for $1 \le i \le n$ be n distinct linear forms with $a_i, b_i \in \mathcal{O}_{\mathbf{K}} \setminus \{0\}, (a_i\mathcal{O}_{\mathbf{K}}, b_i\mathcal{O}_{\mathbf{K}}) = \mathcal{O}_{\mathbf{K}}$ and $(a_i\mathcal{O}_{\mathbf{K}} : 1 \le i \le n) = \mathcal{O}_{\mathbf{K}}$. We further assume that

$$E = \prod_{i=1}^{n} a_i \prod_{1 \le i < j \le n} (a_i b_j - a_j b_i) \quad \text{and} \quad H = \prod_{\mathfrak{N}(\mathfrak{p}) \le n} \mathfrak{p}.$$

For an integral ideal $\mathfrak{b}$, let $\rho_F(\mathfrak{b})$ denote the number of solutions of

$$F(x) = \prod_{i=1}^{n} (a_i x + b_i) \equiv 0 \bmod \mathfrak{b}.$$

Applying the Chinese remainder theorem, we deduce that $\rho_F(\mathfrak{b})$ is a multiplicative function. Further, we observe that for any prime $\mathfrak{p}$, $\rho_F(\mathfrak{p}) < \mathfrak{N}(\mathfrak{p})$ when $\mathfrak{N}(\mathfrak{p}) > n$. Let us define the multiplicative functions

$$f(\mathfrak{b}) = \frac{\mathfrak{N}(\mathfrak{b})}{\rho_F(\mathfrak{b})} \quad \text{and} \quad f_1(\mathfrak{b}) = \sum_{\substack{\mathfrak{a} \mid \mathfrak{b} \\ \mathfrak{a} \subseteq \mathcal{O}_{\mathbf{K}}}} \mu(\mathfrak{a}) f\left(\frac{\mathfrak{b}}{\mathfrak{a}}\right). \tag{4}$$

Let z be a positive real number. We may assume that $\rho_F(\mathfrak{p}) < \mathfrak{N}(\mathfrak{p})$ when $\mathfrak{N}(\mathfrak{p}) \le z$ since otherwise no prime of norm greater than z would be counted in our sum. We have $f_1 > 0$ on the set of non-zero square-free integral ideals coprime to H. Also, $f(\mathcal{O}_{\mathbf{K}}) = 1$. Further, for an ideal $\mathfrak{e}$ of $\mathcal{O}_{\mathbf{K}}$ coprime to H and an integer $n \geq 2$, we define

$$\mathcal{P}_{\mathbf{K}}(z) = \prod_{n < \mathfrak{N}(\mathfrak{p}) \leq z} \mathfrak{p}, \quad S_{\mathfrak{e}}(z) = \sum_{\substack{\mathfrak{N}(\mathfrak{a}) \leq z, \\ (\mathfrak{a}, \mathfrak{e}H) = \mathcal{O}_K}} \frac{\mu^2(\mathfrak{a})}{f_1(\mathfrak{a})},$$

$$G(z) = S_{\mathcal{O}_{\mathbf{K}}}(z), \qquad \lambda_{\mathfrak{e}} = \mu(\mathfrak{e}) \frac{f(\mathfrak{e}) S_{\mathfrak{e}}\left(\frac{z}{\mathfrak{N}(\mathfrak{e})}\right)}{f_1(\mathfrak{e}) G(z)}.$$

Proposition 7. *For any ideal $\mathfrak{b} \,|\, \mathcal{P}_{\mathbf{K}}(z)$, we have $|\lambda_{\mathfrak{b}}| \leq 1$.*

Proof. For an integral ideal $\mathfrak{b}$ dividing $\mathcal{P}_{\mathbf{K}}(z)$, we have

$$\begin{aligned}
S_{\mathcal{O}_{\mathbf{K}}}(z) &= \sum_{\mathfrak{c} | \mathfrak{b}} \mu^2(\mathfrak{c}) / f_1(\mathfrak{c}) \sum_{\substack{\mathfrak{N}(\mathfrak{a}) \leq z / \mathfrak{N}(\mathfrak{c}) \\ (\mathfrak{a}, \mathfrak{b}H) = \mathcal{O}_{\mathbf{K}} \\ \mathfrak{a} \subseteq \mathcal{O}_{\mathbf{K}}}} \frac{\mu^2(\mathfrak{a})}{f_1(\mathfrak{a})} \\
&\geq \sum_{\mathfrak{c} | \mathfrak{b}} \mu^2(\mathfrak{c}) / f_1(\mathfrak{c}) \sum_{\substack{\mathfrak{N}(\mathfrak{a}) \leq z / \mathfrak{N}(\mathfrak{b}) \\ (\mathfrak{a}, \mathfrak{b}H) = \mathcal{O}_{\mathbf{K}} \\ \mathfrak{a} \subseteq \mathcal{O}_{\mathbf{K}}}} \frac{\mu^2(\mathfrak{a})}{f_1(\mathfrak{a})} \\
&= S_{\mathfrak{b}}\left(\frac{z}{\mathfrak{N}(\mathfrak{b})}\right) \sum_{\mathfrak{c} | \mathfrak{b}} \frac{\mu^2(\mathfrak{c})}{f_1(\mathfrak{c})} \\
&= \frac{f(\mathfrak{b})}{f_1(\mathfrak{b})} S_{\mathfrak{b}}\left(\frac{z}{\mathfrak{N}(\mathfrak{b})}\right).
\end{aligned}$$

The last step follows from the fact that $\mathfrak{b}$ is square-free and coprime to H. To see this, note that

$$\sum_{\mathfrak{a} | \mathfrak{b}} \frac{\mu^2(\mathfrak{a})}{f_1(\mathfrak{a})} = \prod_{\mathfrak{p} | \mathfrak{b}} \left(1 + \frac{1}{f_1(\mathfrak{p})}\right) = \frac{\sum_{\mathfrak{a} | \mathfrak{b}} f_1(\mathfrak{a})}{f_1(\mathfrak{b})} = \frac{f(\mathfrak{b})}{f_1(\mathfrak{b})}. \quad \blacksquare$$

We now recall a special case of a result of Lee [10, Theorem 1.1.3].

Theorem 8. *Let* $\mathbf{K}$ *be an imaginary quadratic field and $x \geq 2$. We have*

$$\sum_{\mathfrak{N}(\mathfrak{p}) \leq x} \frac{\log \mathfrak{N}(\mathfrak{p})}{\mathfrak{N}(\mathfrak{p})} = \log x + \mathrm{O}^*\left(2.52 + \frac{e^{28} |d_{\mathbf{K}}|^{1/3} \log |d_{\mathbf{K}}|}{\alpha_{\mathbf{K}}}\right).$$

Using the above theorem, we can prove the following asymptotic.

LEMMA 9. *Let $x \geq 2$ be a real number. Then*

$$\sum_{n < \mathfrak{N}(\mathfrak{p}) \leq x} \frac{\rho_F(\mathfrak{p}) \log \mathfrak{N}(\mathfrak{p})}{\mathfrak{N}(\mathfrak{p})} = n \log x + \mathrm{O}^*\left(n \left(\omega_{\mathbf{K}}(E) + \omega_{\mathbf{K}}(H) + 2.52 + \frac{e^{28} |d_{\mathbf{K}}|^{1/3} \log |d_{\mathbf{K}}|}{\alpha_{\mathbf{K}}} \right) \right).$$

Proof. We first note that since $a_i \mathcal{O}_{\mathbf{K}}$ and $b_i \mathcal{O}_{\mathbf{K}}$ are coprime, we have $\rho_F(\mathfrak{p}) \leq n$. We can replace the sum on the left-hand side of the assertion by

$$\sum_{\mathfrak{N}(\mathfrak{p}) \leq x} \frac{\rho_F(\mathfrak{p}) \log \mathfrak{N}(\mathfrak{p})}{\mathfrak{N}(\mathfrak{p})}$$

if we add an error term

$$\mathrm{O}^*(n \omega_{\mathbf{K}}(H)), \quad \text{where} \quad H = \prod_{\mathfrak{N}(\mathfrak{p}) \leq n} \mathfrak{p}.$$

If (a_i) is not divisible by $\mathfrak{p}$, then the linear congruence $a_i x + b_i \equiv 0 \bmod \mathfrak{p}$ has a unique solution modulo $\mathfrak{p}$. Further, two linear congruences $a_i x + b_i \equiv 0 \bmod \mathfrak{p}$ and $a_j x + b_j \equiv 0 \bmod \mathfrak{p}$ give the same solution modulo $\mathfrak{p}$ if and only if

$$\frac{b_i}{a_i} \equiv \frac{b_j}{a_j} \bmod \mathfrak{p} \quad \text{or in other words,} \quad b_i a_j - a_i b_j \equiv 0 \bmod \mathfrak{p}.$$

Therefore if $(\mathfrak{p}, (E)) = \mathcal{O}_{\mathbf{K}}$ then $\rho_F(\mathfrak{p}) = n$. Hence,

$$\sum_{n < \mathfrak{N}(\mathfrak{p}) \leq x} \frac{\rho_F(\mathfrak{p}) \log \mathfrak{N}(\mathfrak{p})}{\mathfrak{N}(\mathfrak{p})} = \sum_{\mathfrak{N}(\mathfrak{p}) \leq x} \frac{n \log \mathfrak{N}(\mathfrak{p})}{\mathfrak{N}(\mathfrak{p})} + \mathrm{O}^*\big(n(\omega_{\mathbf{K}}(E) + \omega_{\mathbf{K}}(H))\big),$$

where $\omega_{\mathbf{K}}(E)$ denotes the number of distinct prime ideals of $\mathbf{K}$ dividing the ideal (E) in $\mathbf{K}$. This proves Lemma 9. ∎

