ON THE CONVOLUTION OF PERIODIC MULTIPLICATIVE FUNCTIONS

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Abstract. Let f and g be two multiplicative functions that are periodic, non-vanishing at their period and with bounded partial sums. We prove that $\sum_{n\leq x}(f*g)(n)=\Omega(x^{1/4})$ when the periods of f and g, say M_1 and M_2, are squarefree.

1. Introduction

Solving the Erdős-Coons-Tao conjecture and building upon work of Tao [?], Klurman [?] proved that the only multiplicative functions f taking ±1 values and such that

$$\sup_x \left|\sum_{n\leq x} f(n)\right| < \infty$$

are the periodic multiplicative functions with bounded partial sums.

Building upon the referred work of Klurman, the first author proved [?] that if we allow values outside the unit disk, a m-periodic multiplicative function f with bounded partial sums such that $f(m)\neq 0$ satisfies

i. For some prime $q|m$, $\sum_{k=0}^{\infty}\frac{f(q^k)}{q^k}=0$.
ii. For each $p^a\|m$, $f(p^k)=f(p^a)$ for all $k\geq a$.
iii. For each $\gcd(p,m)=1$, $f(p^k)=1$, for all $k\geq 1$.

Conversely, if $f:\mathbb{N}\to\mathbb{C}$ is multiplicative and the three conditions above are satisfied, then f has period m and has bounded partial sums. Therefore, these three conditions above give examples of multiplicative functions with values outside the unit disk with bounded partial sums, despite of the fact that $f(m)$ is zero or not.

Here we are interested in the convolution $f*g$ for f and g satisfying i-ii-iii. If M_1 and M_2 are the periods of f and g respectively, then it was proved [?]

$$\sum_{n\leq x}(f*g)(n) \ll x^{\alpha+\epsilon},$$

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where α is the infimum over the exponents $a > 0$ such that $\Delta(x) \ll x^a$, where $\Delta(x)$ is the classical error term in the Dirichlet divisor problem:

$$\sum_{n\le x}\tau(n) = x\log x + (2\gamma-1)x + \Delta(x).$$

It was conjectured in [?] that these sums are $\Omega(x^{1/4})$. Here we establish this conjecture in some particular cases.

Theorem 1.1. *Let f and g be periodic multiplicative functions satisfying i-ii-iii above, with periods M_1 and M_2 respectively. Assume that M_1 and M_2 are squarefree. Then*

$$\sum_{n\le x}(f*g)(n) = \Omega(x^{1/4}).$$

To proof this result, our starting point is the following formula from [?]:

$$\sum_{n\le x}(f*g)(n) = \sum_{n|M_1M_2}(f*g*\mu*\mu)(n)\Delta(x/n), \tag{1}$$

where μ is the Möbius function.

Our proof of Theorem 1.1 is inspired by an elegant result of Tong [?]:

$$\int_1^X \Delta(x)^2dx = (1+o(1))\left(\sum_{n=1}^{\infty}\frac{\tau(n)^2}{n^{3/2}}\right)X^{3/2}.$$

By Equation (1), the limit

$$\lim_{X\to\infty}\frac{1}{X^{3/2}}\int_1^X\left|\sum_{n\le x}(f*g)(n)\right|^2dx$$

can be expressed as a quadratic form with matrix $(a_{n,m})_{n,m|M_1M_2}$ where

$$a_{n,m} := \lim_{X\to\infty}\frac{1}{X^{3/2}}\int_1^X\Delta(x/n)\Delta(x/m)dx.$$

In the case that M_1 and M_2 are squarefree, we prove that the associated quadratic form is positive. For general values of periods M_1 and M_2 the matrix analysis become very hard and we were not able to describe the eigenvalues of the associated matrix, although we believe that all of them are positive.

2. Notation

We employ both Vinogradov's notation $f \ll g$ or $f = O(g)$ whenever there exists a constant $C > 0$ such that $|f(x)| \le C|g(x)|$, for all x in a set of parameters. When not specified, this set of parameters is $x \in (a,\infty)$ for sufficiently large $a > 0$. We employ

$f = o(g)$ when $\lim_{x\to a} \frac{f(x)}{g(x)} = 0$. In this case a can be a complex number or $\pm\infty$. Finally, $f = \Omega(g)$ when $\limsup_{x\to a} \frac{|f(x)|}{g(x)} > 0$, where a is as in the previous notation.

3. Multiplicative auxiliaries

We begin the proof with the following Lemma.

