On Bombieri's asymptotic sieve

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Abstract

We improve the error term in the Bombieri asympotic sieve when the summation is restricted to integers having at most two prime factors. This results in a refined bilinear decomposition for the characteristic function of the primes that enables us to get a best possible estimate for the trigonometric polynomial over primes.

Key words: Sieve; Bombieri asymptotic sieve; bilinear form.
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1 Introduction

Since the discovery and systematic use of bilinear forms, our understanding of primes has greatly improved. This technique enables one to translate our knowledge of integers to the one of primes, and quite often, problems get reduced to finding a suitable bilinear form with which to write the characteristic function of the primes. Such a bilinear form converts the problem to a corresponding one over integers ranging an interval. However this decomposition of the characteristic function of primes introduces divisor functions, amounting to a loss of some power of $\operatorname{Log} X$ when working with primes $\le X$ (in fact the "trivial bound" is increased by such an amount). We produce here a family of bilinear forms that do not have this feature; more precisely our divisors will have a bounded number of prime factors. Moreover we shall completely explicitate the dependence in this number of prime factors and further reduce their effect by using a preliminary sieving. We first choose an integer $\mathfrak{f}$ and write $\prod^*_{p\le y}$ for a product over primes $\le y$ and prime to $\mathfrak{f}$, and $\sum^*_{d\le y}$ to

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denote a summation over integers $d \leq y$ and prime to $\mathfrak{f}$. We put $\Lambda^{(t)}(n)$ for $\Lambda(n)(\operatorname{Log} n)^{t-1}/(t-1)!$ and study

$$\Sigma_\nu(f, X) = \sum_{n \leq X}{}^{*} \Lambda^{(2\nu)}(n) f(n) + \sum_{n \leq X}{}^{*} \Lambda^{(\nu)} \star \Lambda^{(\nu)}(n) f(n) \tag{1}$$

for a fairly wide class of non-negative functions defined below. The dependence on $\mathfrak{f}$ is easily handled. The non-negativity assumption is much more drastic. However, since we can close the set for which the asymptotic expression holds under linear combination, we can also deal with complex-valued oscillating functions that are bounded by an element of this class. We shall details some applications in Corollary 30, 31 and 32. Let us mention here that Lemma 29 provides us with an asymptotic for the reference quantity $\Sigma_\nu(1, X)$. A close look at the proof will also reveal to the reader that the contribution coming from $\Lambda^{(2\nu)}$ is very close to the contribution of $\Lambda^{(\nu)} \star \Lambda^{(\nu)}$.

Our course of action will be as follows. We will assume a simple "model" f_0 is given for f (in a sense to be precised below) and show in Theorem 1 and 2 that $\Sigma_\nu(f_0, X)$ is a good approximation to $\Sigma_\nu(f, X)$. This part is better sketched in subsection 2.1. In many problems the relevant expression with f_0 may be evaluated by different tools. We fail to evaluate this sum because the function F that appears in (H_1) below is too general for such a purpose.

Theorem 2 is quite sharp to describe the mean value of functions over integers having at most two prime factors. In many cases however, one would like to know the mean value of functions over primes only. Starting from Theorem 1, two courses of actions are possible: using a tauberian argument or handle the bilinear part as a remainder term. This is the path we follow in Theorem 3 in order to bound

$$\sum_{n \leq X} \Lambda(n) e(n\alpha)$$

for a small α close to a rational with a small denominator.

1.1 Description of the properties required on f

The parameters $X \geq e^{20}$ and $\nu \geq 1$ are fixed throughout this paper. The parameters $D_0 = X^{1-\delta}$ and $z = X^\delta$ are also fixed with some $\delta \leq 1/(4\nu)$. We further assume that $\mathfrak{f}$ has only prime factors less than z.

∘ To be able to sieve the sequence $(f(n))$, we need some regularity which we express in the following form. For d prime to $\mathfrak{f}$, define

$$\sum_{n \leq y/d}{}^{*} f(dn) = \sigma(d) F(y) + r_d(f, y)$$

where the $r_d(f,y)$'s are looked upon as error terms, σ is a multiplicative function and F is any function. We suppose given a "simple" model f_0 for f and define $r_d(f_0,y)$ again by

$$\sum_{n\le y/d}^{*} f_0(dn) = \sigma_0(d)F(y) + r_d(f_0,y). \tag{2}$$

These parameters are assumed to verify

$$\begin{cases} |F(y)| \le \hat{F}(X) \quad (y \le X), \qquad 0 \le \sigma(p), \sigma_0(p) < 1, \\ \forall n \le X, \quad f_0(n) \le B_0\hat{F}(X)/X, \\ \prod_{v\le p\le u}^{*} (1-\sigma(p))^{-1} + \prod_{v\le p\le u}^{*} (1-\sigma_0(p))^{-1} \le c\dfrac{\operatorname{Log} u}{\operatorname{Log} v} \quad (2 \le v \le u) \end{cases} \tag{H_1}$$

where c is a constant ≥ 2.

We assume also an inequality in the other direction, namely:

$$V_{\sigma_0}(z) \le c/\operatorname{Log} z \tag{H_2}$$

where we use the definition ($\tilde{\sigma}$ being a generic multiplicative function)

$$V_{\tilde{\sigma}}(z) = \prod_{p\le z}^{*}(1-\tilde{\sigma}(p)). \tag{3}$$

The introduction of σ_0 and f_0 serve two purposes. The first problem we meet is that the main terms are going to be difficult to compute; so much so that we do not even attempt this evaluation but only show how to replace a difficult one (with f) to a supposedly simpler one (with f_0). In [6], the problem is handled in a similar fashion by assuming the main term to simply vanish. The second problem is the one of uniformity which becomes stringent when working with the sequence $(\Lambda(N-p))_{p\le N}$ for instance. The sieve knows how to deal with that, the key observation being that the one sided condition (22) is really what is required: the parameters introduced usually diminish the left-hand side so that c has some uniformity. The proof of Corollary 32 displays on an example how all of that works. This takes care of the uniformity for the primes $\le z$. For the larger ones, the main term with f_0 still has to be computed; this main term is a sum over integers prime to z.

One of the fundamental hypothesis we make concerns positivity

$$f \ge 0 \quad \text{and} \quad f_0 \ge 0. \tag{H_3}$$

This will be crucial to use the sieve argument, but as we remarked above, there is a workaround for oscillating functions; the proof of Theorem 3 uses such a technique.

We quantify the fact that $r_d(f, y)$ (resp. $r_d(f_0, y)$) is an error term with an assumption on

$$\mathrm{R}(f, D, r) = \sum_{d \le D}^{*} \tau_r(d) \max_{y \le X} |r_d(f, y)|, \tag{4}$$

where τ_r is the r-th divisor function. Our precise hypothesis is as follows:

$$\mathrm{R}(f, D_0, 2\nu) + \frac{V_\sigma(z)}{V_{\sigma_0}(z)} \mathrm{R}(f_0, D_0, 2\nu) \le A\hat{F}(X)/\operatorname{Log} X. \tag{H_4}$$

Such an inequality may be tricky to verify when ν is a function of X and we provide in (H_9) a simpler hypothesis.

∘ To handle the difference from f to f_0, we shall use

$$\Delta = \sum_{X \ge p^a, p \ge z} |\sigma(p^a) - \sigma_0(p^a)|. \tag{5}$$

and we control the large values of σ_0 by

$$\prod_{z \le p \le X}^{*} \sum_{a \ge 0} \sigma_0(p^a) \le c \frac{\operatorname{Log} X}{\operatorname{Log} z}. \tag{H_5}$$

In Lemma 21, we deduce from Δ and (H_5) a control on the large values of σ.

1.2 A first result

We start with a fairly raw result taht requires few hypotheses. It still contains a preliminary sieving, which is easily removed in most cases. This of course corresponds to the case $\mathfrak{f} = \prod_{p \le z} p$ and we a priori do not need more notations. However, in the course of the proof, we will write $\sum'$ for a summation restricted to integers having no prime factors below z. To maintain notational consistency, we denote by $\Sigma'_\nu(f, X)$ the sum $\Sigma^*_\nu(f, X)$ when $\mathfrak{f} = \prod_{p \le z} p$.

Theorem 1 *Assuming (H_1)—(H_5), we have*

$$\Sigma'_\nu(f, X) = \frac{V_\sigma(z)}{V_{\sigma_0}(z)} \Sigma'_\nu(f_0, X) + (\rho + \tilde{\rho}) \cdot \frac{V_\sigma(z)}{V_{\sigma_0}(z)} \hat{F}(X) \frac{(\operatorname{Log} X)^{2\nu - 1}}{(2\nu - 1)!}$$

where

$$\begin{cases} |\rho| \le (56\nu^2)^\nu \Big\{ c^2 A + \Big(C_0(c)\delta^{5\nu} + 4^{1/\delta}\Delta \Big)(c/\delta)^{2\nu} \Big\}, \\ |\tilde{\rho}| \le B_0(200\nu^2 \delta \operatorname{Log}(1/\delta))^\nu, \end{cases}$$

provided that

$$\nu^2 \delta \operatorname{Log}(1/\delta) \le \tfrac{1}{6}, \quad 2\nu + 1 \le \delta \operatorname{Log} X, \quad 20 \le \operatorname{Log} X, \quad 4^{1/\delta}\Delta \le \tfrac{1}{2}.$$

$C_0(c)$ is defined in Lemma 3.3.

Note that when ν increases, $\Lambda^{(\nu)}\star\Lambda^{(\nu)}$ tends to give more weight to the points $n=p^hq^k$ with $p^h\sim q^k\sim\sqrt{X}$.

1.3 A result without a preliminary sieving

We now turn to a more refined result, but which requires finer hypotheses. We first need a mild control over the values taken by σ and σ_0 on prime powers :

$$\max(\sigma(p^h)p^h, \sigma_0(p^h)p^h) \le c, \quad (p\le z), \tag{H_6}$$

Such a hypothesis is a consequence when $h=1$ of (H_1) by taking $u=v=p$ there. It is sensible since (H_1) corresponds to a sieve of dimension 1. The hypothesis $\sigma(p^h)\le cp^{-h}$ simply says that individual values do not vary in too vaste a range. For σ_0, we further assume that:

$$V_{\sigma_0}(X^{1/4}) \le c/\operatorname{Log} X. \tag{H_7}$$

Furthermore, the simplest treatment uses the following two bounds:

$$\hat{F}(X) \ge z\sqrt{X}\delta^{-\nu}, \qquad \max(\|f\|_\infty, \|f_0\|_\infty) \le B. \tag{H_8}$$

Theorem 2 *Assuming (H_1)—(H_8), we have*

$$\Sigma_\nu(f,X) = \frac{V_\sigma(z)}{V_{\sigma_0}(z)}\Sigma_\nu(f_0,X) + (\rho+\theta)\cdot\frac{V_\sigma(z)}{V_{\sigma_0}(z)}\hat{F}(X)\frac{(\operatorname{Log} X)^{2\nu-1}}{(2\nu-1)!}$$

where

$$\begin{cases} |\rho| \le (56\nu^2)^\nu\Big\{A + \Big(C_0(c)\delta^{5\nu} + 4^{1/\delta}\Delta\Big)(c/\delta)^{2\nu}\Big\}, \\ |\theta| \ll_c A + (B+B_0)(200\nu^2\delta\operatorname{Log}(1/\delta))^\nu \end{cases}$$

provided that

$$\nu^2\delta\operatorname{Log}(1/\delta) \le \tfrac{1}{6}, \quad 2\nu+1 \le \delta\operatorname{Log} X, \quad 20 \le \operatorname{Log} X, \quad 4^{1/\delta}\Delta \le \tfrac{1}{2}.$$

Comparing with [8], our result is better in that the error term is (essentially) $(\nu^2\delta\operatorname{Log}(1/\delta))^\nu$ while it was (essentially) $\delta\operatorname{Log}(1/\delta)$ in the aforementioned paper.

1.4 A simpler hypothesis on the remainder term

Hypothesis (H_4) may be troublesome due to the uniformity required in ν. We prove in section 6 that it can be replaced by

$$\begin{cases} \forall d \le D_0,\ \max_{y\le X}(|r_d(f,y)|, |r(f_0,y)|) \le C\hat{F}(X)/d, \\ \mathrm{R}(f, D_0, 1) + \dfrac{V_\sigma(z)}{V_{\sigma_0}(z)}\,\mathrm{R}(f_0, D_0, 1) \le A'\hat{F}(X)/\operatorname{Log}^2 X \end{cases} \tag{H_9}$$

in which case we can take $A = \sqrt{C(2\operatorname{Log} X)^{4\nu^2}A'}$.