3.1. An estimate to control the error term

LEMMA 10. *We have*

$$\sum_{\substack{\mathfrak{b}_1, \mathfrak{b}_2 | \mathcal{P}_{\mathbf{K}}(z) \\ \mathfrak{N}(\mathfrak{b}_i) \leq z}} |\lambda_{\mathfrak{b}_1} \lambda_{\mathfrak{b}_2}| \frac{\rho_F([\mathfrak{b}_1, \mathfrak{b}_2])}{\sqrt{\mathfrak{N}([\mathfrak{b}_1, \mathfrak{b}_2])}} \leq (3n)^{4\pi_{\mathbf{K}}(2n)} \zeta_{\mathbf{K}} \left(\frac{3}{2} \right)^{8n} z.$$

Proof. The sum above equals

$$\sum_{\substack{\partial | \mathcal{P}_{\mathbf{K}}(z) \\ \mathfrak{N}(\partial) \leq z}} \frac{\sqrt{\mathfrak{N}(\partial)}}{\rho_F(\partial)} \sum_{\substack{\mathfrak{b}_i | \mathcal{P}_{\mathbf{K}}(z) \\ \partial = (\mathfrak{b}_1, \mathfrak{b}_2) \\ \mathfrak{N}(\mathfrak{b}_i) \leq z}} \frac{|\lambda_{\mathfrak{b}_1} \lambda_{\mathfrak{b}_2}| \rho_F(\mathfrak{b}_1) \rho_F(\mathfrak{b}_2)}{\sqrt{\mathfrak{N}(\mathfrak{b}_1 \mathfrak{b}_2)}}.$$

From the definition of $\lambda_{\mathfrak{b}}$ and substituting $y = z/\mathfrak{N}(\partial)$, we get

$$\begin{aligned}
&\sum_{\substack{\mathfrak{N}(\mathfrak{c})\le y\\(\mathfrak{c},\partial H)=1}}\frac{|\lambda_{\partial\mathfrak{c}}|\rho_F(\mathfrak{c})}{\sqrt{\mathfrak{N}(\mathfrak{c})}}\\
&\qquad = G(z)^{-1}\sum_{\substack{\mathfrak{N}(\mathfrak{c})\le y\\(\mathfrak{c},\partial H)=\mathcal{O}_{\mathbf{K}}}}\mu^2(\mathfrak{c})\frac{\sqrt{\mathfrak{N}(\mathfrak{c})}\,\mathfrak{N}(\partial)}{\rho_F(\partial)f_1(\mathfrak{c}\partial)}\sum_{\substack{\mathfrak{N}(\mathfrak{m})\le y/\mathfrak{N}(\mathfrak{c})\\(\mathfrak{m},\mathfrak{c}\partial H)=\mathcal{O}_{\mathbf{K}}}}\frac{\mu^2(\mathfrak{m})}{f_1(\mathfrak{m})}\\
&\qquad \le \frac{\mathfrak{N}(\partial)}{G(z)\rho_F(\partial)f_1(\partial)}\sum_{\substack{\mathfrak{N}(\mathfrak{m})\le y\\(\mathfrak{m},H)=\mathcal{O}_{\mathbf{K}}}}\frac{\mu^2(\mathfrak{m})}{f_1(\mathfrak{m})}\sum_{\substack{\mathfrak{N}(\mathfrak{c})\le y/\mathfrak{N}(\mathfrak{m})\\(\mathfrak{c},H)=\mathcal{O}_{\mathbf{K}}}}\frac{\mu^2(\mathfrak{c})\sqrt{\mathfrak{N}(\mathfrak{c})}}{f_1(\mathfrak{c})}\\
&\qquad \le \frac{\sqrt{y}\,\mathfrak{N}(\partial)}{G(z)\rho_F(\partial)f_1(\partial)}\sum_{\substack{\mathfrak{N}(\mathfrak{m})\le y\\(\mathfrak{m},H)=\mathcal{O}_{\mathbf{K}}}}\frac{\mu^2(\mathfrak{m})}{f_1(\mathfrak{m})\sqrt{\mathfrak{N}(\mathfrak{m})}}G(y)\\
&\qquad \le \frac{\sqrt{y}\mathfrak{N}(\partial)}{\rho_F(\partial)f_1(\partial)}\prod_{\mathfrak{N}(\mathfrak{p})>n}\left(1+\frac{\rho_F(\mathfrak{p})}{\sqrt{\mathfrak{N}(\mathfrak{p})}\,(\mathfrak{N}(\mathfrak{p})-\rho_F(\mathfrak{p}))}\right).
\end{aligned}$$

Thus

$$\begin{aligned}
&\sum_{\substack{\mathfrak{b}_1,\mathfrak{b}_2|\mathcal{P}_{\mathbf{K}}(z)\\\mathfrak{N}(\mathfrak{b}_i)\le z}}|\lambda_{\mathfrak{b}_1}\lambda_{\mathfrak{b}_2}|\frac{\rho_F([\mathfrak{b}_1,\mathfrak{b}_2])}{\sqrt{\mathfrak{N}([\mathfrak{b}_1,\mathfrak{b}_2])}}\\
&\quad \le \sum_{\substack{\partial|\mathcal{P}_{\mathbf{K}}(z)\\\mathfrak{N}(\partial)\le z}}\frac{\rho_F(\partial)}{\sqrt{\mathfrak{N}(\partial)}}\left(\frac{\sqrt{z\mathfrak{N}(\partial)}}{\rho_F(\partial)f_1(\partial)}\prod_{\mathfrak{N}(\mathfrak{p})>n}\left(1+\frac{\rho_F(\mathfrak{p})}{\sqrt{\mathfrak{N}(\mathfrak{p})}\,(\mathfrak{N}(\mathfrak{p})-\rho_F(\mathfrak{p}))}\right)\right)^2\\
&\quad \le z\prod_{\mathfrak{N}(\mathfrak{p})>n}\left(1+\frac{\rho_F(\mathfrak{p})}{\sqrt{\mathfrak{N}(\mathfrak{p})}\,(\mathfrak{N}(\mathfrak{p})-\rho_F(\mathfrak{p}))}\right)^2\left(1+\frac{\rho_F(\mathfrak{p})\sqrt{\mathfrak{N}(\mathfrak{p})}}{(\mathfrak{N}(\mathfrak{p})-\rho_F(\mathfrak{p}))^2}\right).
\end{aligned}$$

Consider

$$\prod_{\mathfrak{N}(\mathfrak{p})>n}\left(1+\frac{\rho_F(\mathfrak{p})}{\sqrt{\mathfrak{N}(\mathfrak{p})}\,(\mathfrak{N}(\mathfrak{p})-\rho_F(\mathfrak{p}))}\right).$$

We break this into two products, one over prime ideals with norm $\le 2n$ and the other with norm above $2n$. For the first product, we use

$$1+\frac{\rho_F(\mathfrak{p})}{\sqrt{\mathfrak{N}(\mathfrak{p})}\,(\mathfrak{N}(\mathfrak{p})-\rho_F(\mathfrak{p}))}\le 1+n\le 2n.$$

For the second product, since $\rho_F(\mathfrak{p})\le n<2n<\mathfrak{N}(\mathfrak{p})$, we use

$$\mathfrak{N}(\mathfrak{p})-\rho_F(\mathfrak{p})\ge\mathfrak{N}(\mathfrak{p})-n>\frac{\mathfrak{N}(\mathfrak{p})}{2}.$$

Therefore,

$$1+\frac{\rho_F(\mathfrak{p})}{\sqrt{\mathfrak{N}(\mathfrak{p})}\,(\mathfrak{N}(\mathfrak{p})-\rho_F(\mathfrak{p}))}\le 1+\frac{2n}{\mathfrak{N}(\mathfrak{p})^{3/2}}.$$

Finally, by the binomial theorem we have

$$1+\frac{2n}{\mathfrak{N}(\mathfrak{p})^{3/2}}\le\left(1+\frac{1}{\mathfrak{N}(\mathfrak{p})^{3/2}}\right)^{2n}.$$

Hence,

$$\begin{aligned}\prod_{\mathfrak{N}(\mathfrak{p})>n}\left(1+\frac{\rho_F(\mathfrak{p})}{\sqrt{\mathfrak{N}(\mathfrak{p})}\,(\mathfrak{N}(\mathfrak{p})-\rho_F(\mathfrak{p}))}\right)&\le(2n)^{\pi_{\mathbf{K}}(2n)}\prod_{\substack{\mathfrak{p}\\ \mathfrak{N}(\mathfrak{p})>2n}}\left(1+\frac{2n}{\mathfrak{N}(\mathfrak{p})^{3/2}}\right)\\ &\le(2n)^{\pi_{\mathbf{K}}(2n)}\zeta_{\mathbf{K}}\left(\frac{3}{2}\right)^{2n}.\end{aligned}$$

Similarly,

$$\begin{aligned}&\prod_{\mathfrak{N}(\mathfrak{p})>n}\left(1+\frac{\rho_F(\mathfrak{p})\sqrt{\mathfrak{N}(\mathfrak{p})}}{(\mathfrak{N}(\mathfrak{p})-\rho_F(\mathfrak{p}))^2}\right)\\ &\qquad\le\prod_{n<\mathfrak{N}(\mathfrak{p})\le 2n}(1+n\sqrt{\mathfrak{N}(\mathfrak{p})})\prod_{\mathfrak{N}(\mathfrak{p})>2n}\left(1+\frac{4n}{\mathfrak{N}(\mathfrak{p})^{3/2}}\right)\\ &\qquad\le(3n)^{3\pi_{\mathbf{K}}(2n)/2}\zeta_{\mathbf{K}}\left(\frac{3}{2}\right)^{4n}.\end{aligned}$$

This completes the proof of Lemma 10. ∎

3.2. Estimating $G(z)$**.** We repeat the proof of Levin–Faĭnleĭb [6] as described in Halberstam–Richert [8] in the number field setting with the additional condition $\partial\,|\,\mathcal{P}_{\mathbf{K}}(z)$. This can also be done using the methods of [11, Theorem 13.3]. Let

$$G(x,z)=\sum_{\substack{\partial|\mathcal{P}_{\mathbf{K}}(z)\\ \mathfrak{N}(\partial)\le x}}\frac{\mu^2(\partial)}{f_1(\partial)}\quad\text{and}\quad G_{\mathfrak{p}}(x,z)=\sum_{\substack{\partial|\mathcal{P}_{\mathbf{K}}(z)\\ \mathfrak{N}(\partial)\le x\\ (\partial,\mathfrak{p})=\mathcal{O}_{\mathbf{K}}}}\frac{\mu^2(\partial)}{f_1(\partial)}.$$