Lemma 3.1. *Let a, b be positive integers, $\lambda = \gcd(a,b)$, $c = a/\lambda$ and $d = b/\lambda$. Then*

$$\lim_{X\to\infty} \int_1^X \Delta(x/a)\Delta(x/b)\frac{dx}{X^{3/2}} = \frac{1}{2\pi^2\sqrt{\lambda}cd}\sum_{n=1}^{\infty}\frac{\tau(cn)\tau(dn)}{n^{3/2}}.$$

Proof. Let $N > 0$ and $\epsilon > 0$ be a small number that may change from line after line. We proceed with Voronoï's formula for $\Delta(x)$ in the following form (see [?])

$$\Delta(x) = \frac{x^{1/4}}{\pi\sqrt{2}}\sum_{n\le N}\frac{\tau(n)}{n^{3/4}}\cos(\sqrt{nx} - \pi/4) + R_N(x),$$

where, for every positive ϵ, we have

$$R_N(x) \ll x^{\epsilon} + \frac{x^{1/2+\epsilon}}{N^{1/2}}.$$

We select $N = X^{3/4}$. This choice is not optimal but will suffice. This implies that in the range $1 \le x \le X$,

$$\Delta(x/a) = \frac{(x/a)^{1/4}}{\pi\sqrt{2}}\sum_{n\le N}\frac{\tau(n)}{n^{3/4}}\cos(\sqrt{nx/a} - \pi/4) + R_N(x/a) = U_N(x/a) + R_N(x/a)$$

say, where $R_N(x/a) \ll X^{1/8+\epsilon}$.

Now,

$$\begin{aligned}\int_1^X \Delta(x/a)\Delta(x/b)dx &= \int_1^X U_N(x/a)U_N(x/b)dx + \int_1^X U_N(x/a)R_N(x/b)dx\\ &\quad + \int_1^X U_N(x/b)R_N(x/a)dx + \int_1^X R_N(x/a)R_N(x/b)dx\\ &= \int_1^X U_N(x/a)U_N(x/b)dx + O(X^{3/2-1/8+\epsilon}),\end{aligned}$$

where we used the Cauchy-Schwarz inequality in the last equality. Let now $\lambda = \gcd(a,b)$, $c = a/\lambda$ and $d = b/\lambda$. By making the change of variable $u = x/\lambda$, we reach

$$\begin{aligned}\int_1^X U_N(x/a)U_N(x/b)dx &= \lambda\int_1^{X/\lambda} U_N(x/c)U_N(x/d)dx\\ &= \frac{\lambda}{2\pi^2(cd)^{1/4}}\sum_{n,m\leq N}\frac{\tau(n)\tau(m)}{(nm)^{3/4}}\int_1^{X/\lambda} x^{1/2}\cos(\sqrt{nx/c}-\pi/4)\cos(\sqrt{mx/d}-\pi/4)\\ &= \frac{\lambda}{\pi^2(cd)^{1/4}}\sum_{n,m\leq N}\frac{\tau(n)\tau(m)}{(nm)^{3/4}}\int_1^{X^{1/2}/\lambda^{1/2}} x^2\cos(\sqrt{n/c}x-\pi/4)\cos(\sqrt{m/d}x-\pi/4),\end{aligned}$$

where in the last equality above we made a change of variable $u = \sqrt{x}$. We claim now that the main contribution comes when $n/c = m/d$. Since c and d are coprime, the sum over these n and m can be written as

$$\frac{\lambda}{\pi^2 cd}\sum_{n=1}^{\infty}\frac{\tau(cn)\tau(dn)}{n^{3/2}}\int_1^{X^{1/2}/\lambda^{1/2}} x^2\cos^2(\sqrt{n}x-\pi/4)^2dx + O(X^{3/2-3/8+\epsilon}). \tag{2}$$

We recall now that $\cos^2(u) = \frac{1+\cos(2u)}{2}$, and hence the integral above is

$$\int_1^{X^{1/2}/\lambda^{1/2}} x^2\cos^2(\sqrt{n}x-\pi/4)dx = \frac{X^{3/2}}{2\lambda^{3/2}} + O(X), \tag{3}$$

where the big-oh term is uniform in n. Now we will show that the sum over those n and m such that $n/c \neq m/d$ will be $o(X^{3/2})$. With this the proof will be complete by combining (2) and (3).