1.5 A sum over primes

When studying prime numbers, it is often useful to have some information about their distribution in arithmetic progressions. We first introduce a notation: for a sequence f, real numbers X and $X' \in]X, 2X]$ and α on the torus, we put

$$S(f;\alpha) = \sum_{X<n\le X'} f(n)e(n\alpha). \tag{6}$$

By using only combinatorial arguments, we shall prove

Theorem 3 *For $X \ge 1$, $\operatorname{Log} q \le \frac{1}{50}(\operatorname{Log} X)^{1/3}$ and $\alpha = a/q + \beta$ with $|\beta| \le qX^{-1}\exp((\operatorname{Log} X)^{1/3})$ and a prime to q, we have*

$$\sum_{p\le X} \operatorname{Log} p\, e(p\alpha) \ll X\sqrt{q}/\phi(q).$$

All constants are explicit. We now discuss the optimality of Theorem 3. Assuming the Riemann hypothesis for L-functions, one would get

$$|S(\Lambda; a/q)| \ll \left(\frac{\mu^2(q)}{\phi(q)} + \sqrt{q^2/X}\operatorname{Log} X\right) X \tag{7}$$

while by using the prime number Theorem for the modulus q and assuming that there exists a Siegel's zero at $1-\tilde{\delta}$, we get

$$|S(\Lambda; a/q)| \sim \frac{\sqrt{q}}{\phi(q)} X^{1-\tilde{\delta}}.$$

Thus multiplying our bound by a function of q which goes to 0 would improve on our effective knowledge of Siegel's zero, and in this sense it is optimal. It also improves on [3] in the main term as well as in the range in β. Let us note here that in [13] we can already find a proof of an upper bound for

$|S(\Lambda; a/q)|/X$ that has the main characteristic of this one: it is independant of X, goes to zero when q remains smaller than a power of $\operatorname{Log} X$ and is obtained via the bilinear form technique; use Theorem 2b of chapter IX therein with $\varepsilon = 2\operatorname{Log} q/\operatorname{Log}\operatorname{Log} X$ – the upper bound obtained is about $X \cdot (\operatorname{Log} q)^{10}/\sqrt{q}$.

The range in β is amply sufficient for applications but a much shorter one would not do.

1.6 Notations

We write $\sum'$ for a summation restricted to integers free of prime factors $\leq z$. The function f_0 should be a good model for f. More precisely, define

$$\sum_{n\leq y/d}{}' w(n)f(dn) = \frac{V_\sigma(z)}{V_{\sigma_0}(z)} \sum_{n\leq y/d}{}' w(n)f_0(dn) + r_{d,z}(w,y) \tag{8}$$

where w is a weight. We further define

$$\bar{f} = f - \frac{V_\sigma(z)}{V_{\sigma_0}(z)} f_0. \tag{9}$$

For $D \geq 1$, we set

$$\begin{cases} \displaystyle\sum_{d\leq D}{}^{*} \tau_r(d) \max_{y\leq X} |r_{d,z}(w,y)| = \mathrm{R}_z(w,\bar{f},D,r), \\ \displaystyle\sum_{d\leq D}{}' \tau_r(d) \max_{y\leq X} |r_{d,z}(w,y)| = \mathrm{R}'_z(w,\bar{f},D,r). \end{cases} \tag{10}$$

There exist relationships between these quantities, and it is the topic of the third section to show how to control them in terms of some $\mathrm{R}(f_0, D', r)$ and $\mathrm{R}(f, D', r)$.

To avoid typographical work, we set

$$\mathrm{MT} = \hat{F}(X) \frac{(\operatorname{Log} X)^{2\nu-1}}{(2\nu-1)!} \frac{V_\sigma(z)}{V_{\sigma_0}(z)}. \tag{11}$$

As a matter of notations, we shall use either standard ones (in particular p shall always stand for a prime number) or define them when required but for two exceptions : we write $a \equiv b[q]$ to say that a is congruent to b modulo q and the notation $f = \mathcal{O}^*(g)$ to mean that $|f| \leq g$. Though we shall not compute every constant implied in $\mathcal{O}$-symbols, some of them are easy enough to get and displays clearly the dependence of the constants on the various parameters. We finally use Log to denote the natural logarithm.

1.7 *Comments and acknowledgements*

This paper started as a collaboration with Henryk Iwaniec, and we completed the main frame around 1997. It was presented partially in talks, and we somehow forgot about it. The recent papers [4] and [6] made us believe that it would be a good idea to make this work available. Note in particular that in [6], the sequence f is essentially assumed to be of "dimension" 2, where the dimension comes from (H_1): the upper bound we assume is $c(\operatorname{Log} u/\operatorname{Log} v)^\kappa$ where $\kappa = 1$ is (an upper bound of) the dimension.

I then started to update and complete our initial work. Henryk Iwaniec does not wish anymore to appear as an author, since to his opinion, the input to this work is not distributed enough among the former co-authors. I could express thanks to him for interesting discussions, but that would be lessening far too much his contribution to this work.

I end this part by thanking the referee for his/her very thorough reading of this paper.

2 Diamond & Steinig identity and some further notations

In their elementary proof of the Prime Number Theorem with a good remainder term in 1970, Diamond & Steinig introduced a generalisation of Selberg's identity. We are mainly concerned in this part with creating a confortable environment for using these identities as bilinear forms, their main default lying in the fact that they are not easy to write.

Let us fix an integer $\nu \geq 1$.

When $k_1, k_2 \ldots, k_m$ are integers, we define the function $L(k_1, \ldots, k_m)$ by

$$L(k_1, \ldots, k_m) = \operatorname{Log} \star \cdots \star \operatorname{Log} \star \operatorname{Log}^2 \star \cdots \star \operatorname{Log}^2 \star \cdots \star \operatorname{Log}^m \star \cdots \star \operatorname{Log}^m, \quad (12)$$

while μ_k denotes the k-fold convolution power of μ.

Let us denote by $\mathbb{K} = \mathbb{K}(\nu)$ the set of all 2ν-tuples $(k_1, \ldots, k_{2\nu})$ satisfying

$$\begin{cases} k_1 + \cdots + k_{2\nu} = k \leq \nu, \\ k_1 + 2k_2 + \cdots + 2\nu k_{2\nu} = 2\nu. \end{cases} \quad (13)$$

A generic element of $\mathbb{K}$ will be denoted by $\mathbf{k}$ and its length (i.e. $\sum k_i$) by k.

Let us also define the functions c and w over $\mathbb{K}$ by

$$\begin{cases} c(\mathbf{k}) = \dfrac{(-1)^{k-1}(k-1)!}{k_1!\dots k_{2\nu}!}, \\ w(\mathbf{k}) = \dfrac{1}{1!^{k_1}}\cdot\dfrac{1}{2!^{k_2}}\cdots\dfrac{1}{(2\nu)!^{k_{2\nu}}}. \end{cases} \tag{14}$$

We denote by $\mathbb{H}=\mathbb{H}(\nu)$ the set of all 2ν-tuples $(k_1',\dots,k_\nu',k_1'',\dots,k_\nu'')$ satisfying

$$\begin{cases} k_1'+\dots+k_\nu'+k_1''+\dots+k_\nu'' = k'+k'' \le \nu, \\ k_1'+\dots+\nu k_\nu' = k_1''+\dots+\nu k_\nu'' = \nu. \end{cases} \tag{15}$$

A generic element of $\mathbb{H}$ will be denoted by $\mathbf{h}=(\mathbf{k}',\mathbf{k}'')$ and its length by h. The functions $\bar{c}$ and $\bar{w}$ are defined on $\mathbb{H}$ by

$$\begin{cases} \bar{c}(\mathbf{h}) = \dfrac{(-1)^{k'-1}(k'-1)!}{k_1'!\dots k_\nu'!}\,\dfrac{(-1)^{k''-1}(k''-1)!}{k_1''!\dots k_\nu''!}, \\ \bar{w}(\mathbf{h}) = \dfrac{1}{1!^{k_1'}}\cdot\dfrac{1}{2!^{k_2'}}\cdots\dfrac{1}{\nu!^{k_\nu'}}\cdot\dfrac{1}{1!^{k_1''}}\cdot\dfrac{1}{2!^{k_2''}}\cdots\dfrac{1}{\nu!^{k_\nu''}}. \end{cases} \tag{16}$$

After these preparations, the identity under question reads:

Lemma 4 (Diamond & Steinig)

$$\begin{aligned} &\Lambda^{(2\nu)}+\Lambda^{(\nu)}\star\Lambda^{(\nu)} \\ &\quad= 2\nu\sum_{\mathbf{k}\in\mathbb{K}} c(\mathbf{k})w(\mathbf{k})\mu_k\star L(\mathbf{k}) + \nu^2\sum_{\mathbf{h}\in\mathbb{H}}\bar{c}(\mathbf{h})\bar{w}(\mathbf{h})\mu_h\star L(\mathbf{k}')\star L(\mathbf{k}''). \end{aligned}$$

This Lemma follows from an identity of the shape

$$\frac{1}{(2\nu-1)!}\left(\frac{\zeta'}{\zeta}\right)^{(2\nu-1)} + \left\{\frac{1}{(\nu-1)!}\left(\frac{\zeta'}{\zeta}\right)^{(\nu-1)}\right\}^2 = \frac{P(1,\zeta,\dots,\zeta^{(2\nu)})}{\zeta^\nu} \tag{17}$$

where P is a polynomial with integer coefficients. The key here is that ζ appears on the denominator with a power ν and not 2ν.

We shall often use the following short form

$$\Lambda^{(2\nu)}+\Lambda^{(\nu)}\star\Lambda^{(\nu)} = \sum_{\boldsymbol{\ell}} a(\boldsymbol{\ell})\,\mu_\ell\star L(\boldsymbol{\ell}) \tag{18}$$

Finally, we write $\mu_{k,T}$ for the function defined by

$$\mu_{k,T}(m) = \begin{cases} \mu_k(m) & \text{if } m\le T, \\ 0 & \text{otherwise.} \end{cases} \tag{19}$$

We said that our plan is to prove that $\Sigma_\nu(f_0, X)$ is a good approximation to $\Sigma_\nu(f, X)$. As a matter of fact, we use the assumed level of distribution given by hypothesis (H_9) and the fact that σ is close to σ_0 to show that

$$\sum_{n\le X}\sum_{\boldsymbol{\ell}} a(\boldsymbol{\ell})(\mu_{\boldsymbol{\ell},T}\star L(\boldsymbol{\ell}))(n)\bar{f}(n)$$

is small. Here T is a truncation parameter that we will choose later so large that

$$\sum_{n\le X}\sum_{\boldsymbol{\ell}} a(\boldsymbol{\ell})(|\mu_{\boldsymbol{\ell}}-\mu_{\boldsymbol{\ell},T}|\star L(\boldsymbol{\ell}))(n)\Big(f(n)+\frac{V_\sigma(z)}{V_{\sigma_0}(z)}f_0(n)\Big)$$

is indeed an error term. The fact that ℓ is bounded above by ν is essential in this part. If we were to carry out this programm, we would reach an error term for this part of size

$$\ll_\nu \mathrm{MT}\cdot\Big(\frac{\mathrm{Log}^3(X/T)}{\mathrm{Log}\,X}\Big)^\nu$$

thus forcing us to take X/T rather small. However the preliminary sieving reduces the above term (see (29)) to

$$\ll_\nu \mathrm{MT}\cdot\Big(\frac{\mathrm{Log}^3(X/T)}{\delta^2\,\mathrm{Log}^3 X}\Big)^\nu$$

and thus enables us to take X/T as a power of X^δ. (We take X/T to be $(X^\delta)^{\nu^2\,\mathrm{Log}(1/\delta)}$ in (32)). The treatment of the main term, which is already rather intricate due to the iterated convolutions, is made even more intricate by this preliminary sieving. This treatment accounts in fact for most of the length of the paper.

3 Some preliminary estimates

Lemma 5 *For fixed k_0 and $\nu\ge 1$, we have*

$$\sum_{\substack{\mathbf{h}\in\mathbb{H}\\ h=k_0}}|\bar{c}(\mathbf{h})|\le\sum_{\substack{\mathbf{k}\in\mathbb{K}\\ k=k_0}}|c(\mathbf{k})|=\frac{1}{k_0}\binom{2\nu-1}{k_0-1}.$$

Proof. The equality comes from computing the coefficient of $Y^{2\nu}X^{k_0}$ in $(1+XY+XY^2+\cdots+XY^{2\nu})^{2\nu}$ in two different ways (cf between (5.10) and

(5.11) in [5]). Indeed, a direct expansion yields

$$(1+XY+XY^2+\cdots+XY^{2\nu})^{2\nu} = \sum_{0\le k_1,\cdots,k_{2\nu}\le 2\nu} \binom{2\nu}{k_1,\cdots,k_{2\nu}} X^{k_1+k_2+\cdots+k_{2\nu}} Y^{k_1+2k_2+\cdots+2\nu k_{2\nu}}$$

while :

$$(1+X(Y+Y^2+\cdots+Y^{2\nu}))^{2\nu} = \left(1+\frac{XY}{1-Y}\right)^{2\nu} + \mathcal{O}(Y^{2\nu+1})$$
$$= \sum_{k\ge 0}\binom{2\nu}{k}(XY)^k \sum_{j\ge 0}\binom{k-1+j}{k-1} Y^j + \mathcal{O}(Y^{2\nu+1})$$

(with the convention $\binom{-1}{-1}=1$ and $\binom{-1+j}{-1}=0$ when $j\ge 1$). This leads to

$$\sum_{\substack{0\le k_1,\cdots,k_{2\nu}\le 2\nu,\\ k_1+k_2+\cdots+k_{2\nu}=k,\\ k_1+2k_2+\cdots+2\nu k_{2\nu}=2\nu}} \binom{2\nu}{k_1,\cdots,k_{2\nu}} = \binom{2\nu}{k}\binom{2\nu-1}{k-1}$$

from which our claim follows easily. The inequality is easy. ⋄ ⋄ ⋄

Lemma 6 *For* $\mathbf{k}$ *in* $\mathbb{K}$, $w(\mathbf{k})\le 2^{-\nu}$, *and for* $\mathbf{h}$ *in* $\mathbb{H}$, $\bar{w}(\mathbf{h})\le 2^{-\nu}$.