LEMMA 11. *For positive real numbers x, z with $z\le x$, we have*

$$\begin{aligned}&\left(1-\frac{\rho_F(\mathfrak{p})}{\mathfrak{N}(\mathfrak{p})}\right)G_{\mathfrak{p}}\left(\frac{x}{\mathfrak{N}(\mathfrak{p})},z\right)\\ &\qquad=\left(1-\frac{\rho_F(\mathfrak{p})}{\mathfrak{N}(\mathfrak{p})}\right)G\left(\frac{x}{\mathfrak{N}(\mathfrak{p})},z\right)-\frac{\rho_F(\mathfrak{p})}{\mathfrak{N}(\mathfrak{p})}G_{\mathfrak{p}}\left(\frac{x}{\mathfrak{N}(\mathfrak{p})^2},z\right).\end{aligned}$$

Proof. For an integral ideal ∂ coprime to H, let

$$h(\partial) = \frac{\mu^2(\partial)}{f_1(\partial)}. \tag{5}$$

The function h is multiplicative. From the definition of $G(x,z)$, we have

$$G(x,z) = \sum_{\substack{\partial \mid \mathcal{P}_{\mathbf{K}}(z) \\ \mathfrak{N}(\partial)\le x}} h(\partial) = \sum_{\substack{\partial \mid \mathcal{P}_{\mathbf{K}}(z) \\ \mathfrak{N}(\partial)\le x \\ (\partial,\mathfrak{p})=\mathcal{O}_{\mathbf{K}}}} h(\partial) + \sum_{\substack{\partial \mid \mathcal{P}_{\mathbf{K}}(z) \\ \mathfrak{N}(\partial)\le x \\ \mathfrak{p}\mid\partial}} h(\partial)$$
$$= \sum_{\substack{\partial \mid \mathcal{P}_{\mathbf{K}}(z) \\ \mathfrak{N}(\partial)\le x \\ (\partial,\mathfrak{p})=\mathcal{O}_{\mathbf{K}}}} h(\partial) + h(\mathfrak{p}) \sum_{\substack{\partial \mid \mathcal{P}_{\mathbf{K}}(z) \\ \mathfrak{N}(\partial)\le x/\mathfrak{N}(\mathfrak{p}) \\ (\partial,\mathfrak{p})=\mathcal{O}_{\mathbf{K}}}} h(\partial).$$

Multiplying both sides by $1-\rho_F(\mathfrak{p})/\mathfrak{N}(\mathfrak{p})$ we get

$$\left(1-\frac{\rho_F(\mathfrak{p})}{\mathfrak{N}(\mathfrak{p})}\right)G(x,z)$$
$$= \left(1-\frac{\rho_F(\mathfrak{p})}{\mathfrak{N}(\mathfrak{p})}\right)G_{\mathfrak{p}}(x,z) + \left(1-\frac{\rho_F(\mathfrak{p})}{\mathfrak{N}(\mathfrak{p})}\right)h(\mathfrak{p})G_{\mathfrak{p}}\left(\frac{x}{\mathfrak{N}(\mathfrak{p})},z\right).$$

From (5) and (4), we know that

$$h(\mathfrak{p}) = \frac{1}{f_1(\mathfrak{p})} \quad \text{and} \quad f_1(\mathfrak{p}) = f(\mathfrak{p}) - f(\mathcal{O}_{\mathbf{K}}) = f(\mathfrak{p}) - 1.$$

Therefore,

$$\left(1-\frac{\rho_F(\mathfrak{p})}{\mathfrak{N}(\mathfrak{p})}\right)h(\mathfrak{p}) = \left(1-\frac{\rho_F(\mathfrak{p})}{\mathfrak{N}(\mathfrak{p})}\right)\frac{1}{f_1(\mathfrak{p})} \tag{6}$$
$$= \left(1-\frac{\rho_F(\mathfrak{p})}{\mathfrak{N}(\mathfrak{p})}\right)\frac{1}{f(\mathfrak{p})-1} = \frac{1}{f(\mathfrak{p})}.$$

This gives

$$\left(1-\frac{\rho_F(\mathfrak{p})}{\mathfrak{N}(\mathfrak{p})}\right)G(x,z) = \left(1-\frac{\rho_F(\mathfrak{p})}{\mathfrak{N}(\mathfrak{p})}\right)G_{\mathfrak{p}}(x,z) + \frac{1}{f(\mathfrak{p})}G_{\mathfrak{p}}\left(\frac{x}{\mathfrak{N}(\mathfrak{p})},z\right).$$

Replacing x by $\frac{x}{\mathfrak{N}(\mathfrak{p})}$ in $G(x,z)$, we get

$$\left(1-\frac{\rho_F(\mathfrak{p})}{\mathfrak{N}(\mathfrak{p})}\right)G\left(\frac{x}{\mathfrak{N}(\mathfrak{p})},z\right)$$
$$= \left(1-\frac{\rho_F(\mathfrak{p})}{\mathfrak{N}(\mathfrak{p})}\right)G_{\mathfrak{p}}\left(\frac{x}{\mathfrak{N}(\mathfrak{p})},z\right) + \frac{1}{f(\mathfrak{p})}G_{\mathfrak{p}}\left(\frac{x}{\mathfrak{N}(\mathfrak{p})^2},z\right).$$

This yields Lemma 11. ∎

LEMMA 12. *For an integral ideal ∂ and a real number $x > \mathfrak{N}(\partial)$, we have*

$$\sum_{\substack{\sqrt{x/\mathfrak{N}(\partial)} < \mathfrak{N}(\mathfrak{p}) \le x/\mathfrak{N}(\partial) \\ \mathfrak{p} \nmid \partial H}} h(\mathfrak{p}) \le n(\pi_{\mathbf{K}}(2n) + 9),$$

where $\pi_{\mathbf{K}}(x)$ denotes the number of prime ideals of $\mathcal{O}_{\mathbf{K}}$ with norm at most x, and $n \ge 2$ is the number of linear factors of F.

Proof. Let $y := x/\mathfrak{N}(\partial)$. We have

$$\sum_{\substack{\sqrt{y} < \mathfrak{N}(\mathfrak{p}) \le y \\ \mathfrak{p} \nmid \partial H}} h(\mathfrak{p}) \le \sum_{\substack{\sqrt{y} < \mathfrak{N}(\mathfrak{p}) \le y \\ \mathfrak{p} \nmid \partial H}} \frac{n}{\mathfrak{N}(\mathfrak{p}) - \rho_F(\mathfrak{p})} =: \mathrm{T}.$$

Then

$$\mathrm{T} \le \sum_{\substack{\sqrt{y} < \mathfrak{N}(\mathfrak{p}) \le y \\ \mathfrak{p} \nmid \partial H \\ \mathfrak{N}(\mathfrak{p}) < 2n}} \frac{n}{\mathfrak{N}(\mathfrak{p}) - \rho_F(\mathfrak{p})} + \sum_{\substack{\sqrt{y} < \mathfrak{N}(\mathfrak{p}) \le y \\ \mathfrak{p} \nmid \partial H \\ \mathfrak{N}(\mathfrak{p}) \ge 2n}} \frac{n}{\mathfrak{N}(\mathfrak{p}) - \rho_F(\mathfrak{p})}.$$

For the second term above we note that $\rho_F(\mathfrak{p}) \le n < 2n \le \mathfrak{N}(\mathfrak{p})$. Hence,

$$\begin{aligned}\sum_{\substack{\sqrt{y} < \mathfrak{N}(\mathfrak{p}) \le y \\ \mathfrak{p} \nmid \partial H \\ \mathfrak{N}(\mathfrak{p}) \ge 2n}} \frac{n}{\mathfrak{N}(\mathfrak{p}) - \rho_F(\mathfrak{p})} &\le \sum_{\substack{\sqrt{y} < \mathfrak{N}(\mathfrak{p}) \le y \\ \mathfrak{p} \nmid \partial H \\ \mathfrak{N}(\mathfrak{p}) \ge 2n}} \frac{2n}{\mathfrak{N}(\mathfrak{p})} \\ &\le \sum_{\substack{\sqrt{y} < \mathfrak{N}(\mathfrak{p}) \le y \\ \mathfrak{p} \nmid \partial H}} \frac{2n}{\mathfrak{N}(\mathfrak{p})}.\end{aligned}$$

Therefore,

$$\mathrm{T} \le \sum_{\substack{\mathfrak{p} \nmid H \\ \mathfrak{N}(\mathfrak{p}) < 2n}} \frac{n}{\mathfrak{N}(\mathfrak{p}) - \rho_F(\mathfrak{p})} + \sum_{\substack{\sqrt{y} < \mathfrak{N}(\mathfrak{p}) \le y \\ \mathfrak{p} \nmid \partial}} \frac{2n}{\mathfrak{N}(\mathfrak{p})} =: \mathrm{T}_1 + \mathrm{T}_2.$$

For T_1, since $\rho_F(\mathfrak{p}) \le n$ and for $\mathfrak{p} \nmid H$, $\mathfrak{N}(\mathfrak{p}) > n$, we see that $\mathfrak{N}(\mathfrak{p}) - \rho_F(\mathfrak{p})$ is an integer greater than or equal to 1. Consequently,

$$\sum_{\substack{\sqrt{y} < \mathfrak{N}(\mathfrak{p}) \le y \\ \mathfrak{p} \nmid H \\ \mathfrak{N}(\mathfrak{p}) < 2n}} \frac{n}{\mathfrak{N}(\mathfrak{p}) - \rho_F(\mathfrak{p})} < n\pi_{\mathbf{K}}(2n).$$