We recall the identity $2\cos(u)\cos(v) = \cos(u-v)+\cos(u+v)$. Thus, for $\sqrt{n/c} \neq \sqrt{m/d}$, we find that

$$\begin{aligned}&\int_1^{X^{1/2}/\lambda^{1/2}} x^2\cos(\sqrt{n/c}x-\pi/4)\cos(\sqrt{m/d}x-\pi/4)dx\\ &= \int_1^{X^{1/2}/\lambda^{1/2}} x^2\cos((\sqrt{n/c}-\sqrt{m/d})x)dx + \int_1^{X^{1/2}/\lambda^{1/2}} x^2\sin((\sqrt{n/c}+\sqrt{m/d})x)dx\\ &\ll \frac{X}{\sqrt{n/c}-\sqrt{m/d}} \ll \frac{\sqrt{n/c}+\sqrt{m/d}}{nd-mc}X.\end{aligned}$$

Let $\mathbb{1}_P(n)$ be the indicator that n has property P. We find that

$$\sum_{\substack{n,m\le N\\ nd-mc\neq 0}} \frac{\tau(n)\tau(m)}{(nm)^{3/4}} \int_1^{X/\lambda} x^{1/2}\cos(\sqrt{nx/c}-\pi/4)\cos(\sqrt{mx/d}-\pi/4)dx$$
$$\ll XN^{\epsilon} \sum_{\substack{n,m\le N\\ nd-mc\neq 0}} \frac{\sqrt{n/c}+\sqrt{m/d}}{(nm)^{3/4}|nd-mc|}$$
$$= XN^{\epsilon} \sum_{\substack{n,m\le N\\ nd-mc\neq 0}} \frac{\sqrt{n/c}+\sqrt{m/d}}{(nm)^{3/4}|nd-mc|} \sum_{\substack{k=-N\max(c,d)\\ k\neq 0}}^{N\max(c,d)} \mathbb{1}_{nd-mc=k}.$$

On calling this sum S, we readily continue with

$$S \ll XN^{\epsilon} \sum_{k=1}^{N\max(c,d)} \frac{1}{k} \sum_{m\le N} \frac{\sqrt{m}+\sqrt{k}}{((k+mc)m)^{3/4}}$$
$$\ll XN^{\epsilon}\left(O(\log N)^2 + \sum_{k=1}^{N} \frac{1}{\sqrt{k}} \sum_{m\le N} \frac{1}{(km+m^2)^{3/4}}\right)$$
$$\ll XN^{\epsilon}\left(O(\log N)^2 + \sum_{k=1}^{N} \frac{1}{\sqrt{k}} \int_0^{\infty} \frac{dx}{(k+x^2)^{3/4}}\right)$$
$$\ll XN^{\epsilon}\left(O(\log N)^2 + \sum_{k=1}^{N} \frac{1}{\sqrt{k}}\frac{1}{k^{1/4}}\right) \ll XN^{1/4+\epsilon} = X^{1+3/16+\epsilon}.$$

The proof is complete. □

Our next task is to evaluate

$$\sum_{n=1}^{\infty} \frac{\tau(cn)\tau(dn)}{n^{3/2}},$$

for coprime positive integers c and d.

Lemma 3.2. *Let c be fixed positive number and $f(n)$ be a multiplicative function with $f(c)\neq 0$. Then $n\mapsto \frac{f(cn)}{f(c)}$ is multiplicative.*

Proof. We have that for positive integers u, v, we have

$$f(u)f(v) = f(\gcd(u,v))f(\mathrm{lcm}(u,v)).$$

Let $u = cn$, $v = cm$ with $\gcd(n,m) = 1$. Then $f(cn)f(cm) = f(c)f(cnm)$.

Therefore, we obtained

$$\frac{f(cm)}{f(c)}\frac{f(cn)}{f(c)} = \frac{f(cnm)}{f(c)}.$$

□

Lemma 3.3. *Let c, d be two fixed positive integers with* $\gcd(c,d)=1$. *Then*

$$\sum_{n\geq 1}\frac{\tau(cn)\tau(dn)}{n^s}=\tau(cd)\frac{\zeta(s)^4}{\zeta(2s)}\prod_{p^k\|cd}\left(1+p^{-s}\right)^{-1}\left(1-\tfrac{(k-1)}{(k+1)}p^{-s}\right).$$

Proof. Note that $\frac{\tau(cn)}{\tau(c)}$ is a multiplicative function in the variable n, and so is $\frac{\tau(cn)\tau(dn)}{\tau(c)\tau(d)}$. Therefore, for $\Re(s)>1$ we have the following Euler factorization

$$\sum_{n\geq 1}\frac{\tau(cn)\tau(dn)}{\tau(c)\tau(d)n^s}=\prod_{p\nmid cd}\left(1+\sum_{\ell\geq 1}\frac{\tau(p^\ell)^2}{p^{\ell s}}\right)\prod_{p|cd}\left(1+\sum_{\ell\geq 1}\frac{\tau(cp^\ell)\tau(dp^\ell)}{\tau(c)\tau(d)p^{\ell s}}\right).$$