Proof. We want to get the minimum of

$$S=\sum_{j=1}^{2\nu}\sum_{\ell=1}^{j} k_j \operatorname{Log}\ell$$

under $\sum jk_j = 2\nu$ and $\sum k_j \le \nu$. Introducing the variables $K_\ell=\sum_{j=\ell}^{2\nu} k_j$, we solve easily this problem. The other inequality follows from this one. ⋄ ⋄ ⋄

Lemma 7

$$2\nu\sum_{\mathbf{k}\in\mathbb{K}}|c(\mathbf{k})w(\mathbf{k})| + \nu^2\sum_{\mathbf{h}\in\mathbb{H}}|\bar{c}(\mathbf{h})\bar{w}(\mathbf{h})| \le (\nu+2)2^{\nu-1}.$$

Proof. By Lemma 5 and 6, the LHS of Lemma 7 is not more than

$$\frac{2\nu+\nu^2}{2^\nu}\sum_{k_0=1}^{\nu}\binom{2\nu-1}{k_0-1}\frac{1}{k_0} \le \frac{2\nu+\nu^2}{2^\nu}\int_0^1 (1+t)^{2\nu-1}dt \le (\nu+2)2^{\nu-1}$$

hence the result. ⋄ ⋄ ⋄

Lemma 8 *We have* $V_\sigma(z)/V_{\sigma_0}(z) \geq 1/(2c^2)$.

Proof. By (H_1), we get $V_\sigma(z) \geq \frac{\operatorname{Log} 2}{c \operatorname{Log} z} \geq 1/(2c \operatorname{Log} z)$ and (H_2) concludes. $\diamond\diamond\diamond$

Lemma 9 *We have* $n! = (2\pi n)^{1/2}(n/e)^n e^{\theta_+/(12n)}$ *for* $n \geq 1$ *and some* $\theta_+ \in]0,1[$

Cf [5, (2.9)].

Lemma 10 *When* $\nu \geq 1$, *we have* $(2\nu-1)!/(\nu-1)!^2 \leq \sqrt{\nu}2^{2\nu-\frac{3}{2}}$.

This follows readily from [5, (2.10)].

Lemma 11 *We have* $\tau_r(p^a) \leq \frac{1}{2}(e(1+r/A))^A$ *whenever* $a \leq A$.

Proof. Indeed, we have

$$\begin{aligned}\tau_r(p^a) = \binom{r-1+a}{r-1} &\leq \frac{e^{1/12}}{\sqrt{2\pi}} \frac{r}{r+a} \sqrt{\frac{r+a}{ra}} \frac{(r+a)^{r+a}}{r^r a^a} \\ &\leq \frac{e^{1/12}}{\sqrt{2\pi}} \sqrt{\frac{r}{(r+a)a}} (1+r/a)^a (1+a/r)^r \leq e^a (1+r/A)^A/2\end{aligned}$$

since $x \mapsto (1+r/x)^x$ is increasing; indeed the derivative of its logarithm is

$$\operatorname{Log}(1+r/x) - \frac{r/x}{1+(r/x)}$$

and $\operatorname{Log}(1+y) \geq y/(1+y)$ for $y \geq 0$ (the derivative of the difference is ≥ 0). $\diamond\diamond\diamond$

Lemma 12 *We have* $\tau_r(\ell m) \leq \tau_r(\ell)\tau_r(m)$.

Proof. This submultiplicativity is classical. We can establish it by showing that $\tau_r(p^{u+v}) \leq \tau_r(p^u)\tau_r(p^v)$ for every prime p. Let $A(w)$ be the set of r-tuples such that $a_1 + \cdots + a_r = w$. We simply want to build an injective map from $A(u+v)$ into $A(u) \times A(v)$. There are several way to achieve that, and for instance, we can associate to $(a_1, \cdots, a_r) \in A(u+v)$ the two r-tuples $(a'_1, \cdots, a'_r) \in A(u)$ and $(a_1 - a'_1, \cdots, a_r - a'_r) \in A(v)$ where

$$\begin{aligned} a'_1 = \min(a_1, u), a'_2 = \min(a_2, u - a'_1), a'_3 = \min(a_3, u - a'_1 - a'_2), \cdots \\ \cdots \quad a'_r = \min(a_r, u - a'_1 - a'_2 - \cdots - a'_{r-1}). \end{aligned}$$

$\diamond\diamond\diamond$

Lemma 13 *We have*

$$\sum_{n\le N}\tau_r(n)^2/n \le \Big(\sum_{n\le N}1/n\Big)^{r^2}.$$

Proof. Indeed and on using Lemma 12, we find that

$$\sum_{n\le N}\frac{\tau_r(n)^2}{n}=\sum_{n\le N}\frac{\tau_r(n)}{n}\sum_{n_1n_2\cdots n_r=n}1\le\sum_{n_1n_2\cdots n_r\le N}\frac{\tau_r(n_1)\tau_r(n_2)\cdots\tau_r(n_r)}{n_1n_2\cdots n_r}$$
$$\le\Big(\sum_{n\le N}\tau_r(n)/n\Big)^r\le\Big(\sum_{n\le N}1/n\Big)^{r^2}$$

and the Lemma readily follows. ⋄ ⋄ ⋄

The following Lemma is not essential but will help in keeping our estimates explicit, and thus, hopefully, more understandable.

Lemma 14 *For* $300\le a\le b$, *we have*

$$\prod_{a<p\le b}(1-1/p)^{-1}\le 1.04(\operatorname{Log} b)/\operatorname{Log} a.$$

Proof. We use inequalities (3.25) and (3.28) from [12], to get that

$$\prod_{a<p\le b}(1-1/p)^{-1}\le c_1(\operatorname{Log} b)/\operatorname{Log} a$$

with

$$c_1=\left(1+\frac{1}{2\operatorname{Log}^2 b}\right)\left(1-\frac{1}{2\operatorname{Log}^2 a}\right)^{-1}.$$

A numerical application concludes. ⋄ ⋄ ⋄

Let us recall the following lemma (this is th 01 of [9]):

Lemma 15 *Let* H *be a non-negative multiplicative function verifying*

(1) $\sum_{p\le y}H(p)\operatorname{Log} p\le \alpha y$ *for* $y\ge 0$,
(2) $\sum_p\sum_{a\ge 2}H(p^a)p^{-a}\operatorname{Log}(p^a)\le\beta$.

Then, for $x>1$, *we have*

$$\sum_{n\le x}H(n)\le(\alpha+\beta+1)\frac{x}{\operatorname{Log} x}\sum_{n\le x}H(n)/n.$$

We use it to prove:

Lemma 16 *For $r \geq 1$ and $z \geq 300$, we have*

$$\sum_{n \leq N}{}' \tau_r(n) \leq N\, 3^r (\operatorname{Log} N)^r / (\operatorname{Log} z)^r, \textit{ and } \quad \sum_{n \leq N}{}' \frac{\tau_r(n)}{n} \leq 3^r (\operatorname{Log} N)^{r+1} / (\operatorname{Log} z)^r.$$

This Lemma gives the correct order of magnitude in N and z.

Proof. To use Lemma 15, note that $\tau_r(p) = \ell$ while $\tau_r(p^a) = \binom{r-1+a}{r-1}$. This leads to

$$\begin{aligned}
\sum_{a \geq 2} \tau_\ell(p^a) p^{-a} \operatorname{Log}(p^a) &= \frac{\operatorname{Log} p}{p} \sum_{a \geq 2} r \binom{r+a-1}{r} p^{-(a-1)} \\
&= \frac{\operatorname{Log} p}{p} r \left(\frac{1}{(1-p^{-1})^{r+1}} - 1 \right) \\
&\leq \frac{\operatorname{Log} p}{p^2} \frac{r(r+1)}{(1-p^{-1})^{r+1}} \leq r(r+1) 2^{r+1} \frac{\operatorname{Log} p}{p^2}.
\end{aligned}$$

As a consequence, we reach

$$\sum_{n \leq N}{}' \tau_r(n) \ll \frac{N}{\operatorname{Log} N} 2^r r^2 \prod_{z < p \leq X} (1 - 1/p)^{-r} \leq \frac{N}{\operatorname{Log} N} r^2 (2.08)^r \delta^{-r}$$

where we used Lemma 14 to estimate the product over the primes. We conclude by noticing that $r^2(2.08)^r \leq 3^r$. The second estimates follows readily from the above lines. $\diamond \diamond \diamond$

Lemma 17 *Let $m \leq M$ be an integer with no prime factor $\leq z$. The following upper bound holds true for $\boldsymbol{\ell}$ in $\mathbb{K}$ or in $\mathbb{H}$:*

$$L(\boldsymbol{\ell})(m) \leq \tau_\ell(m) \left(\frac{e \operatorname{Log} m}{2\nu} \right)^{2\nu} / w(\boldsymbol{\ell}).$$

(Replace $w(\boldsymbol{\ell})$ by $\bar{w}(\boldsymbol{\ell})$ in case $\boldsymbol{\ell} \in \mathbb{H}$).

Proof. Indeed, we have to write m as a product

$$m = \prod_{1 \leq i \leq \nu} \prod_{1 \leq j \leq \ell_i} m_{i,j}$$

where $\ell_1, \ell_2, \cdots, \ell_{2\nu}$ define $\boldsymbol{\ell}$ and thus verify

$$\sum_{1 \leq i \leq 2\nu} i\ell_i = 2\nu, \qquad \sum_{1 \leq i \leq 2\nu} \ell_i = \ell \leq \nu.$$

Since $\operatorname{Log} 1 = 0$, we have to count only the $m_{i,j}$'s that are > 1 and thus $\geq z$. We set $m_{i,j} = z^{\alpha_{i,j}}$ so that

$$(\operatorname{Log} z)^{-2\nu} L(\boldsymbol{\ell})(m)/\tau_{\boldsymbol{\ell}}(m) \leq \prod_{1\leq i\leq 2\nu} \prod_{1\leq j\leq \ell_i} \alpha_{i,j}^i = \exp \sum_{1\leq i\leq 2\nu} \sum_{1\leq j\leq \ell_i} i \operatorname{Log} \alpha_{i,j}$$

and we should maximize the sum containing the $\operatorname{Log} \alpha_{i,j}$'s under the conditions

$$\alpha_{i,j} \geq 1, \quad \sum_{i,j} \alpha_{i,j} = A \leq (\operatorname{Log} m)/\operatorname{Log} z.$$

We can use Lagrange multipliers, forgetting the individual lower bounds, and prove that we should have $\alpha_{i,j} = i/\lambda$ for a fixed λ. We readily discover that $\lambda = 2\nu/A$ and thus that

$$L(\boldsymbol{\ell})(m)/\tau_{\boldsymbol{\ell}}(m) \leq \left(\frac{\operatorname{Log} m}{2\nu}\right)^{2\nu} \prod_i i^{i\ell_i} \leq \left(\frac{\operatorname{Log} m}{2\nu}\right)^{2\nu} \prod_i (e^i i!)^{\ell_i}$$

since Lemma 9 implies that $(i/e)^i \leq i!$. We conclude easily. $\diamond\diamond\diamond$

When using the above Lemma, we will replace Lemma 7 by the following one, which we prove in the same way:

Lemma 18

$$2\nu \sum_{\mathbf{k}\in\mathbb{K}} |c(\mathbf{k})| + \nu^2 \sum_{\mathbf{h}\in\mathbb{H}} |\bar{c}(\mathbf{h})| \leq (\nu+2)2^{2\nu-1}.$$

4 Inter-relations between different error terms

Our aim is to derive estimates for

$$\sum_{n\leq X/d}{}' w(n) f(dn)$$

where w is a weight to be precised. When this weight is intricate, obtaining the main term for the above sum in terms of similar expression for f is difficult, hence the introduction of f_0. Recall that $\bar{f} = f - \frac{V_\sigma(z)}{V_{\sigma_0}(z)} f_0$.