For T_2, since $\mathbf{K}$ is a quadratic field, $\mathfrak{N}(\mathfrak{p})$ is either p or p^2. If $\mathfrak{N}(\mathfrak{p}) = p$, then $\sqrt{y} < p \le y$. On the other hand, if $\mathfrak{N}(\mathfrak{p}) = p^2$, then $p \le \sqrt{y}$. Further, if $\mathfrak{N}(\mathfrak{p}) = p$, there are at most two primes above $p\mathbb{Z}$, and if $\mathfrak{N}(\mathfrak{p}) = p^2$, there is exactly one such prime. This gives us

$$\sum_{\substack{\sqrt{y}<\mathfrak{N}(\mathfrak{p})\leq y\\ \mathfrak{p}\nmid\partial H}} h(\mathfrak{p}) \leq n\pi_{\mathbf{K}}(2n) + 2\sum_{\sqrt{y}<p\leq y}\frac{2n}{p} + \sum_{p\leq\sqrt{y}}\frac{2n}{p^2}.$$

Note that

$$\sum_{\sqrt{y}<p\leq y}\frac{2n}{p} = \sum_{p\leq y}\frac{2n}{p} - \sum_{p\leq\sqrt{y}}\frac{2n}{p}.$$

The first sum on the right is estimated using a result of Debaene [3] (see Lemma 5) and the second using a result of Rosser–Schoenfeld [12] (see Lemma 6). This shows, for $x \geq 16\mathfrak{N}(\partial)$, i.e. $y \geq 16$, that

$$\begin{aligned}\sum_{p\leq y}\frac{2n}{p} - \sum_{p\leq\sqrt{y}}\frac{2n}{p} &\leq 2n\left(0.666 + \log\log\frac{x}{\mathfrak{N}(\partial)} - \log\log\sqrt{\frac{x}{\mathfrak{N}(\partial)}}\right)\\ &\leq 2n(0.666+\log 2) \leq 2.8n.\end{aligned}$$

If $x < 16\mathfrak{N}(\partial)$, i.e. $y < 16$, then

$$\sum_{p\leq y}\frac{2n}{p} - \sum_{p\leq\sqrt{y}}\frac{2n}{p} \leq \sum_{p<16}\frac{2n}{p} \leq 2.7n.$$

Finally, since

$$2\sum_{p}\frac{1}{p^2} \leq \frac{\pi^2}{3} \leq 3.29,$$

we get

$$\sum_{\substack{\sqrt{y}<\mathfrak{N}(\mathfrak{p})\leq y\\ \mathfrak{p}\nmid\partial H}} h(\mathfrak{p}) \leq n\pi_{\mathbf{K}}(2n) + 5.6n + 3.29n \leq n(\pi_{\mathbf{K}}(2n)+9). \blacksquare$$

Let

$$T(x,z) = \int_1^x G(t,z)\,\frac{dt}{t}.$$

It follows from the definition of $G(t,z)$ that

$$T(x,z) = \int_1^x \sum_{\substack{\mathfrak{N}(\partial)\leq t\\ \partial|\mathcal{P}_{\mathbf{K}}(z)}} h(\partial)\,\frac{dt}{t} = \sum_{\substack{\mathfrak{N}(\partial)\leq x\\ \partial|\mathcal{P}_{\mathbf{K}}(z)}} h(\partial)\int_{\mathfrak{N}(\partial)}^x \frac{dt}{t} = \sum_{\substack{\mathfrak{N}(\partial)\leq x\\ \partial|\mathcal{P}_{\mathbf{K}}(z)}} h(\partial)\log\frac{x}{\mathfrak{N}(\partial)}.$$

LEMMA 13. *Let $z \geq 1$ be a real number. Then*

$$\sum_{\substack{\mathfrak{N}(\partial)\leq x\\ \partial|\mathcal{P}_{\mathbf{K}}(z)}} h(\partial)\log\mathfrak{N}(\partial) = nT(x,z) - nT\left(\frac{x}{z},z\right) + \mathrm{O}^*\left(\left(\omega_{\mathbf{K}}(E)+\omega_{\mathbf{K}}(H)+10n^2+\frac{e^{28}|d_{\mathbf{K}}|^{1/3}\log|d_{\mathbf{K}}|}{\alpha_{\mathbf{K}}}\right)nG(x,z)\right).$$

Proof. We have

$$S = \sum_{\substack{\mathfrak{N}(\partial)\leq x\\ \partial|\mathcal{P}_{\mathbf{K}}(z)}} h(\partial)\sum_{\mathfrak{p}|\partial}\log\mathfrak{N}(\mathfrak{p}) = \sum_{n<\mathfrak{N}(\mathfrak{p})\leq z} h(\mathfrak{p})\log\mathfrak{N}(\mathfrak{p}) \sum_{\substack{\mathfrak{N}(\mathfrak{m})\leq x/\mathfrak{N}(\mathfrak{p})\\ \mathfrak{m}|\mathcal{P}_{\mathbf{K}}(z)\\ (\mathfrak{m},\mathfrak{p})=\mathcal{O}_{\mathbf{K}}}} h(\mathfrak{m})$$
$$= \sum_{n<\mathfrak{N}(\mathfrak{p})\leq z} h(\mathfrak{p})G_{\mathfrak{p}}\left(\frac{x}{\mathfrak{N}(\mathfrak{p})},z\right)\log\mathfrak{N}(\mathfrak{p}).$$

Applying Lemma 11 and using (6), we get

$$S = \sum_{n<\mathfrak{N}(\mathfrak{p})\leq z} \frac{\rho_F(\mathfrak{p})\log\mathfrak{N}(\mathfrak{p})}{\mathfrak{N}(\mathfrak{p})} G\left(\frac{x}{\mathfrak{N}(\mathfrak{p})},z\right) + \sum_{n<\mathfrak{N}(\mathfrak{p})\leq z}\frac{\rho_F(\mathfrak{p})h(\mathfrak{p})}{\mathfrak{N}(\mathfrak{p})}\log\mathfrak{N}(\mathfrak{p}) \sum_{\substack{x/\mathfrak{N}(\mathfrak{p})^2<\mathfrak{N}(\mathfrak{m})\leq x/\mathfrak{N}(\mathfrak{p})\\ \mathfrak{m}|\mathcal{P}_{\mathbf{K}}(z)\\ (\mathfrak{m},\mathfrak{p})=\mathcal{O}_{\mathbf{K}}}} h(\mathfrak{m}).$$

Using the definition of $G(x,z)$ in the first sum and interchanging the summations, we get

$$S = \sum_{\substack{\mathfrak{N}(\partial)\leq x\\ \partial|\mathcal{P}_{\mathbf{K}}(z)}} h(\partial) \sum_{n<\mathfrak{N}(\mathfrak{p})\leq \min(x/\mathfrak{N}(\partial),z)} \frac{\rho_F(\mathfrak{p})}{\mathfrak{N}(\mathfrak{p})}\log\mathfrak{N}(\mathfrak{p}) + \sum_{\substack{x/z^2<\mathfrak{N}(\partial)\leq x\\ \partial|\mathcal{P}_{\mathbf{K}}(z)}} h(\partial) \sum_{\substack{\sqrt{x/\mathfrak{N}(\partial)}<\mathfrak{N}(\mathfrak{p})\leq\min(x/\mathfrak{N}(\partial),z)\\ (\mathfrak{p},\partial H)=\mathcal{O}_{\mathbf{K}}}} \frac{\rho_F(\mathfrak{p})h(\mathfrak{p})\log\mathfrak{N}(\mathfrak{p})}{\mathfrak{N}(\mathfrak{p})}.$$

Applying Lemma 12, we get

$$\sum_{\substack{\sqrt{x/\mathfrak{N}(\partial)}<\mathfrak{N}(\mathfrak{p})\leq\min(x/\mathfrak{N}(\partial),z)\\ (\mathfrak{p},\partial H)=\mathcal{O}_{\mathbf{K}}}} \frac{\rho_F(\mathfrak{p})h(\mathfrak{p})\log\mathfrak{N}(\mathfrak{p})}{\mathfrak{N}(\mathfrak{p})} \leq n \sum_{\substack{\sqrt{x/\mathfrak{N}(\partial)}<\mathfrak{N}(\mathfrak{p})\leq\min(x/\mathfrak{N}(\partial),z)\\ \mathfrak{p}\nmid\partial H}} h(\mathfrak{p}) \leq n^2(\pi_{\mathbf{K}}(2n)+9).$$

Combining the above, we get

$$S = \sum_{\substack{\mathfrak{N}(\partial)\le x\\ \partial|\mathcal{P}_{\mathbf{K}}(z)}} h(\partial) \sum_{n<\mathfrak{N}(\mathfrak{p})\le \min(x/\mathfrak{N}(\partial),z)} \frac{\rho_F(\mathfrak{p})}{\mathfrak{N}(\mathfrak{p})}\log \mathfrak{N}(\mathfrak{p}) + \mathrm{O}^*\big(n^2(\pi_{\mathbf{K}}(2n)+9)G(x,z)\big).$$

For the first term, we get

$$\sum_{\substack{\mathfrak{N}(\partial)\le x\\ \partial|\mathcal{P}_{\mathbf{K}}(z)}} h(\partial) \sum_{n<\mathfrak{N}(\mathfrak{p})\le \min(x/\mathfrak{N}(\partial),z)} \frac{\rho_F(\mathfrak{p})}{\mathfrak{N}(\mathfrak{p})}\log \mathfrak{N}(\mathfrak{p}) = \sum_{\substack{\mathfrak{N}(\partial)\le x/z\\ \partial|\mathcal{P}_{\mathbf{K}}(z)}} h(\partial) \sum_{n<\mathfrak{N}(\mathfrak{p})\le z} \frac{\rho_F(\mathfrak{p})}{\mathfrak{N}(\mathfrak{p})}\log \mathfrak{N}(\mathfrak{p}) + \sum_{\substack{x/z<\mathfrak{N}(\partial)\le x\\ \partial|\mathcal{P}_{\mathbf{K}}(z)}} h(\partial) \sum_{n<\mathfrak{N}(\mathfrak{p})\le x/\mathfrak{N}(\partial)} \frac{\rho_F(\mathfrak{p})}{\mathfrak{N}(\mathfrak{p})}\log \mathfrak{N}(\mathfrak{p}).$$