For $|x|<1$, we know that

$$\sum_{\ell\geq 0}(\ell+1)x^\ell=\frac{1}{(1-x)^2},\qquad \sum_{\ell\geq 0}(\ell+1)^2x^\ell=\frac{(1+x)}{(1-x)^3},$$

from which we also derive that

$$\sum_{\ell\geq 0}\ell(\ell+1)x^\ell=\frac{2x}{(1-x)^3}.$$

Now,

$$\begin{aligned}\prod_{p\nmid cd}\left(1+\sum_{\ell\geq 1}\frac{\tau(p^\ell)^2}{p^{\ell s}}\right)&=\prod_{p}\left(1+\sum_{\ell\geq 1}\frac{(\ell+1)^2}{p^{\ell s}}\right)\prod_{p|cd}\left(1+\sum_{\ell\geq 1}\frac{(\ell+1)^2}{p^{\ell s}}\right)^{-1}\\&=\prod_{p}\frac{(1+p^{-s})}{(1-p^{-s})^3}\prod_{p|cd}\frac{(1-p^{-s})^3}{(1+p^{-s})}\\&=\frac{\zeta(s)^4}{\zeta(2s)}\prod_{p|cd}\frac{(1-p^{-s})^3}{(1+p^{-s})}.\end{aligned}$$

If $\gcd(c,d)=1$

$$\begin{aligned}\prod_{p|cd}\left(1+\sum_{\ell\geq 1}\frac{\tau(cp^\ell)^2\tau(dp^\ell)^2}{\tau(c)\tau(d)p^{\ell s}}\right)&=\prod_{p^k\|cd}\left(1+\sum_{\ell\geq 1}\frac{(k+1+\ell)(\ell+1)}{(k+1)p^{\ell s}}\right)\\&=\prod_{p^k\|cd}\left(1+\sum_{\ell\geq 1}\frac{(\ell+1)}{p^{\ell s}}+\frac{1}{k+1}\sum_{\ell\geq 1}\frac{\ell(\ell+1)}{p^{\ell s}}\right)\\&=\prod_{p^k\|cd}\left(1-p^{-s}\right)^{-3}\left(1-\tfrac{(k-1)}{(k+1)}p^{-s}\right).\end{aligned}$$

□

4. Quadratic forms auxiliaries

The main proof will lead to considering the quadratic form attached with a matrix of the form

$$M_{S,f} = \left(f\left(\frac{\mathrm{lcm}(a,b)}{\gcd(a,b)}\right)\right)_{a,b\in S} \tag{4}$$

where S is some finite set of integers while f is a *non-negative multiplicative function such that* $f(p^k) \leq$. So we stray somewhat from the main line and investigate this situation. Our initial aim is to find conditions under which the associated quadratic form is positive definite, but we shall finally restrict our scope. GCD-matrices have received quite some attention, but it seems the matrices occuring in (4) have not been explored. We obtain results in two specific contexts.

Completely multiplicative case. Here is our first result.

Lemma 4.1. *When f is completely multiplicative, the matrix $M_{S,f}$ is non-negative. When $f(p) \in (0,1)$ and S is divisor closed, this matrix is positive definite. The determinant in that case is given by the formula*

$$\det\left(f\left(\frac{\mathrm{lcm}(a,b)}{\gcd(a,b)}\right)\right)_{a,b\in S} = \prod_{d\in S} f(d)^2\Big(\mu * \frac{1}{f^2}\Big)(d).$$

By *divisor closed*, we mean that every divisor of an element of S also belongs to S.

Proof. We write $f(\mathrm{lcm}(a,b)/\gcd(a,b)) = f(ab/\gcd(a,b)^2) = f(a)f(b)g(\gcd(a,b)^2)$ where $g(n) = f(1/n)^2$ is another non-negative multiplicative function. We use introduce the auxiliary function $h = \mu * g$. Please notice that this function is multiplicative and non-negative, as $g(p) \geq 1$. We use Selberg's diagonalization process to write

$$\begin{aligned}\sum_{a,b\in S} f\left(\frac{\mathrm{lcm}(a,b)}{\gcd(a,b)}\right)x_a x_b &= \sum_{a,b\in S} g(\gcd(a,b))f(a)x_a f(b)x_b \\ &= \sum_{a,b\in S}\sum_{d|(a,b)} h(d)f(a)x_a f(b)x_b = \sum_d h(d)\Bigg(\sum_{\substack{a\in S\\ d|a}} f(a)x_a\Bigg)^2\end{aligned}$$