If g is a C^1-function over $[1, X]$, we define

$$\|g'\|_1 = \int_1^X |g'(t)|dt, \quad \|g\|_\infty = \max_{1\leq t\leq X} |g(t)|, \quad \|g\| = \max(\|g'\|_1, \|g\|_\infty). \quad (20)$$

4.1 Smooth weights

Lemma 19 *When w is a C^1-function over $[1,X]$, we have*

$$\begin{cases} \mathrm{R}_z(w,\bar f,D,r) \le 3\|w\|\,\mathrm{R}_z(1,\bar f,D,r), \\ \mathrm{R}'_z(w,\bar f,D,r) \le 3\|w\|\,\mathrm{R}'_z(1,\bar f,D,r). \end{cases}$$

Proof. By summation by parts we get

$$\begin{aligned}\sum_{n\le y/d}{}' w(n)\bar f(dn) &= \int_1^{y/d} w'(t)\sum_{t<n\le y/d}{}' \bar f(dn)dt + w(1)\sum_{n\le y/d}{}'\bar f(dn) \\ &\le \left(|w(1)| + 2\int_1^X |w'(t)|dt\right)\max_{t\le X}|r_{d,z}(1,t)|\end{aligned}$$

as required. $\diamond\diamond\diamond$

4.2 Divisor-like weights

Let $w_1,\dots,w_k$ be k C^1-functions over $[1,X]$. We consider the weight $w = w_1\star\dots\star w_k$. We put

$$\|w\| = \|w_1\|\dots\|w_k\|.$$

Lemma 20 *We have, when $1\le D\le X$,*

$$\begin{cases} \mathrm{R}_z(w,\bar f,X(D/X)^k,r) \le 3(2^k-1)\cdot\mathrm{R}_z(1,\bar f,D,r+k-1)\|w\|, \\ \mathrm{R}'_z(w,\bar f,X(D/X)^k,r) \le 3(2^k-1)\cdot\mathrm{R}'_z(1,\bar f,D,r+k-1)\|w\| \end{cases}$$

Proof. Put $D(w) = X(D/X)^k$. We use induction over $k\ge 1$. The case $k=1$ is treated in Lemma 19. Let us prove Lemma 20 to hold for k if it holds for $k-1$. We put $w = w_1\star\dots\star w_{k-1}$ and apply Dirichlet hyperbola formula to get, with $MN = y/d$,

$$\begin{aligned}\sum_{mn\le y/d}{}' w(n)w_k(m)\bar f(dnm) &= \sum_{n\le N}{}' w(n)\sum_{m\le y/dn}{}' w_k(m)\bar f(dnm) \\ &\quad + \sum_{m\le M}{}' w_k(m)\sum_{N<n\le y/dm}{}' w(n)\bar f(dmn).\end{aligned}$$

Assuming $dN\le D$ and $dM\le D(w)$, we get

$$r_{d,z}(w\star w_k,y) = \sum_{n\le N}{}' w(n)r_{dn,z}(w_k,y) + \sum_{m\le M}{}' w_k(m)(r_{dm,z}(w,y) - r_{dm,z}(w,Ndm))$$

We take $M = D(w)/d$ and get

$$\sum_{d\le D(w\star w_k)}^{*} \tau_r(d) \max_{y\le X} |r_{d,z}(w\star w_k, y)| \le$$
$$\sum_{d\le D(w\star w_k)}^{*} \tau_r(d) \sum_{n\le X/D(w)}^{*} |w(n)| \max_{y\le X} |r_{dn,z}(w_k, y)|$$
$$+ 2 \sum_{d\le D(w\star w_k)}^{*} \tau_r(d) \sum_{m\le D(w)/d}^{*} |w_k(m)| \max_{y\le X} |r_{dm,z}(w, y)|$$

which in turn is not more than

$$\sum_{\ell\le D}^{*} \tau_r\star|w|(\ell) \max_{y\le X} |r_{\ell,z}(w_k,y)| + 2 \sum_{\ell\le D(w)}^{*} \tau_r\star|w_k|(\ell) \max_{y\le X} |r_{\ell,z}(w,y)|.$$

Now note that

$$\tau_r\star|w|(\ell) \le \|w_1\|_\infty \ldots \|w_{k-1}\|_\infty \tau_{r+k-1}(\ell)$$

so that our remainder term is not more than

$$3\left\{\|w_k\|\|w_1\|\ldots\|w_{k-1}\| + 2(2^k-1)\|w_1\|\ldots\|w_{k-1}\|\|w_k\|_\infty\right\} \mathrm{R}_z(1,\bar{f},D,r+k-1)$$

from which the first inequality of the Lemma follows readily. The proof of the second inequality is similar. ⋄ ⋄ ⋄

4.3 Sieve weights

We first need a Lemma to connect $\sigma(d)$ and $\sigma_0(d)$.

Lemma 21 *When $\Delta \le (e+er\delta)^{-1/\delta}$, where Δ is defined in (5), we have*

$$\sum_{d\le X}{}' \tau_r(d)|\sigma(d)-\sigma_0(d)| \le 2\Delta(e+er\delta)^{1/\delta}(c/\delta)^r$$

Proof. We get the following inequality by mimicking the proof given section 15 of [8]:

$$|\sigma(d)-\sigma_0(d)| \le \sum_{p^a\|d} |\sigma(p^a)-\sigma_0(p^a)| \max(\sigma(d/p^a), \sigma_0(d/p^a)).$$

We then write

$$\begin{aligned}U_r(|\sigma-\sigma_0|) &= \sum_{d\le D}{}' \tau_r(d)|\sigma(d)-\sigma_0(d)| \\ &\le \sum_{d\le D}{}' \tau_r(d) \sum_{p^a\|d} |\sigma(p^a)-\sigma_0(p^a)| \max(\sigma(d/p^a), \sigma_0(d/p^a)) \\ &\le \Delta \max \tau_r(p^a) \sum_{d\le D/z}{}' \tau_r(d) \max(\sigma(d), \sigma_0(d)).\end{aligned}$$

Notice that, by Lemma 11 and on noticing that $a \le 1/\delta$:

$$\tau_r(p^a) = \binom{r-1+a}{r-1} \le (e+er\delta)^{1/\delta}/2.$$

We furthermore write

$$\max(\sigma(d), \sigma_0(d)) \le |\sigma(d)-\sigma_0(d)| + \sigma_0(d)$$

so that

$$U_r(|\sigma-\sigma_0|) \le \Delta(e+er\delta)^{1/\delta} U_r(|\sigma-\sigma_0|) + \Delta(e+er\delta)^{1/\delta} U_r(\sigma_0).$$

Our bound on Δ enables to infer from the above that

$$U_r(|\sigma-\sigma_0|) \le 2\Delta(e+er\delta)^{1/\delta} U_r(\sigma_0).$$

Next we write

$$U_r(\sigma_0) \le \prod_{z<p\le D} (1+\sum_{a\ge 1} \sigma_0(p^a))^r \le (c/\delta)^r \tag{21}$$

by (H_5) and the Lemma follows readily. ⋄ ⋄ ⋄

Let $\tilde{z} \ge 1$ be a real number and $P(\tilde{z}) = P(\tilde{z}, \mathfrak{f}) = \prod^*_{p\le\tilde{z}} p$. We take $w(n) = 1$ when $(n, P(\tilde{z})) = 1$ and 0 otherwise. We first recall a well-known Lemma (cf for instance [7, Lemma 5]).

Lemma 22 (Fundamental Lemma) *Let $M \ge 2$ and $\tilde{z} \ge 1$ be two real parameters. There exist two sequences (λ_m^+), (λ_m^-) with the following properties:*

$$\lambda_1^+ = \lambda_1^- = 1, \qquad |\lambda_m^+|, |\lambda_m^-| \le 1, \qquad \lambda_m^+ = \lambda_m^- = 0 \quad \text{when } m > M.$$

For any n for which $(n, P(\tilde{z})) \ne 1$,

$$\sum_{m|n} \lambda_m^- \le 0 \le \sum_{m|n} \lambda_m^+,$$

while $\sum_{m|n} \lambda_m^- = \sum_{m|n} \lambda_m^+ = 1$ when $(n, P(\tilde{z})) = 1$. For any multiplicative function $\tilde{\sigma}$ verifying $0 \le \tilde{\sigma} < 1$ and

$$\prod_{v\le p\le u}{}^{*} (1-\tilde{\sigma}(p))^{-1} \le \tilde{c}\frac{\operatorname{Log} u}{\operatorname{Log} v} \quad (2 \le v \le u), \tag{22}$$

we have

$$\sum_{m|P(\tilde z)} \lambda_m^+ \tilde\sigma(m) \le (1 + C_0(\tilde c) e^{-(\operatorname{Log} M)/\operatorname{Log}\tilde z}) \prod_{p\le\tilde z}{}^{*}(1-\tilde\sigma(p)),$$
$$\sum_{m|P(\tilde z)} \lambda_m^- \tilde\sigma(m) \ge (1 - C_0(\tilde c) e^{-(\operatorname{Log} M)/\operatorname{Log}\tilde z}) \prod_{p\le\tilde z}{}^{*}(1-\tilde\sigma(p)),$$

where $C_0(\tilde c)$ is a number which depends only on the constant $\tilde c$.

Lemma 23 *For $D_0 \ge Dz$ and when $\Delta(e+er\delta)^{1/\delta} \le 1/2$, the quantity $\mathrm{R}'_z(1,\bar f, D, r)$ is not more than*

$$\mathrm{R}(f, D_0, r+1) + \frac{V_\sigma(z)}{V_{\sigma_0}(z)}\,\mathrm{R}(f_0, D_0, r+1) + \Bigl(2C_0(c) e^{-\frac{\operatorname{Log} D_0/D}{\operatorname{Log} z}} \hat F(X) + 2\Delta(e+er\delta)^{1/\delta}\Bigr) V_\sigma(z)\,(c/\delta)^r\,\hat F(X).$$

Proof. We employ a sieve of level $M = D_0/D \ge z$ to get

$$\pm \sum_{n\le y/d}{}' f(dn) \le \sum_{n\le y/d}{}^{*} \sum_{m|n} \lambda_m^\pm f(dn) = \sum_{m|P(z)} \lambda_m^\pm \Bigl\{\sigma(dm)F(y) + r_{dm}(f,y)\Bigr\}.$$

We find that

$$\sum_{d\le D_0/M}{}' \tau_r(d) \Biggl|\sum_{m|P(z)} \lambda_m^\pm r_{dm}(f,y)\Biggr| \le \sum_{\ell\le D_0}{}^{*} \tau_{r+1}(\ell) \max_{y\le X} |r_\ell(f,y)| = \mathrm{R}(f, D_0, r+1).$$

Furthermore

$$\sum_{n\le y/d}{}' f(dn) = \sigma(d)V_\sigma(z)F(y)(1 + \mathcal{O}^*(C_0(c)e^{-\frac{\operatorname{Log} M}{\operatorname{Log} z}})) + \mathcal{O}^*\Biggl(\sum_{\substack{m\le M\\ m|P(z)}} |r_{dm}(f,y)|\Biggr),$$

and a similar estimate holds for f_0. To compare both, we use Lemma 21 and bound

$$\Biggl|\sum_{n\le y/d}{}' f(dn) - \frac{V_\sigma(z)}{V_{\sigma_0}(z)} \sum_{n\le y/d}{}' f_0(dn)\Biggr|$$

by

$$|\sigma(d) - \sigma_0(d)| V_\sigma(z) F(y) + \Bigl(\sigma(d) + \sigma_0(d)\Bigr) V_\sigma(z) F(y) C_0(c) e^{-\frac{\operatorname{Log} M}{\operatorname{Log} z}} + \sum_{\substack{m\le M\\ m|P(z)}} |r_{dm}(f,y)| + \frac{V_\sigma(z)}{V_{\sigma_0}(z)} \sum_{\substack{m\le M\\ m|P(z)}} |r_{dm}(f_0,y)|.$$

The first sum over d is treated in Lemma 21, while we treat the two next ones as in (21). The Lemma follows readily. ⋄ ⋄ ⋄

4.4 *Divisor-like weights with preliminary sieving*

Let $w_1, \ldots, w_k$ be C^1-functions over $[1, X]$. We consider the weight $w = w_1 \star \cdots \star w_k$.

Lemma 24 *If $D_0/D \geq z$ and when $\Delta(e + er\delta)^{1/\delta} \leq 1/2$, then the remainder term $\mathrm{R}'_z(w, \bar{f}, X(D/X)^k, r)$ is not more than*

$$3 \cdot 2^k \|w\| \biggl\{ \mathrm{R}(f, D_0, r+k) + \frac{V_\sigma(z)}{V_{\sigma_0}(z)} \mathrm{R}(f_0, D_0, r+k) + \Bigl(2C_0(c) e^{-\frac{\operatorname{Log} D_0/D}{\operatorname{Log} z}} + 2\Delta(e + er\delta)^{1/\delta} \Bigr) V_\sigma(z) \, (c/\delta)^{r+k-1} \hat{F}(X) \biggr\}.$$

Proof. We combine Lemma 20 and Lemma 23. ⋄ ⋄ ⋄

5 Proof of Theorem 1

Recall that $\Sigma'_\nu(f, X)$ denotes the sum $\Sigma^*_\nu(f, X)$ when $\mathfrak{f} = \prod_{p \leq z} p$. We use the following succession of approximations

$$\begin{aligned} \Sigma'_\nu(f, X) &= \sum_\ell a(\ell) \Bigl(S_1(f, T, \ell) + \mathcal{O}^*(S_2(f, T, \ell)) \Bigr) \\ &= \sum_\ell a(\ell) \Bigl(\frac{V_\sigma(z)}{V_{\sigma_0}(z)} S_1(f_0, T, \ell) \\ &\qquad + \mathcal{O}^* \Bigl(\frac{V_\sigma(z)}{V_{\sigma_0}(z)} S_2(f_0, T, \ell) + S_2(f, T, \ell) \Bigr) \Bigr) + \text{Remainder} \Bigr) \\ &= \frac{V_\sigma(z)}{V_{\sigma_0}(z)} \Sigma'_\nu(f_0, X) + \text{Remainder} \end{aligned}$$

5.1 *Evaluation of the truncated sums*

We have to approximate

$$S_1(f, T, \ell) = \sum_{n \leq T}{}' \mu_\ell(n) \sum_{m \leq X/n}{}' L(\ell)(m) f(mn) \tag{23}$$

by $\frac{V_\sigma(z)}{V_{\sigma_0}(z)} S_1(f_0, T, \ell)$. In doing so, we get a remainder term which is not more than

$$\sum_{n \leq T}{}' \tau_\ell(n) \max_{y \leq X} |r_{n,z}(L(\ell), y)| = \mathrm{R}'_z(L(\ell), \bar{f}, T, \ell). \tag{24}$$

Let D be a parameter to be chosen later and such that $1 \le D \le D_0/z$. We then take $T = X(D/X)^\nu$, use Lemma 24, hypothesis (H_2) and (H_4). Note that $\|L(\ell)\| = (\operatorname{Log} X)^{2\nu}$. We get that the difference

$$S_1(f, T, \ell) - \frac{V_\sigma(z)}{V_{\sigma_0}(z)} S_1(f_0, T, \ell)$$

is not more in absolute value than

$$3{\cdot}2^{\ell+1}\hat{F}(X)(\operatorname{Log} X)^{2\nu}\left\{\frac{A}{2\operatorname{Log} X}+\left(C_0(c)e^{-\frac{\operatorname{Log}\frac{D_0}{D}}{\operatorname{Log} z}} + \Delta(e+e\ell\delta)^{1/\delta}\right)V_\sigma(z)\,(c/\delta)^{\ell+\ell-1}\right\}$$

where it is worthwhile recalling that $\ell \le \nu$. Summing over ℓ and using Lemma 7, we get a remainder term which is not more than

$$3 \cdot 2^{2\nu}(\nu+2)\hat{F}(X)(\operatorname{Log} X)^{2\nu-1}\frac{V_\sigma(z)}{V_{\sigma_0}(z)}\times$$
$$\left\{\frac{A}{2(V_\sigma(z)/V_{\sigma_0}(z))}+\left(C_0(c)e^{-\frac{\operatorname{Log} D_0/D}{\operatorname{Log} z}} + \Delta(e+e\ell\delta)^{1/\delta}\right)V_{\sigma_0}(z)\operatorname{Log} X\,(c/\delta)^{2\nu-1}\right\}$$

We invoke Lemma 8 and (H_2) together with Lemma 9 to get the majorant

$$\mathrm{MT}\cdot(37\nu^2)^\nu\left\{Ac^2+\left(C_0(c)e^{-\frac{\operatorname{Log} D_0/D}{\operatorname{Log} z}} + \Delta(e+e\nu\delta)^{1/\delta}\right)(c/\delta)^{2\nu}\right\} \tag{25}$$

since $3\frac{\nu+2}{\nu}(4\nu/e)^{2\nu}e^{\frac{1}{24}}\sqrt{\pi\nu} \le (37\nu^2)^\nu$.