We now apply Lemma 9 to deduce

$$\frac{1}{n}\sum_{\substack{\mathfrak{N}(\partial)\le x\\ \partial|\mathcal{P}_{\mathbf{K}}(z)}} h(\partial) \sum_{n<\mathfrak{N}(\mathfrak{p})\le \min(x/\mathfrak{N}(\partial),z)} \frac{\rho_F(\mathfrak{p})}{\mathfrak{N}(\mathfrak{p})}\log \mathfrak{N}(\mathfrak{p}) = \sum_{\substack{\mathfrak{N}(\partial)\le x/z\\ \partial|\mathcal{P}_{\mathbf{K}}(z)}} h(\partial)\log z + \sum_{\substack{x/z<\mathfrak{N}(\partial)\le x\\ \partial|\mathcal{P}_{\mathbf{K}}(z)}} h(\partial)\log\frac{x}{\mathfrak{N}(\partial)} + \mathrm{O}^*(L_1 G(x,z)),$$

where

$$L_1 = \omega_{\mathbf{K}}(E)+\omega_{\mathbf{K}}(H)+2.52+\frac{e^{28}|d_{\mathbf{K}}|^{1/3}\log|d_{\mathbf{K}}|}{\alpha_{\mathbf{K}}}.$$

Combining the above, we get

$$\begin{aligned}\frac{S}{n} &= \sum_{\substack{\mathfrak{N}(\partial)\le x\\ \partial|\mathcal{P}_{\mathbf{K}}(z)}} h(\partial)\log\frac{x}{\mathfrak{N}(\partial)} - \sum_{\substack{\mathfrak{N}(\partial)\le x/z\\ \partial|\mathcal{P}_{\mathbf{K}}(z)}} h(\partial)\log\frac{x/z}{\mathfrak{N}(\partial)} \\ &\quad + \mathrm{O}^*\left(\left(\omega_{\mathbf{K}}(E)+\omega_{\mathbf{K}}(H)+2.52+n(\pi_{\mathbf{K}}(2n)+9) + \frac{e^{28}|d_{\mathbf{K}}|^{1/3}\log|d_{\mathbf{K}}|}{\alpha_{\mathbf{K}}}\right)G(x,z)\right) \\ &= T(x,z) - T\left(\frac{x}{z},z\right) \\ &\quad + \mathrm{O}^*\left(\left(\omega_{\mathbf{K}}(E)+\omega_{\mathbf{K}}(H)+10n^2+\frac{e^{28}|d_{\mathbf{K}}|^{1/3}\log|d_{\mathbf{K}}|}{\alpha_{\mathbf{K}}}\right)G(x,z)\right). \quad \blacksquare\end{aligned}$$

Note that $G(z,z)=G(z)$ and $T(z,z)=T(z)$.

Corollary 14. *For any real number $y\geq 1$, we have*

$$G(y)\log y=(n+1)T(y)+G(y)r(y)\log y,$$

where

$$|r(y)|\leq\left(\omega_{\mathbf{K}}(E)+\omega_{\mathbf{K}}(H)+10n^2+\frac{e^{28}|d_{\mathbf{K}}|^{1/3}\log|d_{\mathbf{K}}|}{\alpha_{\mathbf{K}}}\right)\frac{n}{\log y}.$$

Proof. Using Lemma 13 and adding $T(x,z)$ to both sides, we get

$$G(x,y)\log x=(n+1)T(x,y)-nT\left(\frac{x}{y},y\right)$$
$$+\mathrm{O}^*\left(n\left(\omega_{\mathbf{K}}(E)+\omega_{\mathbf{K}}(H)+10n^2+\frac{e^{28}|d_{\mathbf{K}}|^{1/3}\log|d_{\mathbf{K}}|}{\alpha_{\mathbf{K}}}\right)G(x,y)\right).$$

Putting $x=y$, we get the corollary. ∎

From now onwards, for any real number $y>3$, we denote

$$U_{\mathbf{K}}(y)=\log\left(\frac{n+1}{\log^{n+1}y}T(y)\right)$$

and

$$L_{\mathbf{K}}=n\left(\omega_{\mathbf{K}}(E)+\omega_{\mathbf{K}}(H)+10n^2+\frac{e^{28}|d_{\mathbf{K}}|^{1/3}\log|d_{\mathbf{K}}|}{\alpha_{\mathbf{K}}}\right).\tag{7}$$

Lemma 15. *For a real number z with $\log z\geq 3(n+1)L_{\mathbf{K}}$, we have*

$$G(z)=c_{\mathbf{K},F}\log^n z\left(1+\mathrm{O}^*\left(\frac{9(n+1)L_{\mathbf{K}}}{\log z}\right)\right)$$

for some positive constant $c_{\mathbf{K},F}$ depending on $\mathbf{K}$ and F.

Proof. We first observe that for $\log z\geq 3(n+1)L_{\mathbf{K}}$ and any real number $y\geq z$, we have

$$|U'_{\mathbf{K}}(y)|=\left|-\frac{n+1}{y\log y}+\frac{T'(y)}{T(y)}\right|=\left|-\frac{n+1}{y\log y}+\frac{G(y)}{yT(y)}\right|=\left|\frac{r(y)}{1-r(y)}\frac{n+1}{y\log y}\right|$$
$$\leq\frac{2(n+1)L_{\mathbf{K}}}{y\log^2 y}.$$

This implies that the integral of $U'_{\mathbf{K}}(y)$ from z to ∞ is convergent. Further,

$$\left|-\int_z^\infty U'_{\mathbf{K}}(y)\,dy\right|\leq\frac{2(n+1)L_{\mathbf{K}}}{\log z}<1.$$

Recall that

$$\frac{n+1}{\log^{n+1}z}T(z)=\exp(U_{\mathbf{K}}(z))=c_{\mathbf{K},F}\exp\left(-\int_z^\infty U'_{\mathbf{K}}(y)\,dy\right)$$

for some constant $c_{\mathbf{K},F}$. We now observe that

$$\exp\Big(-\int_z^\infty U'_{\mathbf{K}}(y)\,dy\Big) = 1 - \int_z^\infty U'_{\mathbf{K}}(y)\,dy + \frac{1}{2!}\Big(\int_z^\infty U'_{\mathbf{K}}(y)\,dy\Big)^2 - \cdots$$

and therefore

$$\begin{aligned}\exp\Big(-\int_z^\infty U'_{\mathbf{K}}(y)\,dy\Big) &= 1 + \mathrm{O}^*\bigg(\frac{2(n+1)L_{\mathbf{K}}}{\log z} + \Big(\frac{2(n+1)L_{\mathbf{K}}}{\log z}\Big)^2 + \cdots\bigg)\\ &= 1 + \mathrm{O}^*\bigg(\frac{2(n+1)L_{\mathbf{K}}}{\log z - 2(n+1)L_{\mathbf{K}}}\bigg)\\ &= 1 + \mathrm{O}^*\bigg(\frac{6(n+1)L_{\mathbf{K}}}{\log z}\bigg).\end{aligned}$$

Further, we have

$$\frac{1}{1-r(z)} = 1 + \frac{r(z)}{1-r(z)} = 1 + \mathrm{O}^*\bigg(\frac{L_{\mathbf{K}}}{\log z - L_{\mathbf{K}}}\bigg) = 1 + \mathrm{O}^*\bigg(\frac{2L_{\mathbf{K}}}{\log z}\bigg)$$

since $\log z \geq 3L_{\mathbf{K}}$. Applying Corollary 14 and combining the above, we get

$$\begin{aligned}G(z) &= \frac{n+1}{(1-r(z))\log z}T(z)\\ &= c_{\mathbf{K},F}\log^n z\bigg(1 + \mathrm{O}^*\Big(\frac{2L_{\mathbf{K}}}{\log z}\Big)\bigg)\bigg(1 + \mathrm{O}^*\Big(\frac{6(n+1)L_{\mathbf{K}}}{\log z}\Big)\bigg)\\ &= c_{\mathbf{K},F}\log^n z\bigg(1 + \mathrm{O}^*\Big(\frac{9(n+1)L_{\mathbf{K}}}{\log z}\Big)\bigg). \ \blacksquare\end{aligned}$$

REMARK 16. If one wants a lower bound for $G(z)$ in the case $n = 1$, one can use a simpler method that avoids relying on the sum $\rho_F(\mathfrak{p})(\log \mathfrak{N}(\mathfrak{p}))/\mathfrak{N}(\mathfrak{p})$ as in [4, Theorem 30].

We conclude this section by computing the constant $c_{\mathbf{K},F}$.

LEMMA 17. *We have*

$$c_{\mathbf{K},F} = \frac{\alpha_{\mathbf{K}}^n}{n!}\prod_{\mathfrak{N}(\mathfrak{p})\leq n}\Big(1 - \frac{1}{\mathfrak{N}(\mathfrak{p})}\Big)^n \prod_{\mathfrak{p}\nmid H}(1 + h(\mathfrak{p}))\Big(1 - \frac{1}{\mathfrak{N}(\mathfrak{p})}\Big)^n.$$

Proof. For a real parameter $s > 0$, consider the series

$$M = \sum_{\substack{\partial\subseteq\mathcal{O}_{\mathbf{K}}\\ \partial\neq(0)\\ (\partial,H)=\mathcal{O}_{\mathbf{K}}}} \frac{h(\partial)}{\mathfrak{N}(\partial)^s}.$$

In the region $\Re s > 0$, we have $M = \prod_{\mathfrak{p}\nmid H}(1 + h(\mathfrak{p})/\mathfrak{N}(\mathfrak{p})^s)$. Applying partial summation, we obtain

$$\begin{aligned} M &= \lim_{x\to\infty}\left(\frac{\sum_{\mathfrak{N}(\partial)\le x,\, (\partial,H)=\mathcal{O}_{\mathbf{K}}} h(\partial)}{x^s} + s\int_1^x \frac{\sum_{\mathfrak{N}(\partial)\le t,\, (\partial,H)=\mathcal{O}_{\mathbf{K}}} h(\partial)}{t^{s+1}}\,dt\right)\\ &= \lim_{x\to\infty}\left(\frac{G(x)}{x^s} + s\int_1^x \frac{G(t)}{t^{s+1}}\,dt\right). \end{aligned}$$