from which the non-negativity follows readily. When f verifies the more stringent condition that $f(p) \in (0,1)$, we know that both f and h are strictly positive. Let us define $y_d = \sum_{\substack{a\in S\\ d|a}} f(a)x_a$. The variable d varies in the set D of divisors of S. We assume that S is divisor closed, so that $D = S$. We can readily invert the triangular system giving the y_d's as functions of the x_a's into

$$f(a)x_a = \sum_{a|b} \mu(b/a)y_b$$

Indeed, the fact that the mentioned system is triangular ensures that a solution y is unique if it exists. We next verify that the proposed expression is indeed a solution by:

$$\sum_{\substack{a\in S\\ d|a}} f(a)x_a = \sum_{\substack{a\in S\\ d|a}} \sum_{a|b} \mu(b/a)y_b = \sum_{\substack{b\in S\\ d|b}} y_b \sum_{d|a|b} \mu(b/a) = y_d$$

as the last inner sum vanishes when $d \neq b$. We thus have a writing as a linear combination of squares of independant linear forms. In a more pedestrian manner, if our quadratic form vanishes, then all y_d's do vanish, hence so do the x_a's. □

Here is a corollary.

Lemma 4.2. *When the set S contains solely squarefree integers, the matrix $M_{S,f}$ is non-negative.*

Proof. Simply apply Lemma 4.1 to completely multiplicative function f' that have the same values on primes as f. □

An additive-like situation. Here, we restrict our attention to the case when $S = \{1, p, p^2, \cdots, p^K\}$. In that case, the matrix we get is simply a symmetric Toeplitz matrix. These have been thoroughly studied and we cannot get general results like Lemma 4.2 is that case. We may however work out some criterium that is simple to verify in our case. We first recall the following lemma of Frobenius.

Lemma 4.3. *A hermitian complex valued matrix $M = (m_{i,j})$ defines a positive definite form if and only if all its principal minors*

$$\det(m_{i,j})_{i,j\leq m}$$

are positive

So in our case, here is the list of conditions to verify:

- $1 - f(p)^2 > 0$
- $(1 - f(p^2))(f(p^2) - 2f(p)^2 + 1) > 0$.
- $(f(p)^2 - (1 - 2f(p^2) + f(p^3))(1 + f(p)))(f(p)^2 + (1 - 2f(p^2) - f(p^3))(1 + f(p))) > 0$.

We conclude to the next lemma.

Lemma 4.4. *Recall that $f(1) = 1$, that $f(p) \in (0,1)$ and that $f(p^2) \in (0,1)$. we have*

(1) The matrix $(f(p^{\max(i,j)-\min(i,j)}))_{i,j\leq 1}$ is positive definite.
(2) The matrix $(f(p^{\max(i,j)-\min(i,j)}))_{i,j\leq 2}$ is positive definite if and only if $f(p^2) - 2f(p)^2 + 1$.

(3) The matrix $(f(p^{\max(i,j)-\min(i,j)}))_{i,j\le 3}$ *is positive definite if and only if* $f(p^2)-2f(p)^2+1$ *and*

$$\left[\frac{f(p)^2}{1+f(p)}-f(p^3)-1+2f(p^2)\right]\left[\frac{f(p)^2}{1+f(p)}-f(p^3)+1-2f(p^2)\right]>0.$$

A tensor product-like situation. Lemma 4.2 is enough to solve our main problem when M_1 and M_2 are coprime. We need to go somewhat further. Let S be a divisor closed set. We consider the quadratic form

$$\sum_{a,b\in S} f\left(\frac{\operatorname{lcm}(a,b)}{\gcd(a,b)}\right)x_a x_b \tag{5}$$

where the variables x_a's are also multiplicatively split, i.e.

$$x_a = \prod_{p^k\|a} x_{p^k}. \tag{6}$$

Let $S(p)$ the subset of S made only of 1 and of prime powers. We extend S so that it contains every products of integers from any collection of distinct $S(p)$*. We then find that

$$\sum_{a,b\in S} f\left(\frac{\operatorname{lcm}(a,b)}{\gcd(a,b)}\right)x_a x_b = \prod_{p\in S}\left(\sum_{p^k,p^\ell\in S(p)} f\left(p^{\max(k,\ell)-\min(k,\ell)}\right)x_{p^k}x_{p^\ell}\right). \tag{7}$$

We check this identity simply by opening the right-hand side and seeing that every summand from the left-hand side appears one and only one time. Then Lemma 4.4 applies.

5. Proof of the main result

References

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*This is not automatically the case, as the example $S=\{1,2,3,5,6,10\}$ shows, since 30 does not belong to S