5.2 *Error term due to truncation: from $S_2(f, T, \ell)$ to $S_2(f_0, T, \ell)$*

We have to approximate

$$S_2(f, T, \ell) = \sum_{m \le X/T}{}' L(\ell)(m) \sum_{T<n\le X/m}{}' \tau_\ell(n) f(mn) \tag{26}$$

by $\frac{V_\sigma(z)}{V_{\sigma_0}(z)} S_2(f_0, T, \ell)$. We can use Lemma 24, hypothesis (H_2) and (H_4) to get

$$S_2(f, T, \ell) - \frac{V_\sigma(z)}{V_{\sigma_0}(z)} S_2(f_0, T, \ell)$$

in not more in absolute value than

$$3 \cdot 2^{\ell+2}\hat{F}(X)\left(\operatorname{Log}\frac{X}{T}\right)^{2\nu}\frac{V_\sigma(z)}{V_{\sigma_0}(z)}\times$$
$$\left\{\frac{Ac^2}{\operatorname{Log} 2\operatorname{Log} X}+\left(C_0(c)e^{-\frac{\operatorname{Log} D_0/D}{\operatorname{Log} z}} + \Delta(e+e\ell\delta)^{1/\delta}\right)V_{\sigma_0}(z)\,(c/\delta)^{\ell+\ell-1}\right\} \tag{27}$$

provided that $X(D/X)^\nu \geq X/T$. This will be ensured by $T^2 \geq X$. We follow the same path as above, and get the very same majorant (see (25)) but multiplied by

$$2\Big(\frac{\operatorname{Log}(X/T)}{\operatorname{Log} X}\Big)^{2\nu} \leq 2(1/2)^{2\nu} \leq 1/2. \tag{28}$$

5.3 *Error term due to truncation: bounding $S_2(f_0, T, \boldsymbol{\ell})$*

Our last task consists in finding an upper bound for $S_2(f_0, T, \boldsymbol{\ell})$ defined in (26). On using hypothesis (H_1), we can bound above f_0 by $B_0\hat{F}(X)/X$, so that we are reduced to finding an upper bound for $S_2(\mathbb{1}, T, \boldsymbol{\ell})$ (and subsequently multiply it by $B_0\hat{F}(X)/X$). We follow an easy path, by using Lemma 16 (since X/m is at least $T \geq \sqrt{X}$), followed by Lemma 17:

$$\begin{aligned} S_2(\mathbb{1}, T, \boldsymbol{\ell}) &\leq \frac{2\cdot 3^\ell X}{\delta^\ell \operatorname{Log} X} \sum_{m\leq X/T}{}' L(\boldsymbol{\ell})(m)/m \\ &\leq \frac{2\cdot 3^\ell X}{\delta^\ell \operatorname{Log} X}\left(\frac{e\operatorname{Log}(X/T)}{2\nu}\right)^{2\nu} \frac{3^\ell \operatorname{Log}^{\ell+1}(X/T)}{(\delta \operatorname{Log} X)^\ell}/w(\ell) \\ &\leq \left(\frac{17\operatorname{Log}^3(X/T)}{\nu^2\delta^2\operatorname{Log} X}\right)^\nu /w(\ell) \end{aligned}$$

since $\ell \leq \nu$, and where, as usual, one should replace $w(\ell)$ by $\overline{w}(\ell)$ in case ℓ belongs to $\mathbb{H}$. We then use Lemma 6 to sum over $\boldsymbol{\ell}$ and get a total contribution of all $S_2(f_0, T, \boldsymbol{\ell})$'s which is not more than

$$\begin{aligned} &\frac{B_0\hat{F}(X)}{\operatorname{Log}^{1+\nu} X}(\nu+2)2^{2\nu-1}17^\nu\left(\frac{\operatorname{Log}^3(X/T)}{\nu^2\delta^2\operatorname{Log} X}\right)^\nu \\ &\qquad\qquad \leq c_2\cdot B_0\,\mathrm{MT}\cdot\left(\frac{\operatorname{Log}^3(X/T)}{\delta^2\operatorname{Log}^3 X}\right)^\nu \end{aligned} \tag{29}$$

where the constant is, by Lemma 9,

$$c_2 = \sqrt{\pi/\nu}(2/e)^{2\nu}e^{1/(24\nu)}17^\nu \leq 31^\nu.$$

5.4 *Choice of the parameters to prove Theorem 1*

On taking $T = X(D/X)^\nu$, we derive an equality of the shape

$$\Sigma'_\nu(f, X) = \frac{V_\sigma(z)}{V_{\sigma_0}(z)}\Sigma'_\nu(f_0, X) + (\rho + \tilde{\rho})\cdot\mathrm{MT}$$

where we expect ρ to be $o(1)$. We have precisely (see (25) and (28))

$$|\rho| \leq \tfrac{3}{2}(37\nu^2)^\nu \Big\{ c^2 A + \Big(C_0(c) e^{-\frac{\operatorname{Log} D_0/D}{\operatorname{Log} z}} + \Delta(e + e\nu\delta)^{1/\delta} \Big) \Big(\frac{c}{\delta}\Big)^{2\nu} \Big\}$$

with (see (29))

$$\tilde{\rho} = B_0 \Big(\frac{31 \operatorname{Log}^3(X/T)}{\delta^2 \operatorname{Log}^3 X} \Big)^\nu .$$

Let us gather the hypotheses on z, T and D :

$$X \geq T \geq X^{1/2} \geq z \geq e^{2\nu+1}, \quad T = X(D/X)^\nu, \quad D_0 \geq Dz.$$

We write

$$\begin{cases} \operatorname{Log} z = \delta \operatorname{Log} X, \\ \operatorname{Log} D_0 = (1-\delta) \operatorname{Log} X, \\ \operatorname{Log} T = (1-\nu\varphi) \operatorname{Log} X, \end{cases} \tag{30}$$

which gives $\operatorname{Log} D = (1-\varphi) \operatorname{Log} X$ and the conditions are

$$\tfrac{1}{2} \geq \nu\varphi, \qquad \varphi - \delta \geq \delta, \qquad \delta \geq \frac{2\nu+1}{\operatorname{Log} X}.$$

We infer from the above the following somewhat simplified upper bound for $|\rho|$:

$$\tfrac{3}{2}(37\nu^2)^\nu \Big\{ c^2 A + \Big(C_0(c) e^{-\frac{\varphi-\delta}{\delta}} + \Delta(e+e\nu\delta)^{1/\delta} \Big) (c/\delta)^{2\nu} \Big\} \tag{31}$$

Since Δ will be typically extremely small, we concentrate on the other terms. We take

$$\varphi = 6\nu\delta \operatorname{Log}(1/\delta) \tag{32}$$

provided that

$$\tfrac{1}{6} \geq \nu^2 \delta \operatorname{Log}(1/\delta), \qquad \delta \geq \frac{2\nu+1}{\operatorname{Log} X} \tag{33}$$

and we further simplify the expression by using $e + e/4 \leq 4$. Lemma 23 implies also a condition on Δ which we simplify in a similar manner. Note finally that

$$\frac{\varphi - \delta}{\delta} = 6\Big(1 - \frac{1}{6\nu \operatorname{Log}(1/\delta)} \Big) \nu \operatorname{Log}(1/\delta) \geq 5\nu \operatorname{Log}(1/\delta)$$

since $\delta \leq 1/4$.

6 Simplifying (H_4)

We prove that (H_9) implies (H_4). To achieve that in the simplest manner use Hölder's inequality:

$$\mathrm{R}(f,D_0,2\nu)^2 \le \sum_{d\le D_0} \tau_{2\nu}(d)^2 d^{-1} \sum_{d\le D_0}^{*} \max_{y\le X} d|r_d(f,y)|^2$$
$$\le C\hat{F}(X) \sum_{d\le D_0} \tau_{2\nu}(d)^2 d^{-1} \sum_{d\le D_0}^{*} \max_{y\le X} |r_d(f,y)|$$

We use Lemma 13 and note that $\sum_{n\le x} 1/n \le 2\operatorname{Log} x$ as soon as $x\ge e$, to reach

$$A = \sqrt{C(2\operatorname{Log} X)^{4\nu^2} A'} \tag{34}$$

which indeed is what we have announced.

7 Removal of the preliminary sieving: proof of Theorem 2

In this section, we give an upper bound for

$$\sum_{\substack{n\le X\\(n,P(z))\neq 1}} \Lambda^{(2\nu)}(n)g(n) + \sum_{\substack{n\le X\\(n,P(z))\neq 1}} \Lambda^{(\nu)}\star\Lambda^{(\nu)}(n)g(n) \tag{35}$$

when g is f or f_0. We use (H_8), $\delta \operatorname{Log} X \ge (2\nu+1)$ and $z\ge e^{2\nu+1}$.

The argument is fairly simple. For the first sum we use the fact that it contains few summands. As for the second one we proceed in three steps. We write $n = p^h m$ with $p\le z$ and distinguish several cases: when $n\le X^{1/2}$, we use a maximum for f and f_0; we proceed also in this way when m is a power of a prime $\le X^{1/2}$. However, when $n = p^h m > X^{1/2}$ with m prime $> X^{1/2}$ (and thus $p^h \le X^{1/2}$) we have to be more cautious since taking the maximum on f will not be enough in applications: a typical example is when f is the characteristic function of an arithmetic progression modulo q where using a maximum would loose a factor q.

To keep our estimates precises, we shall use two results of [12] : $\psi(X)\le 1.04X$ for all $X>0$ ((3.35) of the aforementioned paper) and (see (3.24) and (3.6) therein) :

$$\sum_{p\le X} \frac{\operatorname{Log} p}{p} < \operatorname{Log} X, \qquad \pi(X) \le 1.26X/\operatorname{Log} X \qquad (X\ge 1). \tag{36}$$

Here is another preliminary estimate :

Lemma 25 *We have*

$$\sum_{h\ge 1} \frac{h^{\nu-1}}{p^{h-1}} \le \frac{(\nu-1)!}{(1-1/p)^{\nu}} \le 2^{\nu}(\nu-1)!$$

Proof. Put $\beta_h = h^{\nu-1}/[h(h+1)\dots(h+\nu)] \leq 1$. We have

$$\sum_{h\geq 1} \frac{h^{\nu-1}}{p^{h-1}} \leq \max_{h\geq 1}(\beta_h) \sum_{k\geq 0} \frac{(k+1)\dots(k+\nu-1)}{p^k} \leq \frac{(\nu-1)!}{(1-1/p)^\nu}$$

hence the result. ⋄⋄⋄

7.1 The part with the primes

We first have

$$\begin{aligned}\sum_{p\leq z} \sum_{h\leq \frac{\operatorname{Log} X}{\operatorname{Log} p}} (\operatorname{Log} p)^{2\nu} h^{2\nu-1} &\leq \sum_{p\leq z} (\operatorname{Log} p)^{2\nu} \frac{1}{2\nu} \left(\frac{2\operatorname{Log} X}{\operatorname{Log} p}\right)^{2\nu} \\ &\leq 1.26 \cdot 2^{2\nu-1} (\operatorname{Log} X)^{2\nu} \frac{z}{\operatorname{Log} z}.\end{aligned}$$

We divide this estimate by $(2\nu-1)!$ and use Lemma 8. This gives us an error term of size at most

$$\mathrm{MT} \cdot B \cdot c^2 \cdot \frac{1.26}{\operatorname{Log} 2}\, 2^{2\nu} \frac{z}{\hat{F}(X)\delta}. \tag{37}$$