By Lemma 15, we have $G(x) \ll \log^{n+1} x$ and hence $M = s\int_1^\infty \frac{G(t)}{t^{s+1}}\,dt$. We now split the integral into two parts. Let $z_1 = 3(n+1)L_{\mathbf{K}}$, where $L_{\mathbf{K}}$ is as in (7). Then

$$M = s\int_1^{z_1} \frac{G(t)}{t^{s+1}}\,dt + s\int_{z_1}^{\infty} \frac{G(t)}{t^{s+1}}\,dt.$$

To estimate the first integral, we observe that for real $s > 0$, we have

$$s\int_1^{z_1} \frac{G(t)}{t^{s+1}}\,dt \le s\int_1^{z_1} \frac{G(t)}{t}\,dt = sT(z_1).$$

Recall that

$$T(z_1) = \sum_{\substack{\mathfrak{N}(\partial)\le z_1\\ \partial|\mathcal{P}_{\mathbf{K}}(z_1)}} h(\partial)\log\frac{z_1}{\mathfrak{N}(\partial)} \le \log z_1 \sum_{\substack{\mathfrak{N}(\partial)\le z_1\\ \partial|\mathcal{P}_{\mathbf{K}}(z_1)}} \frac{1}{f_1(\partial)} = \mathrm{O}(1).$$

For the second integral, applying Lemma 15, we have

$$\begin{aligned} s\int_{z_1}^{\infty} \frac{G(t)}{t^{s+1}}\,dt &= s\int_{z_1}^{\infty} \frac{c_{\mathbf{K},F}\log^n t + \mathrm{O}(\log^{n-1} t)}{t^{s+1}}\,dt\\ &= s\int_1^{\infty} \frac{c_{\mathbf{K},F}\log^n t + \mathrm{O}(\log^{n-1} t)}{t^{s+1}}\,dt + \mathrm{O}(s). \end{aligned}$$

We now use the fact that for $s > 0$,

$$\int_1^\infty \frac{\log^n t}{t^{s+1}}\,dt = \frac{\Gamma(n+1)}{s^{n+1}}.$$

Therefore

$$M = \prod_{\mathfrak{p}\nmid H}\left(1 + \frac{h(\mathfrak{p})}{\mathfrak{N}(\mathfrak{p})^s}\right) = c_{\mathbf{K},F}\frac{\Gamma(n+1)}{s^n} + \mathrm{O}\left(\frac{\Gamma(n)}{s^{n-1}}\right) + \mathrm{O}(s).$$

It immediately follows that

$$c_{\mathbf{K},F} = \frac{1}{\Gamma(n+1)} \lim_{s\to 0^+} s^n \prod_{\mathfrak{p}\nmid H}\left(1+\frac{h(\mathfrak{p})}{\mathfrak{N}(\mathfrak{p})^s}\right)$$
$$= \frac{\alpha_{\mathbf{K}}^n}{n!}\prod_{\mathfrak{p}\mid H}\left(1-\frac{1}{\mathfrak{N}(\mathfrak{p})}\right)^n \lim_{s\to 0^+}\prod_{\mathfrak{p}\nmid H}\left(1+\frac{h(\mathfrak{p})}{\mathfrak{N}(\mathfrak{p})^s}\right)\left(1-\frac{1}{\mathfrak{N}(\mathfrak{p})^{s+1}}\right)^n.$$

This completes the proof of Lemma 17. ■

4. Proof of the main theorem. Let z be a real number such that $z \geq 4$. We use $\mathfrak{N}((\alpha))$ to denote the absolute norm of the principal ideal (α) and $\mathcal{Q}$ to denote the set of all prime elements of $\mathcal{O}_{\mathbf{K}}$. Recall that

$$f_i(x) = a_i x + b_i \quad \text{for } 1 \leq i \leq n \quad \text{and} \quad F(x) = \prod_{i=1}^n f_i(x).$$

We want to estimate

$$D = \sum_{\substack{\mathfrak{N}((\alpha))\leq u\\ f_i(\alpha)\in\mathcal{Q}\text{ for all } i}} 1 \leq \sum_{\substack{\mathfrak{N}((\alpha))\leq z\\ f_i(\alpha)\in\mathcal{Q}\text{ for all } i}} 1 + \sum_{j=1}^n \sum_{\substack{\mathfrak{N}((f_j(\alpha)))\leq z\\ f_i(\alpha)\in\mathcal{Q}\text{ for all } i}} 1 + \sum_{\substack{z<\mathfrak{N}((\alpha))\leq u\\ (F(\alpha),\mathcal{P}_{\mathbf{K}}(z))=1}} 1$$
$$\leq \sum_{\substack{\mathfrak{N}((\alpha))\leq z\\ f_i(\alpha)\in\mathcal{Q}\text{ for all } i}} 1 + 2|\mu_{\mathbf{K}}|nz + \sum_{\substack{z<\mathfrak{N}((\alpha))\leq u\\ (F(\alpha),\mathcal{P}_{\mathbf{K}}(z))=1}} 1,$$

where $|\mu_{\mathbf{K}}|$ is the number of roots of unity in $\mathcal{O}_{\mathbf{K}}$.

To estimate the first sum, we observe that for $u, v \in \mathcal{O}_{\mathbf{K}} \setminus \{0\}$, the norms of u, v are positive and

$$N_{\mathbf{K}/\mathbb{Q}}(u+v) = N_{\mathbf{K}/\mathbb{Q}}(u) + \mathrm{Tr}_{\mathbf{K}/\mathbb{Q}}(u\bar{v}) + N_{\mathbf{K}/\mathbb{Q}}(v),$$

where $\bar{v}$ denotes the complex conjugate of v. If $\mathcal{O}_{\mathbf{K}} = \mathbb{Z}[\sqrt{-d}]$ and $u\bar{v} = a + b\sqrt{-d}$, then

$$\mathrm{Tr}_{\mathbf{K}/\mathbb{Q}}(u\bar{v}) = 2a \leq 2(a^2 + b^2 d) \leq 2N_{\mathbf{K}/\mathbb{Q}}(u\bar{v}).$$

Similarly, if $\mathcal{O}_{\mathbf{K}} = \mathbb{Z}\left[\frac{1+\sqrt{-d}}{2}\right]$ and $u\bar{v} = a + \frac{b}{2} + \frac{b\sqrt{-d}}{2}$, we have

$$\mathrm{Tr}_{\mathbf{K}/\mathbb{Q}}(u\bar{v}) = 2\left(a+\frac{b}{2}\right) \leq 2\left(\left(a+\frac{b}{2}\right)^2 + \frac{b^2 d}{4}\right) \leq 2N_{\mathbf{K}/\mathbb{Q}}(u\bar{v}).$$

Indeed, this is clearly true when $a+\frac{b}{2} \leq 0$ or $a+\frac{b}{2} \geq 1$. Now if $0 < a+\frac{b}{2} < 1$, then $a+\frac{b}{2} = \frac{1}{2}$ and $b \neq 0$ and therefore $1 \leq 2\left(\frac{1}{4} + \frac{b^2 d}{4}\right)$. Thus in both cases $N_{\mathbf{K}/\mathbb{Q}}(u+v) \leq 4N_{\mathbf{K}/\mathbb{Q}}(uv)$. Therefore, the first sum under consideration satisfies

$$\sum_{\substack{\mathfrak{N}((\alpha))\le z\\ f_i(\alpha)\in\mathcal{Q}\text{ for all } i}} 1 \le \sum_{\substack{\mathfrak{N}((f_{i_0}(\alpha)))\le 4\mathfrak{N}((a_{i_0}b_{i_0}))z\\ f_i(\alpha)\in\mathcal{Q}\text{ for all } i}} 1 \le 8|\mu_{\mathbf{K}}|\mathfrak{N}((a_{i_0}b_{i_0}))nz,$$

where $\mathfrak{N}((a_{i_0}b_{i_0})) = \min\{\mathfrak{N}((a_ib_i)) : 1\le i\le n\}$. Hence,

$$D \le 10|\mu_{\mathbf{K}}|\mathfrak{N}((a_{i_0}b_{i_0}))nz + \sum_{\substack{z<\mathfrak{N}((\alpha))\le u\\ (F(\alpha),\mathcal{P}_{\mathbf{K}}(z))=1}} 1.$$

Let us now consider the sum

$$\sum_{\substack{\mathfrak{N}((\alpha))\le u\\ (F(\alpha),\mathcal{P}_{\mathbf{K}}(z))=1}} 1 = \sum_{\mathfrak{N}((\alpha))\le u}\Big(\sum_{\mathfrak{b}|(F(\alpha),\mathcal{P}_{\mathbf{K}}(z))}\mu(\mathfrak{b})\Big) \le \sum_{\mathfrak{N}((\alpha))\le u}\Big(\sum_{\mathfrak{b}|(F(\alpha),\mathcal{P}_{\mathbf{K}}(z))}\lambda_{\mathfrak{b}}\Big)^2.$$

Rearranging the terms, we get

$$\sum_{\mathfrak{N}((\alpha))\le u}\Big(\sum_{\mathfrak{b}|(F(\alpha),\mathcal{P}_{\mathbf{K}}(z))}\lambda_{\mathfrak{b}}\Big)^2 = \sum_{\substack{\mathfrak{b}_1,\mathfrak{b}_2|\mathcal{P}_{\mathbf{K}}(z)\\ \mathfrak{N}(\mathfrak{b}_i)\le z}}\lambda_{\mathfrak{b}_1}\lambda_{\mathfrak{b}_2}\sum_{\substack{\mathfrak{N}((\alpha))\le u\\ [\mathfrak{b}_1,\mathfrak{b}_2]|F(\alpha)}} 1.$$