An appeal to (H_8) reduces this term to

$$\mathrm{MT} \cdot B \cdot c^2 \cdot \frac{1.26}{\operatorname{Log} 2}\, 2^{2\nu} \delta^\nu \frac{\operatorname{Log} X}{\sqrt{X}(2\nu+1)} \leq 0.001\ \mathrm{MT}\ B\ c^2\ (4\delta)^\nu. \tag{38}$$

7.2 The bilinear part: small n's

First note that by Lemma 25

$$\begin{aligned}&\sum_{p\leq z} \sum_{h\leq \frac{\operatorname{Log} X}{2\operatorname{Log} p}} (\operatorname{Log} p)^\nu h^{\nu-1} \sum_{m\leq \sqrt{X}/p^h} \Lambda(m)(\operatorname{Log} m)^{\nu-1} \\ &\leq 1.26(\tfrac{1}{2}\operatorname{Log} X)^{\nu-1}\sqrt{X} \sum_{p\leq z} \frac{(\operatorname{Log} p)^\nu}{p} \sum_{h\leq \frac{\operatorname{Log} X}{2\operatorname{Log} p}} \frac{h^{\nu-1}}{p^{h-1}} \\ &\leq 1.26(\tfrac{1}{2}\operatorname{Log} X)^{\nu-1}\sqrt{X} \sum_{p\leq z} \frac{(\operatorname{Log} p)^\nu}{p} \frac{(\nu-1)!}{(1-1/p)^\nu} \\ &\leq 2.52(\nu-1)!\sqrt{X}(\operatorname{Log} X)^{\nu-1}(\operatorname{Log} z)^\nu.\end{aligned}$$

This gives us (use Lemma 8) an error term of size at most

$$\mathrm{MT}\cdot B\cdot c^2\cdot\frac{5.04}{\operatorname{Log}2}\frac{(2\nu)!}{2\,\nu!}\frac{\sqrt{X}\delta^\nu}{\hat{F}(X)}. \tag{39}$$

An appeal to (H_8) and to Lemma 9 reduces this term to

$$\mathrm{MT}\cdot B\cdot c^2\cdot 5.4(4\nu\delta^2/e)^\nu/z\leq 0.01\cdot\mathrm{MT}\cdot B\cdot c^2(\nu\delta)^\nu. \tag{40}$$

7.3 The bilinear part: m's with a small prime factor

Next, we have

$$\begin{aligned}\sum_{p\leq z}\sum_{h\leq\frac{\operatorname{Log}X}{\operatorname{Log}p}}&(\operatorname{Log}p)^\nu h^{\nu-1}\sum_{p'\leq\sqrt{X}}\sum_{k\leq\frac{\operatorname{Log}X}{\operatorname{Log}p'}}(\operatorname{Log}p')^\nu k^{\nu-1}\\ &\leq\sum_{p\leq z}\left(\frac{2\operatorname{Log}X}{\operatorname{Log}p}\right)^\nu(\operatorname{Log}p)^\nu\sum_{p'\leq\sqrt{X}}\left(\frac{2\operatorname{Log}X}{\operatorname{Log}p'}\right)^\nu(\operatorname{Log}p')^\nu\\ &\leq(2\operatorname{Log}X)^{2\nu-2}8\cdot(1.26)^2 z\sqrt{X}/\delta.\end{aligned}$$

We divide this bound by $(\nu-1)!^2$ and invoke Lemma 10. We next appeal to (H_8), Lemma 8 and $\delta\operatorname{Log}X\geq(2\nu+1)$ to obtain an error term of size at most

$$2\mathrm{MT}\cdot B\cdot\frac{(1.26)^2c^2}{\operatorname{Log}2}\cdot 2^{2\nu-\frac{1}{2}}\sqrt{\nu}2^{2\nu}\frac{\delta^\nu}{2\nu+1}\leq\mathrm{MT}\cdot B\cdot c^2\cdot(16\delta)^\nu. \tag{41}$$

7.4 The bilinear part: m a large prime

We are left with

$$\sum_{\substack{p^h\leq\sqrt{X}\\ p\leq z}}(\operatorname{Log}p)^\nu h^{\nu-1}\sum_{\sqrt{X}<p'\leq X/p^h}(\operatorname{Log}p')^\nu f(p'p^h) \tag{42}$$

where we can simply use an upper sieve up to $X^{1/4}\leq D_0/\sqrt{X}$. This means we invoke the Fundamental Lemma (Lemma 22) with $M=\tilde{z}=X^{1/4}$, $\tilde{\sigma}=\sigma$ and we include p inside $\mathfrak{f}$. The constante $\tilde{c}=c$ is unchanged. Hypothesis (H_4) is of course more than enough for such a simple task. The quantity in (42) is thus bounded above by:

$$(\operatorname{Log} X)^{2\nu} R(f, D_0, 1) \\ + (\operatorname{Log} X)^{\nu} \sum_{p^h \le \sqrt{X}} (\operatorname{Log} p)^{\nu} h^{\nu-1} \sigma(p^h) \frac{V_\sigma(X^{1/4})}{1-\sigma(p)} \hat{F}(X)(1 + C_0(c)).$$

Hypothesis (H_1) implies, on taking $u = v = p$, that $(1-\sigma(p))^{-1} \le c$. Now invoke (H_6) and (H_2) and get that the above is at most

$$(\operatorname{Log} X)^{2\nu} R(f, D_0, 1) \\ + 2^{2-\nu} (\operatorname{Log} X)^{\nu} V_\sigma(X^{1/4}) \sum_{p \le z} \frac{\operatorname{Log}^{\nu} p}{p} \sum_{h \ge 1} \frac{h^{\nu-1}}{p^{h-1}} c^2 \hat{F}(X)(1 + C_0(c)). \quad (43)$$

The treatment of $V_\sigma(X^{1/4})$ is different from the one for $V_{\sigma_0}(X^{1/4})$. For this latter we have (H_7) at our disposal, and $\frac{V_\sigma(z)}{V_{\sigma_0}(z)}$ comes from the factor in front. For σ, we first extract $\frac{V_\sigma(z)}{V_{\sigma_0}(z)}$ and are left with

$$V_{\sigma_0}(z) \prod_{z < p \le X^{1/4}} (1 - \sigma(p)) \le V_{\sigma_0}(X^{1/4}) \exp\left(\sum_{z < p \le X^{1/4}} \frac{\sigma_0(p) - \sigma(p)}{1 - \sigma_0(p)} \right). \quad (44)$$

On appealing to (H_1) and recalling (5), we get via (H_7):

$$V_\sigma(X^{1/4}) \le \frac{V_\sigma(z)}{V_{\sigma_0}(z)} \frac{c e^{c\Delta}}{\operatorname{Log} X}. \quad (45)$$

As a conclusion, these estimates introduce a factor $u = c$ (for σ_0) and $u = ce^{c\Delta}$ (for σ).

We use Lemma 25 for the sum over h and reach the upper bound

$$2^{1-2\nu} (\operatorname{Log} X)^{2\nu-1} R(f, D_0, 1) \\ + 2^2 (\operatorname{Log} X)^{\nu-1} (\operatorname{Log} z)^{\nu} (\nu-1)! c^2 u \hat{F}(X)(1 + C_0(c))$$

or also after dividing by $(\nu-1)!^2$ (use Lemma 8)

$$\frac{2^{1-2\nu}}{(\nu-1)!^2} (\operatorname{Log} X)^{2\nu-1} R(f, D_0, 1) + \mathrm{MT}\ 4uc^2 \frac{(2\nu-1)!}{(\nu-1)!} (1 + C_0(c)) \delta^{\nu}.$$

This is not more by Lemma 9 than

$$\frac{2^{1-2\nu}}{(\nu-1)!^2} (\operatorname{Log} X)^{2\nu-1} R(f, D_0, 1) + \mathrm{MT}\ 8uc^2 e^{1/24} (4\nu\delta/e)^{\nu} (1 + C_0(c)) \\ \ll \frac{2^{1-2\nu}}{(\nu-1)!^2} (\operatorname{Log} X)^{2\nu-1} R(f, D_0, 1) + \mathrm{MT}\ 9uc^2 (2\nu\delta)^{\nu} (1 + C_0(c)). \quad (46)$$

We add up the contribution of f and of f_0. The error term is not more than $(\theta_1+\theta_2)\mathrm{MT}$ where (adding (38), (40), (41) and (46))

$$|\theta_1| \le 11Bc^2(1+c(1+e^{c\Delta})(1+C_0(c)))(8\nu\delta)^\nu, \tag{47}$$

and by Lemma 10

$$|\theta_2| \le A\frac{2^{1-2\nu}(2\nu-1)!}{(\nu-1)!^2}(\operatorname{Log} X)^{-1} \le 0.6\sqrt{\nu}A/\operatorname{Log} X. \tag{48}$$

We finally use $\frac14 \ge \nu\delta \ge 2\nu/\operatorname{Log} X$ to reach

$$|\theta_2| \le 0.1\ A. \tag{49}$$

8 Examples

The following three examples are standard. It is better to start with two preliminary Lemmas. They require two definitions. First, we set for any integer $\kappa \ge 1$

$$\mathscr{F}_\kappa(X) = \int_1^X \operatorname{Log}^{\kappa-1} t\,dt, \tag{50}$$

and second, we define

$$\mathfrak{L} = \exp\Big\{-\frac{\operatorname{Log}^{3/5} X}{\operatorname{Log}\operatorname{Log}^{1/5} X}\Big\}. \tag{51}$$

An integration by parts yields:

Lemma 26

$$\mathscr{F}_\kappa(X) = X\sum_{1\le k\le\kappa}(-1)^{k-1}\frac{(\kappa-1)!}{(\kappa-k)!}\operatorname{Log}^{\kappa-k} X + (-1)^\kappa(\kappa-1)!.$$

When $2\kappa \le \operatorname{Log} X$, we have $\mathscr{F}_\kappa(X) \asymp X\operatorname{Log}^{\kappa-1} X$.

Proof. We check this formula by recursion. Indeed, it holds when $\kappa=1$ and we easily check that $\mathscr{F}_\kappa(X) = X\operatorname{Log}^{\kappa-1} X - (\kappa-1)\mathscr{F}_{\kappa-1}(X)$. We obtain the order of magnitude by first noticing that the series is alternated. Secondly, when κ is even and we are looking for a lower bound, the last term can be discarded, while, when $\kappa\ge 3$ is odd and we are again searching for a lower bound, we can join this last term with the term with $k=\kappa$. We proceed similarly for the upper bound and deal directly with $\kappa=1,2,3$. ⋄ ⋄ ⋄

Lemma 27 *There is a positive constant c such that, for $X \geq 3$ and $X \geq \mathfrak{f} \geq 1$, one has*

$$\sum_{n \leq X}{}^{*} \Lambda^{(\kappa)}(n) = \frac{\mathscr{F}_\kappa(X)}{(\kappa-1)!}(1 + \mathcal{O}(\mathfrak{L}^c))$$

provided $2\kappa \leq \operatorname{Log} X$.

Proof. Indeed when $\kappa \geq 2$ we have, since $\operatorname{Log}^{\kappa-1} n = \mathscr{F}'_\kappa(n)$

$$\sum_{n \leq X}{}^{*} \Lambda(n) \operatorname{Log}^{\kappa-1} n = \psi^*(X) \operatorname{Log}^{\kappa-1} X - \int_1^X \psi^*(t)\, \mathscr{F}''_\kappa(t) dt$$

where

$$\psi^*(X) = \sum_{\substack{n \leq X,\\ (n,\mathfrak{f})=1}} \Lambda(n) = \psi(X) + \mathcal{O}(\omega(\mathfrak{f}) \operatorname{Log} X).$$

We thus reduce the expression above to the case $\mathfrak{f} = 1$ up to an error term of order $\mathcal{O}(\omega(\mathfrak{f}) \operatorname{Log}^\kappa X)$. We recall that there exists $c_1 > 0$ such that (see [10, Corollary 8.30]) , when $y \geq 10$,

$$\psi(y) = y + y \exp\biggl\{-c_1 \frac{\operatorname{Log}^{3/5} y}{\operatorname{Log}\operatorname{Log}^{1/5} y}\biggr\}. \tag{52}$$

This leads to, for a positive constant c_2,

$$\sum_{\sqrt{X} < n \leq X} \Lambda(n) \operatorname{Log}^{\kappa-1} n = X \operatorname{Log}^{\kappa-1} X - \int_{\sqrt{X}}^X t\, \mathscr{F}''_\kappa(t) dt$$
$$+ \mathcal{O}(\sqrt{X} \operatorname{Log}^{\kappa-1} X + X\, \mathfrak{L}^{c_2} \operatorname{Log}^\kappa X\bigr).$$

The first $\mathcal{O}$-term is of lower order than the second one. On using the conditions $\kappa \leq \operatorname{Log} X$ and $\mathfrak{f} \leq X$, we reach

$$\sum_{n \leq X}{}^{*} \Lambda(n) \operatorname{Log}^{\kappa-1} n = X \operatorname{Log}^{\kappa-1} X - \int_1^X t\, \mathscr{F}''_\kappa(t) dt + \mathcal{O}\Bigl(X\, \mathfrak{L}^{c_2} \operatorname{Log}^\kappa X\Bigr).$$

The main term is indeed $\mathscr{F}_\kappa(X)$ and Lemma 26 enables us to put this main term in factor. ⋄ ⋄ ⋄

Lemma 28 *When ν_1 and ν_2 are positive integers we have, for $X \geq 1$:*

$$\int_1^X \frac{\mathscr{F}_{\nu_1}(X/t)\, \mathscr{F}'_{\nu_2}(t) dt}{(\nu_1 - 1)!(\nu_2 - 1)!} = \frac{\mathscr{F}_{\nu_1+\nu_2}(X)}{(\nu_1 + \nu_2 - 1)!}.$$

Proof. We appeal to Lemma 26 with $\kappa = \nu_1$ to get that the LHS above, say E, is

$$E = X \sum_{1\le k\le \nu_1} (-1)^{k-1}\frac{(\nu_1-1)!}{(\nu_1-k)!}\int_1^X \frac{\operatorname{Log}^{\nu_1-k}(X/t)\operatorname{Log}^{\nu_2-1} t\,dt}{t} + (-1)^{\nu_1}(\nu_1-1)!\,\mathscr{F}_{\nu_2}(X).$$

The inner integral is $(\operatorname{Log} X)^{\nu_1+\nu_2-k}(\nu_1-k)!(\nu_2-1)!/(\nu_1+\nu_2-k)!$ (this is for instance [5, (2.7)]). This gives us

$$E = X \sum_{1\le k\le \nu_1} (-1)^{k-1}\frac{(\nu_1-1)!(\nu_2-1)!}{(\nu_1+\nu_2-k)!}(\operatorname{Log} X)^{\nu_1+\nu_2-k} + (-1)^{\nu_1}(\nu_1-1)!\,\mathscr{F}_{\nu_2}(X).$$

We use Lemma 26 with $\kappa=\nu_2$ and check (by using the same Lemma with this time $\kappa=\nu_1+\nu_2$) that the RHS is $\mathscr{F}_{\nu_1+\nu_2}(X)(\nu_1-1)!(\nu_2-1)!/(\nu_1+\nu_2-1)!$, as required. $\diamond\diamond\diamond$

Lemma 29 *There is a positive constant c such that, for $X\ge 3$ and $X\ge \mathfrak{f}\ge 1$, one has*

$$\sum_{n\le X}{}^{*}\Lambda^{(2\nu)}(n) + \sum_{n\le X}{}^{*}\Lambda^{(\nu)}\star\Lambda^{(\nu)}(n) = \frac{2\,\mathscr{F}_{2\nu}(X)}{(2\nu-1)!}(1+\mathcal{O}(\mathfrak{L}^{c}))$$

provided $4\nu\le \operatorname{Log} X$.

Proof. We invoke Lemma 27 and check that, for a positive c_3:

$$\sum_{\substack{\ell\le\sqrt{X},\\ m\le X/\ell}}{}^{*}\Lambda^{(\nu)}(\ell)\Lambda^{(\nu)}(m) = \sum_{\ell\le\sqrt{X}}{}^{*}\Lambda^{(\nu)}(\ell)\frac{\mathscr{F}_{\nu}(X/\ell)}{(\nu-1)!}(1+\mathcal{O}(\mathfrak{L}^{c_3})).$$

On using Lemma 26 together with Lemma 9, we readily check that the remainder term is

$$\ll X\frac{\operatorname{Log}^{2\nu-1}X}{(\nu-1)!^2}\,\mathfrak{L}^{c_3} \ll \frac{\mathscr{F}_{2\nu}(X)}{(2\nu-1)!}\,\mathfrak{L}^{c_4}$$

for some positive c_4. The main term reads

$$\sum_{\ell\le\sqrt{X}}{}^{*}\Lambda^{(\nu)}(\ell)\frac{\mathscr{F}_{\nu}(X/\ell)}{(\nu-1)!} = \int_1^X \sum_{\ell\le\min(X/t,\sqrt{X})}{}^{*}\Lambda^{(\nu)}(\ell)\,\mathscr{F}'_{\nu}(t)dt/(\nu-1)!$$

which up to an admissible error term equals

$$\int_1^X \mathscr{F}_{\nu}(\min(X/t,\sqrt{X}))\,\mathscr{F}'_{\nu}(t)dt/(\nu-1)!^2.$$

We use Dirichlet hyperbola principle and the estimates above to infer that

$$\sum_{n\le X}^{*}\Lambda^{(\nu)}\star\Lambda^{(\nu)}(n)=2\int_{\sqrt{X}}^{X}\frac{\mathscr{F}_\nu(X/t)\,\mathscr{F}'_\nu(t)dt}{(\nu-1)!^2}+\frac{\mathscr{F}_\nu(\sqrt{X})^2}{(\nu-1)!^2}+\mathcal{O}\left(\frac{\mathscr{F}_{2\nu}(X)}{(2\nu-1)!}\,\mathfrak{L}^{c_4}\right).$$

We integrate one $\int_{\sqrt{X}}^{X}\cdots$ by parts, change t by X/t and finally check that the main term above is

$$\int_1^X\frac{\mathscr{F}_\nu(X/t)\,\mathscr{F}'_\nu(t)dt}{(\nu-1)!^2}=\mathscr{F}_{2\nu}(X)/(2\nu-1)!^2 \tag{53}$$

by Lemma 28. This concludes the proof. $\diamond\diamond\diamond$

8.1 *Primes in progressions*

We take $f(n)=1$ when $n\equiv a[q]$ (where $(a,q)=1$) and $f(n)=0$ otherwise. Then we select $\mathfrak{f}=q$. We select $f_0(n)=1/\phi(q)$ when $(n,q)=1$ and $f_0(n)=0$ otherwise. This gives us $\sigma(d)=\sigma_0(d)=1/d$ (so that $\Delta=0$), furthermore $\hat{F}(X)=F(X)=X/q$, $B_0=q/\phi(q)$ and c is bounded. We easily get:

$$\begin{aligned}R(f,D_0,2\nu)+R(f_0,D_0,2\nu)&\le 2\sum_{d\le D_0}\tau_{2\nu}(d)\le 2D_0\sum_{d\le D_0}\tau_{2\nu}(d)/d\\&\le 2D_0(2\operatorname{Log}X)^{2\nu}\le 2\frac{qD_0}{X}(2\operatorname{Log}X)^{2\nu}\hat{F}(X)\end{aligned} \tag{54}$$

when $X\ge 3$. This leads to

Corollary 30 *There exists a positive constant c such that*

$$\Big(\sum_{\substack{n\le X\\ n\equiv a[q]}}\Lambda^{(2\nu)}(n)+\sum_{\substack{n\le X\\ n\equiv a[q]}}\Lambda^{(\nu)}\star\Lambda^{(\nu)}(n)\Big)\Big/\Big(\frac{2\,\mathscr{F}_{2\nu}(X)}{\phi(q)(2\nu-1)!}\Big)=1+\mathcal{O}\Big(\mathfrak{L}^{c}+q\exp\{-\tfrac{1}{2}\delta\operatorname{Log}X\}+\frac{q}{\phi(q)}\delta^{\nu/2}\Big)$$

provided

$$3\le q\le X,\quad \nu\le\frac{\delta\operatorname{Log}X}{6\operatorname{Log}\operatorname{Log}X},\quad \nu^2\delta^{3/2}\operatorname{Log}(1/\delta)\le 1/200$$

and δ is small enough.

Proof. We write the error term as $\varepsilon+\rho+\theta$. First ε comes from the main term, see Lemma 29, and $|\varepsilon|\ll\mathfrak{L}^{c_5}$ for some positive constant c_5; secondly, ρ

comes from the first part of the error term in Theorem 2:

$$|\rho| \le (56\nu^2)^\nu \Big\{ 2\frac{q}{z}(2\operatorname{Log} X)^{2\nu} + C_0(c)(c^2\delta^3)^\nu \Big\}.$$

Finally θ depends on δ and is bounded by (recall that $D_0 = X^{1-\delta}$)

$$|\theta| \ll 2\frac{q}{X^\delta}(2\operatorname{Log} X)^{2\nu} + C_0(c)(56c^2\nu^2\delta^3)^\nu + q(200\nu^2\delta\operatorname{Log}(1/\delta))^\nu.$$

This leads to

$$|\rho| + |\theta| \ll \frac{q}{X^\delta}(112\nu\operatorname{Log} X)^{2\nu} + q(200\nu^2\delta\operatorname{Log}(1/\delta))^\nu$$

provided δ be small enough. Our hypotheses imply that $\nu \le \frac{1}{112}\operatorname{Log} X$ when X is large enough. This enables us to reduce the above bound to

$$|\rho| + |\theta| \ll qX^{-\delta}\operatorname{Log}^{3\nu} X + q\delta^{-\nu/2} \ll q\exp\Big(-\tfrac{1}{2}\delta\operatorname{Log} X\Big) + q\delta^{-\nu/2}.$$

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8.2 An oscillatory function

We want to study $g_a(n) = e(n\alpha)$ with $\alpha = \frac{a}{q} + \beta$, $(a,q) = 1$. We take $\mathfrak{f} = q$ and assume $|D_0 q\beta| \le \frac{1}{2}$. We notice that, when $(d,\mathfrak{f}) = 1$ and $d \le D_0$:

$$\sum_{\substack{n \le y/d \\ (n,q)=1}} g_a(dn) = \frac{\mu(q)}{q}\frac{y}{d}\frac{e(y\beta)-1}{2i\pi y\beta} + \mathcal{O}(q^2/\phi(q)) = \frac{G(y)}{d} + \mathcal{O}(q^2/\phi(q)). \quad (55)$$

Proof. Indeed, we treat the condition $(n,q) = 1$ via the Moebius function:

$$\begin{aligned}\sum_{\substack{n \le y/d \\ (n,q)=1}} g_a(dn) &= \sum_{\delta|q} \mu(\delta) \sum_{\substack{m \le y/(d\delta) \\ (n,q)=1}} g_a(d\delta m) \\ &= \mu(q) \sum_{\substack{m \le y/(dq) \\ (n,q)=1}} g_a(dqm) + \mathcal{O}\Bigg(\sum_{\substack{\delta|q, \\ \delta \neq q}} \mu^2(\delta) / \Big\| \frac{ad+\beta dq}{q/\delta} \Big\| \Bigg).\end{aligned}$$

The $\mathcal{O}$-tem is $\mathcal{O}(q^2/\phi(q))$. The main term does not depend on a anymore, but we should still separate y and d. We achieve that in two steps. We first write this main term as follows:

$$\frac{e\Big(\beta dq[y/(dq)]\Big)-1}{e(\beta dq)-1} = \frac{e(\beta y)-1}{e(\beta dq)-1} + e(\beta y)\frac{e(\beta dq\eta)-1}{e(\beta dq)-1}$$

with $-\eta$ being the fractionnal part of $y/(dq)$. The second summand on the RHS is $\mathcal{O}(1)$. We further write

$$\frac{e(\beta y)-1}{e(\beta dq)-1} = \frac{e(\beta y)-1}{2i\pi\beta dq} + \frac{e(\beta y)-1}{2i\pi\beta dq}\Big(\frac{2i\pi\beta dq}{e(\beta dq)-1} - 1\Big)$$

where the second summand on the RHS is again $\mathcal{O}(1)$. This proves our claim. $\diamond\diamond\diamond$

We apply Theorem 1 to the functions $1+\Re g_a$, $1+\Im g_a$ and 1, the model function for g_a being g_1. In these three cases, we proceed as for (54) and get

$$\mathrm{R}(f, D_0, 2\nu) + \mathrm{R}(f_0, D_0, 2\nu) \ll \frac{q}{\phi(q)}\frac{qD_0}{X}(2\operatorname{Log} X)^{2\nu}\cdot X,$$

and we take $\hat{F}(X) = X$ and $B_0 = 2$. We thus get the same results as in the preceding case, save that A is to be multiplied by $\mathcal{O}(q/\phi(q))$. This leads in this case to

$$|\rho| + |\theta| \ll \frac{q^2}{\phi(q)X^\delta}\operatorname{Log}^{3\nu} X + \delta^{\nu/2} \ll \frac{q}{X^\delta}\operatorname{Log}^{4\nu} X + \delta^{\nu/2}$$

on assumptions very similar to the ones of Corollary 30, namely:

$$3 \le q \le X, \quad 11\nu \le \delta\frac{\operatorname{Log} X}{\operatorname{Log}\operatorname{Log} X}, \quad \nu^2\delta^{3/2}\operatorname{Log}(1/\delta) \le 1/200.$$

To remove the model g_1, we sum $\Sigma_\nu(g_a, X)$ over a modulo q (but invertible modulo q) and majorize the resulting sum by taking $\beta = 0$ therein and appealing to Lemma 29. Let us state formally:

Corollary 31 *Let $\alpha = (a/q) + \beta$. We have*

$$\Sigma_\nu(e(n\alpha), X) \ll \frac{X\operatorname{Log}^{2\nu-1} X}{(2\nu-1)!}\left(\frac{1}{\phi(q)} + q\exp\{-\tfrac{1}{2}\delta\operatorname{Log} X\} + \delta^{\nu/2}\right)$$

with $\mathfrak{f} = q$, provided

$$3 \le q \le X, \quad |X^{1-\delta}q\beta| \le \tfrac{1}{2}, \quad 11\nu \le \delta\frac{\operatorname{Log} X}{\operatorname{Log}\operatorname{Log} X}, \quad \nu^2\delta^{3/2}\operatorname{Log}(1/\delta) \le 1/200$$

and δ is small enough.