Let $\mathfrak{b} = [\mathfrak{b}_1,\mathfrak{b}_2]$. To estimate the inner sum, we need to count $\alpha\in\mathcal{O}_{\mathbf{K}}$ such that α lies in one of the $\rho_F(\mathfrak{b})$ classes in $\mathcal{O}_{\mathbf{K}}/\mathfrak{b}$. If $\mathfrak{b}\,|\,\mathcal{P}_{\mathbf{K}}(z)$ and $\mathfrak{b}_0$ is the largest divisor of $\mathfrak{b}$ which is coprime to $E=\prod_{i=1}^n a_i\prod_{1\le i<j\le n}(a_ib_j-a_jb_i)$, then we can write $\rho_F(\mathfrak{b}) = n^{\omega(\mathfrak{b}_0)}\rho_F(\mathfrak{b}/\mathfrak{b}_0)$. Applying Theorem 3 with $\mathfrak{a}=\mathcal{O}_{\mathbf{K}}$, $\mathfrak{q}=\mathfrak{b}$, we deduce for $z\le\sqrt{u}$ that

$$(8)\qquad \sum_{\substack{\mathfrak{b}_1,\mathfrak{b}_2|\mathcal{P}_{\mathbf{K}}(z)\\ \mathfrak{N}(\mathfrak{b}_i)\le z}}\lambda_{\mathfrak{b}_1}\lambda_{\mathfrak{b}_2}\sum_{\substack{\mathfrak{N}((\alpha))\le u\\ [\mathfrak{b}_1,\mathfrak{b}_2]|F(\alpha)}} 1$$

$$= \sum_{\substack{\mathfrak{b}_1,\mathfrak{b}_2|\mathcal{P}_{\mathbf{K}}(z)\\ \mathfrak{N}(\mathfrak{b}_i)\le z}}\lambda_{\mathfrak{b}_1}\lambda_{\mathfrak{b}_2}\bigg(\frac{c_{\mathbf{K}}\rho_F([\mathfrak{b}_1,\mathfrak{b}_2])u}{\mathfrak{N}([\mathfrak{b}_1,\mathfrak{b}_2])} + \mathrm{O}^*\bigg(10^{14}\rho_F([\mathfrak{b}_1,\mathfrak{b}_2])\sqrt{\frac{u}{\mathfrak{N}([\mathfrak{b}_1,\mathfrak{b}_2])}}\bigg)\bigg),$$

where $c_{\mathbf{K}} = 2\pi/\sqrt{|d_{\mathbf{K}}|}$. Note that the main term is

$$\sum_{\mathfrak{b}_1,\mathfrak{b}_2|\mathcal{P}_{\mathbf{K}}(z)}\frac{\lambda_{\mathfrak{b}_1}\lambda_{\mathfrak{b}_2}}{f([\mathfrak{b}_1,\mathfrak{b}_2])},$$

where f is as defined in (4). Hence,

$$\sum_{\mathfrak{b}_1,\mathfrak{b}_2|\mathcal{P}_{\mathbf{K}}(z)}\frac{\lambda_{\mathfrak{b}_1}\lambda_{\mathfrak{b}_2}f((\mathfrak{b}_1,\mathfrak{b}_2))}{f(\mathfrak{b}_1)f(\mathfrak{b}_2)} = \sum_{\mathfrak{b}_1,\mathfrak{b}_2|\mathcal{P}_{\mathbf{K}}(z)}\frac{\lambda_{\mathfrak{b}_1}\lambda_{\mathfrak{b}_2}}{f(\mathfrak{b}_1)f(\mathfrak{b}_2)}\sum_{\mathfrak{a}|(\mathfrak{b}_1,\mathfrak{b}_2)} f_1(\mathfrak{a})$$

$$= \sum_{\mathfrak{a}|\mathcal{P}_{\mathbf{K}}(z)} f_1(\mathfrak{a})\Big(\sum_{\substack{\mathfrak{a}|\mathfrak{c}\\ \mathfrak{c}|\mathcal{P}_{\mathbf{K}}(z)}}\frac{\lambda_{\mathfrak{c}}}{f(\mathfrak{c})}\Big)^2.$$

Further, we observe that

$$\sum_{\substack{\mathfrak{c}|\mathcal{P}_{\mathbf{K}}(z)\\ \mathfrak{a}|\mathfrak{c}}} \frac{\lambda_{\mathfrak{c}}}{f(\mathfrak{c})} = G(z)^{-1} \sum_{\substack{\mathfrak{c}|\mathcal{P}_{\mathbf{K}}(z)\\ \mathfrak{a}|\mathfrak{c}}} \frac{\mu(\mathfrak{c})}{f_1(\mathfrak{c})} \sum_{\substack{\mathfrak{N}(\mathfrak{g})\le z/\mathfrak{N}(\mathfrak{c})\\ (\mathfrak{g},\mathfrak{c}H)=\mathcal{O}_{\mathbf{K}}}} \frac{\mu^2(\mathfrak{g})}{f_1(\mathfrak{g})}.$$

Writing $\mathfrak{c} = \mathfrak{h}\mathfrak{a}$ with $(\mathfrak{h},\mathfrak{a}) = \mathcal{O}_{\mathbf{K}}$, we get

$$\begin{aligned}
&\frac{\mu(\mathfrak{a})}{f_1(\mathfrak{a})} G(z)^{-1} \sum_{\substack{\mathfrak{N}(\mathfrak{h})\le z/\mathfrak{N}(\mathfrak{a})\\ \mathfrak{h}|\mathcal{P}_{\mathbf{K}}(z)\\ (\mathfrak{h},\mathfrak{a})=\mathcal{O}_{\mathbf{K}}}} \frac{\mu(\mathfrak{h})}{f_1(\mathfrak{h})} \sum_{\substack{\mathfrak{N}(\mathfrak{g})\le z/\mathfrak{N}(\mathfrak{h}\mathfrak{a})\\ (\mathfrak{g},\mathfrak{h}\mathfrak{a}H)=\mathcal{O}_{\mathbf{K}}}} \frac{\mu^2(\mathfrak{g})}{f_1(\mathfrak{g})}\\
&\qquad = \frac{\mu(\mathfrak{a})}{f_1(\mathfrak{a})} G(z)^{-1} \sum_{\substack{\mathfrak{N}(\mathfrak{h})\le z/\mathfrak{N}(\mathfrak{a})\\ \mathfrak{h}|\mathcal{P}_{\mathbf{K}}(z)\\ (\mathfrak{h},\mathfrak{a})=\mathcal{O}_{\mathbf{K}}}} \sum_{\substack{\mathfrak{N}(\mathfrak{g})\le z/\mathfrak{N}(\mathfrak{h}\mathfrak{a})\\ (\mathfrak{g},\mathfrak{h}\mathfrak{a}H)=\mathcal{O}_{\mathbf{K}}}} \mu(\mathfrak{h}) \frac{\mu^2(\mathfrak{g}\mathfrak{h})}{f_1(\mathfrak{g}\mathfrak{h})}.
\end{aligned}$$

Setting $\mathfrak{a}_1 = \mathfrak{g}\mathfrak{h}$ gives

$$\sum_{\substack{\mathfrak{c}|\mathcal{P}_{\mathbf{K}}(z)\\ \mathfrak{a}|\mathfrak{c}}} \frac{\lambda_{\mathfrak{c}}}{f(\mathfrak{c})} = \frac{\mu(\mathfrak{a})}{f_1(\mathfrak{a})} G(z)^{-1} \sum_{\substack{\mathfrak{N}(\mathfrak{a}_1)\le z/\mathfrak{N}(\mathfrak{a})\\ (\mathfrak{a}_1,\mathfrak{a}H)=\mathcal{O}_{\mathbf{K}}}} \frac{\mu^2(\mathfrak{a}_1)}{f_1(\mathfrak{a}_1)} \sum_{\mathfrak{h}|\mathfrak{a}_1} \mu(\mathfrak{h}) = G(z)^{-1} \frac{\mu(\mathfrak{a})}{f_1(\mathfrak{a})}.$$

Therefore the main term in (8) is $c_{\mathbf{K}} u G(z)^{-1}$. Applying Lemmas 15 and 17, for $\log z \ge 18(n+1)L_{\mathbf{K}}$ we have

$$\begin{aligned}
G(z) = \frac{\alpha_{\mathbf{K}}^n}{n!} \prod_{\mathfrak{p}|H} \left(1 - \frac{1}{\mathfrak{N}(\mathfrak{p})}\right)^n \prod_{\mathfrak{p}\nmid H} (1 + h(\mathfrak{p}))\left(1 - \frac{1}{\mathfrak{N}(\mathfrak{p})}\right)^n\\
\times \log^n z \left(1 + \mathrm{O}^*\left(\frac{9(n+1)L_{\mathbf{K}}}{\log z}\right)\right).
\end{aligned}$$

To deal with the error term, we use Lemma 10.

Combining everything, for $z \le \sqrt{u}$ we get

$$D \le c_{\mathbf{K}} u G(z)^{-1} + 10|\mu_{\mathbf{K}}|\mathfrak{N}((a_{i_0} b_{i_0}))nz + 10^{14}(3n)^{4\pi_{\mathbf{K}}(2n)} \zeta_{\mathbf{K}}\left(\frac{3}{2}\right)^{8n} z\sqrt{u}.$$

We now simplify the above expression to get

$$D \le \frac{2\pi}{\sqrt{|d_{\mathbf{K}}|}} u G(z)^{-1} + 3^{61n} n^{16n} \mathfrak{N}((a_{i_0} b_{i_0})) z\sqrt{u}.$$

Therefore, if we choose

$$z = \frac{\pi\sqrt{u} G(u)^{-1}}{2\sqrt{|d_{\mathbf{K}}|} 3^{61n} n^{16n} \mathfrak{N}((a_{i_0} b_{i_0}))} \le \frac{\pi\sqrt{u} G(z)^{-1}}{2\sqrt{|d_{\mathbf{K}}|} 3^{61n} n^{16n} \mathfrak{N}((a_{i_0} b_{i_0}))},$$

for $\log z \ge 18(n+1)L_{\mathbf{K}}$ we have

$$D \le \frac{5}{4} \cdot \frac{2\pi}{\sqrt{|d_{\mathbf{K}}|}} G(z)^{-1} u.$$

We now compute the lower bound for u. Note that

$$(4n)^n \frac{\sqrt{u}}{\log^n u} \geq u^{1/4}.$$

Hence, for

$$u^{1/4} \geq \frac{(4n)^n \sqrt{|d_{\mathbf{K}}|}\, 3^{62n} n^{16n} \mathfrak{N}((a_{i_0} b_{i_0})) \alpha_{\mathbf{K}}^n}{n! \pi \exp(-18(n+1) L_{\mathbf{K}})} \times \prod_{\mathfrak{p} | H} \left(1 - \frac{1}{\mathfrak{N}(\mathfrak{p})}\right)^n \prod_{\mathfrak{p} \nmid H} (1 + h(\mathfrak{p})) \left(1 - \frac{1}{\mathfrak{N}(\mathfrak{p})}\right)^n,$$

we have

$$D \leq \frac{5\left(\prod_{\mathfrak{p}|H} \left(1 - \frac{1}{\mathfrak{N}(\mathfrak{p})}\right)^{-n} \prod_{\mathfrak{p} \nmid H} (1 + h(\mathfrak{p}))^{-1} \left(1 - \frac{1}{\mathfrak{N}(\mathfrak{p})}\right)^{-n}\right) n! |\mu_{\mathbf{K}}| u}{2\alpha_{\mathbf{K}}^{n-1} h_{\mathbf{K}} \log^n \dfrac{\pi \sqrt{u}\, G(u)^{-1}}{2\sqrt{|d_{\mathbf{K}}|}\, 3^{61n} n^{16n} \mathfrak{N}((a_{i_0} b_{i_0}))}}.$$

Now we consider the product

$$\prod_{\mathfrak{p} \nmid H} (1 + h(\mathfrak{p})) \left(1 - \frac{1}{\mathfrak{N}(\mathfrak{p})}\right)^n = \prod_{\mathfrak{p} \nmid H} \left(1 + \frac{\rho_F(\mathfrak{p})}{\mathfrak{N}(\mathfrak{p}) - \rho_F(\mathfrak{p})}\right) \left(1 - \frac{1}{\mathfrak{N}(\mathfrak{p})}\right)^n$$
$$\leq \prod_{\mathfrak{p} \nmid H} \left(\left(1 + \frac{1}{\mathfrak{N}(\mathfrak{p}) - \rho_F(\mathfrak{p})}\right) \left(1 - \frac{1}{\mathfrak{N}(\mathfrak{p})}\right)\right)^n.$$

We see that

$$\left(1 + \frac{1}{\mathfrak{N}(\mathfrak{p}) - \rho_F(\mathfrak{p})}\right) \left(1 - \frac{1}{\mathfrak{N}(\mathfrak{p})}\right) = \left(1 + \frac{\rho_F(\mathfrak{p}) - 1}{\mathfrak{N}(\mathfrak{p})(\mathfrak{N}(\mathfrak{p}) - \rho_F(\mathfrak{p}))}\right)$$
$$\leq \left(1 + \frac{1}{\mathfrak{N}(\mathfrak{p})(\mathfrak{N}(\mathfrak{p}) - \rho_F(\mathfrak{p}))}\right)^{n-1}.$$

This gives

$$\prod_{\mathfrak{p} \nmid H} (1 + h(\mathfrak{p})) \left(1 - \frac{1}{\mathfrak{N}(\mathfrak{p})}\right)^n \leq \prod_{\mathfrak{p} \nmid H} \left(1 + \frac{1}{\mathfrak{N}(\mathfrak{p})(\mathfrak{N}(\mathfrak{p}) - \rho_F(\mathfrak{p}))}\right)^{n(n-1)}.$$

Finally,

$$\prod_{\mathfrak{p} \nmid H} \left(1 + \frac{1}{\mathfrak{N}(\mathfrak{p})(\mathfrak{N}(\mathfrak{p}) - \rho_F(\mathfrak{p}))}\right)$$
$$\leq \prod_{\substack{\mathfrak{N}(\mathfrak{p}) < 2n \\ \mathfrak{p} \nmid H}} \left(1 + \frac{1}{\mathfrak{N}(\mathfrak{p})(\mathfrak{N}(\mathfrak{p}) - \rho_F(\mathfrak{p}))}\right) \prod_{\mathfrak{p}} \left(1 + \frac{1}{\mathfrak{N}(\mathfrak{p})^2}\right)^2.$$

Therefore the constant satisfies

$$\prod_{\mathfrak{p}\nmid H}(1+h(\mathfrak{p}))\Big(1-\frac{1}{\mathfrak{N}(\mathfrak{p})}\Big)^n \le 2^{n(n-1)\pi_{\mathbf{K}}(2n)}\zeta_{\mathbf{K}}(2)^{2n(n-1)} \le 2^{6n^3}.$$

Thus, for

$$u^{1/4} \ge \exp(18(n+1)L_{\mathbf{K}})\bigg(\frac{\sqrt{|d_{\mathbf{K}}|}\,3^{23n^3}n^{17n}\mathfrak{N}((a_{i_0}b_{i_0}))\alpha_{\mathbf{K}}^n}{n!\pi}\bigg),$$

we have

$$D \le \frac{5\big(\prod_{\mathfrak{p}|H}\big(1-\frac{1}{\mathfrak{N}(\mathfrak{p})}\big)^{-n}\prod_{\mathfrak{p}\nmid H}(1+h(\mathfrak{p}))^{-1}\big(1-\frac{1}{\mathfrak{N}(\mathfrak{p})}\big)^{-n}\big)n!|\mu_{\mathbf{K}}|u}{2\alpha_{\mathbf{K}}^{n-1}h_{\mathbf{K}}\log^n\dfrac{n!\pi\sqrt{u}}{\sqrt{|d_{\mathbf{K}}|}\,3^{22n^3}n^{16n}\mathfrak{N}((a_{i_0}b_{i_0}))\alpha_{\mathbf{K}}^n\log^n u}}.$$

Further, since $u^{1/(4n)} > \log u^{1/(4n)}$, we get

$$u^{1/4} \ge \exp(18(n+1)L_{\mathbf{K}})\frac{\sqrt{|d_{\mathbf{K}}|}\,3^{23n^3}n^{17n}\mathfrak{N}((a_{i_0}b_{i_0}))\alpha_{\mathbf{K}}^n}{n!\pi},$$

and consequently

$$D \le \frac{5\big(\prod_{\mathfrak{p}|H}\big(1-\frac{1}{\mathfrak{N}(\mathfrak{p})}\big)^{-n}\prod_{\mathfrak{p}\nmid H}(1+h(\mathfrak{p}))^{-1}\big(1-\frac{1}{\mathfrak{N}(\mathfrak{p})}\big)^{-n}\big)n!|\mu_{\mathbf{K}}|u}{2\alpha_{\mathbf{K}}^{n-1}h_{\mathbf{K}}\log^n\dfrac{n!\pi u^{1/4}}{\sqrt{|d_{\mathbf{K}}|}\,3^{23n^3}n^{17n}\mathfrak{N}((a_{i_0}b_{i_0}))\alpha_{\mathbf{K}}^n}}.$$

Note that by relabelling a_i's and b_i's for $1 \le i \le n$, we can choose i_0 to be equal to 1.

Acknowledgements. We thank the referee for her/his suggestions which made the exposition of the article better.

Research of this article was partially supported by Indo-French Program in Mathematics (IFPM). All authors would like to thank IFPM for financial support. The first author would also like to acknowledge SPARC project 445 and DAE number theory plan project for partial financial support.

References

[1] P. T. Bateman and A. R. Horn, *A heuristic asymptotic formula concerning the distribution of prime numbers*, Math. Comp. 16 (1962), 363–367.

[2] P. T. Bateman and A. R. Horn, *Primes represented by irreducible polynomials in one variable*, in: Theory of Numbers, Proc. Sympos. Pure Math. 8, Amer. Math. Soc., Providence, RI, 119–135, 1965.

[3] K. Debaene, *Explicit counting of ideals and a Brun–Titchmarsh inequality for the Chebotarev density theorem*, Int. J. Number Theory 15 (2019), 883–905.

[4] J.-M. Deshouillers, S. Gun, O. Ramaré and J. Sivaraman, *Representing ideal classes of ray class groups by product of prime ideals of small size*, arXiv:2210.11051 (2022).

[5] L. E. Dickson, *A new extension of Dirichlet's theorem on prime numbers*, Messenger of Math. 33 (1903/1904), 155–161.
[6] B. V. Levin and A. S. Faĭnleĭb, *On a method of summation of multiplicative functions*, Izv. Akad. Nauk SSSR Ser. Mat. 31 (1967), 697–710 (in Russian).
[7] S. Gun, O. Ramaré and J. Sivaraman, *Counting ideals in ray classes*, J. Number Theory 243 (2023), 13–37.
[8] H. Halberstam and H.-E. Richert, *Sieve Methods*, London Math. Soc. Monogr. 4, Academic Press, London, 1974.
[9] N. Kandhil, R. Lunia and J. Sivaraman, *Explicit upper bounds on the avarage of Euler–Kronecker constants of narrow ray class fields*, Res. Number Theory 9 (2023), no. 4, art. 69, 32 pp, 2023.
[10] E. S. Lee, *Explicit Mertens' theorems for number fields*, PhD thesis, Univ. of South Wales Canberra of the Australian Defence Force Academy.
[11] O. Ramaré, *Excursions in Multiplicative Number Theory*, Birkhäuser/Springer, Basel, 2022.
[12] J. B. Rosser and L. Schoenfeld, *Approximate formulas for some functions of prime numbers*, Illinois J. Math. 6 (1962), 64–94.

Sanoli Gun
Institute of Mathematical Sciences
A CI of Homi Bhabha National Institute
CIT Campus, Taramani
Chennai 600113, India
E-mail: sanoli@imsc.res.in

Jyothsnaa Sivaraman
Statistics and Mathematics Unit
Indian Statistical Institute
8th Mile, Mysore Rd, RVCE Post
Bengaluru 560059, India
E-mail: jyothsnaa.s@gmail.com

Olivier Ramaré
CNRS/Institut de Mathématiques de Marseille
Aix Marseille Université
U.M.R. 7373, Site Sud
Campus de Luminy, Case 907
13288 Marseille Cedex 9, France
E-mail: olivier.ramare@univ-amu.fr