8.3 Representation of an integer as a sum of a prime and product of two primes

Our result is also adapted to the Goldbach problem by selecting $f(n) = \Lambda(N-n)$. This example is interesting since it shows that our hypotheses on σ and σ_0

have been properly tailored; this part is almost entirely copied from [8]. We select the following parameters:

$$\mathfrak{f} = 1,\ \sigma(d) = \frac{\mathbb{1}_{(d,N)=1}}{\phi(d)},\ f_0(n) = 1,\ \sigma_0(d) = \frac{1}{d},\ \hat{F}(X) = X,\ F(y) = y.$$

This enables us to take $B_0 = 1$. We consider the hypothesis:

$$\sum_{\substack{d \le N^{1-\delta},\\ (d,N)=1}} \max_{y \le X} \Big|\psi(N; d, N) - \frac{X}{\phi(d)}\Big| \le A'' N (11 \operatorname{Log} N)^{-5\nu^2-2} \tag{H_{10}}$$

with the aim of producing an asymptotic formula for the weighted representation number

$$R_\nu(N) = \sum_{n+m=N} \Lambda(n)(\Lambda^{(2\nu)} + \Lambda^\nu \star \Lambda^\nu)(m). \tag{56}$$

Bombieri [1] considered a similar question for the problem of twin primes and with δ tending to zero; Iwaniec & Friedlander in [7] extended this work in several directions, and in particular were able to handle the problem of uniformity to produce a similar result in the case of Goldbach's conjecture. In both cases, the asymptotic formula is obtained by letting δ go to zero. The result we state below has this same feature, but we can increase ν to get a better error term, thus allowing larger δ's. Let us recall that Chen [2] proved that every large enough integer is indeed a sum of a prime and an integer having at most two prime factors, but his method, or any subsequent improvement (see for instance [14]), are not able to produce any asymptotic.

We define also

$$\mathfrak{S}(N) = 2 \prod_{\substack{p|N,\\ p>2}} \frac{p-1}{p-2} \prod_{p>2} \left(1 - \frac{1}{(p-1)^2}\right). \tag{57}$$

Corollary 32 *There exists a constant $c > 0$ such that, for any large enough even integer N and when (H_{10}) holds for some small enough $\delta > 0$ and some $\nu \ge 1$, we have*

$$R_\nu(N) = \frac{2\,\mathscr{F}_{2\nu}(N)\mathfrak{S}(N)}{(2\nu-1)!}\Big(1 + \mathcal{O}(\mathfrak{L}^c + \sqrt{A''} + (\nu^4\delta)^{\nu/2})\Big)$$

provided that

$$\delta > 100/\sqrt{\operatorname{Log} N}, \quad \nu^2\delta \operatorname{Log}(1/\delta) \le 1/8, \quad A'' \ge N^{-\delta/2}.$$

Proof. We note that (H_1) is verified for some constant $c > 0$ independent of N: the coprimality condition at most diminishes the product $\prod_{v \le p \le u}(1 -$

$\sigma(p))^{-1}$. Hypotheses (H_2), (H_5) and (H_7) hold true since, this time, σ_0 does not depend on N. Hypothesis (H_9) follows on one side from the Brun-Titchmarsh Theorem (see [11] for instance) with $C = \mathcal{O}(\operatorname{Log}\operatorname{Log} N)$ and on the other side from (H_{10}) with any A' chosen so that

$$A' \geq A''(11 \operatorname{Log} N)^{-5\nu^2} + \prod_{p|N, p\neq 2} \frac{p}{p-2} N^{-\delta} \operatorname{Log}^2 N.$$

Since we suppose N large enough, we can assume that

$$\delta \operatorname{Log} N \geq 2 \frac{66}{\delta \operatorname{Log}(1/\delta)} \operatorname{Log}\operatorname{Log} N$$

which ensures us that $11(5\nu^2+4)\operatorname{Log}\operatorname{Log} N \leq \frac{1}{2}\delta \operatorname{Log} N$. This implies that $A'' \geq N^{-\delta/2} \geq (11 \operatorname{Log} N)^{5\nu^2+4} N^{-\delta}$. As a consequence, we can take $A' = 2A''(11 \operatorname{Log} N)^{-5\nu^2}$. It is thus enough to have $A \gg \sqrt{(2/11)^{5\nu^2} A''}$. We finally take

$$A = c_1 (56\nu^2)^{-\nu} \sqrt{A''}$$

for a suitable constant $c_1 > 0$. The reader will check that (see [8, (2.4)])

$$\Delta \ll (\operatorname{Log} N)/N^{\delta}.$$

Hypothesis (H_6) is trivial to verify in our case, and (H_8) holds true with $B = \operatorname{Log} N$ (we defined f on the integers $n < N$ only but readily extend it by 0 out of this range). We are thus in a position to use Theorem 2. The quantity $\sum_{\nu}^{*}(f_0, N)$ is evaluated in Lemma 29. The quotient $\frac{V_\sigma(z)}{V_{\sigma_0}(z)}$ is given by

$$\begin{aligned}\frac{V_\sigma(z)}{V_{\sigma_0}(z)} &= 2 \prod_{2<p\leq z} \Big(1 - \frac{1}{p-1}\Big)\Big(1-\frac{1}{p}\Big)^{-1} \prod_{\substack{2<p\leq z,\\ p|N}} \Big(1 - \frac{1}{p-1}\Big)^{-1} \\ &= \mathfrak{S}(N) + \mathcal{O}\left(\frac{\delta^{-1} + \operatorname{Log}\operatorname{Log} N}{N^{\delta}}\right) = \mathfrak{S}(N) + \mathcal{O}\left(\sqrt{A''}\right).\end{aligned}$$

We further have, following notations of Theorem 2:

$$|\rho| + |\theta| \ll \sqrt{A''} + (56\nu^2)^{\nu}\Big(\delta^{5\nu} + 4^{1/\delta}\Delta\Big)(c/\delta)^{2\nu} + (200\nu^2\delta \operatorname{Log}(1/\delta))^{\nu} \operatorname{Log} N.$$

The assumption $\delta > 2/\sqrt{\operatorname{Log} N}$ ensures that $4^{1/\delta}\Delta < e^{1/\delta} \leq e^{5/(8\nu\delta)} \leq \delta^{5\nu}$ when N is large enough. When δ is small enough, we reach

$$|\rho| + |\theta| \ll \sqrt{A''} + (200\nu^2\delta \operatorname{Log}(1/\delta))^{\nu} \operatorname{Log} N \ll \sqrt{A''} + (\nu^4\delta)^{\nu/2} \operatorname{Log} N$$

At this level, we have proved the estimate of the Corollary with a $(\nu^4\delta)^{\nu/2} \operatorname{Log} N$ instead of a $(\nu^4\delta)^{\nu/2}$. This additional $\operatorname{Log} N$ is due to our simplistic hypothesis on the maximum of f in (H_8). In fact, we have here $\hat{F}(X) \geq z\sqrt{X}\delta^{-\nu} \operatorname{Log} X$ (since X is N here); f_0 is bounded by 1. As for f, the estimates (38), (40) still hold divided by $\operatorname{Log} X$, while we can sieve more efficiently to reach both (41) and (46) multiplied by an additional factor of order $\mathfrak{S}(N)/\operatorname{Log} N$. ⋄⋄⋄

9 Proof of Theorem 3

We first note that a bound for $S(\Lambda^{(k)}; \alpha)$ (notation defined in (6)) yields directly a bound for $S(\Lambda; \alpha)$ by

$$|S(\Lambda; \alpha)| \leq 2(k-1)! \frac{\max_{X<X'\leq 2X} |S(\Lambda^{(k)}; \alpha)|}{\operatorname{Log}^{k-1} X} \qquad (X \geq 2). \tag{58}$$

We then write $S(\Lambda^{(2\nu)}; \alpha)$ as

$$\Sigma_\nu(e(n\alpha), X') - \Sigma_\nu(e(n\alpha), X) - \sum_{X<mn\leq X'} \Lambda^{(\nu)}(m)\Lambda^{(\nu)}(n)e(mn\alpha) \tag{59}$$

where we have taken $\mathfrak{f} = 1$. This does not change anything in the bounds given for $\Sigma_\nu(e(n\alpha), X)$ given in Corollary 31. We first reduce the last summand of (59).

Lemma 33 *Let T be a real number $\geq \sqrt{X}$ and not more than $X/2$. Let $\nu \geq 1$ be an integer. We have for $X \geq 10$*

$$\begin{aligned} S(\Lambda^{(\nu)} * \Lambda^{(\nu)}; \alpha) = 2 \sum_{\sqrt{X}<p\leq T} \Lambda^{(\nu)}(p) \sum_{\substack{\ell<p \\ X/p<\ell\leq X'/p}} \Lambda^{(\nu)}(\ell)e(\alpha p\ell) \\ + \mathcal{O}\left(X^{\frac{3}{4}} \operatorname{Log}^{2\nu-1} X + X \operatorname{Log}^{\nu-1} \frac{X}{T} \operatorname{Log}^\nu X\right)/(\nu-1)!^2. \end{aligned}$$

Proof. We do not assume $X \geq \exp(50)$ for the estimates that follow. We have

$$S(\Lambda^{(\nu)} \star \Lambda^{(\nu)}; \alpha) = \sum_{\substack{\ell_1,\ell_2 \\ X<\ell_1\ell_2\leq X'}} \Lambda^{(\nu)}(\ell_1)\Lambda^{(\nu)}(\ell_2)e(\alpha\ell_1\ell_2)$$

which now equals

$$2\sum_{\substack{\ell_2<\ell_1\\ X<\ell_1\ell_2\le X'}}\Lambda^{(\nu)}(\ell_1)\Lambda^{(\nu)}(\ell_2)e(\alpha\ell_1\ell_2)+\mathcal{O}^*\left(\frac{1.04}{2^{2\nu-1}}\sqrt{2X}\frac{\operatorname{Log}^{2\nu-1}2X}{(\nu-1)!^2}\right)$$

$$=2\sum_{\sqrt{X}<p\le X}\Lambda^{(\nu)}(p)\sum_{\substack{\ell_2<p\\ \frac{X}{p}<\ell_2\le\frac{X'}{p}}}\Lambda^{(\nu)}(\ell_2)e(\alpha p\ell_2)$$

$$+\mathcal{O}^*\left(\frac{1.04}{2^{2\nu-1}}\sqrt{2X}\operatorname{Log}^{2\nu-1}2X+(2\times 1.04)^2X^{3/4}\frac{\operatorname{Log}^{2\nu-1}X}{\operatorname{Log}2}\right)/(\nu-1)!^2,$$

since $\psi(X)\le 1.04X$ for all $X>0$ (cf [12, (3.35)]). We now need to remove the large p's, which does not give rise to any problem, whence the result. $\diamond\diamond\diamond$

To handle the remaining bilinear part, we use a diadic decomposition and the following lemma.

Lemma 34 *When $\sqrt{X}\le P\le P'\le 2P$, $X/P\ge q^2$, $|q\beta|\le X^{-0.55}$, $X\le X'\le 2X$ and $X\ge 2$, we have*

$$\sum_{P<p\le P'}\Lambda^{(\nu)}(p)\sum_{\substack{n<p\\ X<np\le X'}}\Lambda^{(\nu)}(n)e\left(pn\alpha\right)\ll\frac{\operatorname{Log}^{\nu-1}(2X/P)}{(\nu-1)!}\frac{\operatorname{Log}^{\nu-1}(2P)}{(\nu-1)!}\frac{\sqrt{q}X}{\phi(q)},$$

the constant in the $\ll$-symbol being absolute.

Proof. For simplicity, we put $\rho=1/(\nu-1)!$ and call Σ our sum. We apply the Cauchy inequality and we relax the condition "p prime" into "p survives a sieve of level M". We use the Fundamental Lemma with M (which we shall later choose as $M=\frac14X^{1/20}$), $z=\sqrt{M}$, $\mathfrak{f}=q$. We assume $Mq\operatorname{Log}q\le P/\operatorname{Log}P$ and $M\ge X^{\varepsilon}$ for some $\varepsilon>0$. Then $|\Sigma|^2$ is not more than

$$\sum_{P<p\le P'}\Lambda^{(\nu)}(p)^2\sum_{P<m\le 2P}\sum_{d|m}\lambda_d^+\sum_{\substack{n,\tilde n\\ \frac{X}{m}<n,\tilde n\le\frac{X}{m}}}\Lambda^{(\nu)}(n)\Lambda^{(\nu)}(\tilde n)e(\alpha m(n-\tilde n)).$$

The first sum is $\ll\rho^2P\operatorname{Log}^{2\nu-1}(2P)$ and we now study the second one.

∘ When $n\equiv\tilde n[q]$, and on using the Brun-Titchmarsh Theorem (see [11]) the contribution is

$$\ll\frac{qP}{\phi(q)\operatorname{Log}P}\sum_{\substack{n\equiv\tilde n[q]\\ \frac{X}{2P}<n,\tilde n\le\frac{2X}{P}}}\Lambda^{(\nu)}(n)\Lambda^{(\nu)}(\tilde n)\ll_\varepsilon\frac{\rho^2qP}{\phi(q)^2\operatorname{Log}P}\left(\frac{X}{P}\right)^2\operatorname{Log}^{2\nu-2}\left(\frac{2X}{P}\right).$$

∘ When $n\not\equiv\tilde n[q]$, the contribution is at most

$$\sum_{\substack{n\not\equiv\tilde n[q]\\ \frac{X}{2P}<n,\tilde n\le\frac{X'}{P}}}\Lambda^{(\nu)}(n)\Lambda^{(\nu)}(\tilde n)\sum_{d\le M}\lambda_d^+\sum_{\substack{m\in I(P,n,\tilde n)\\ d|m}}e(\alpha m(n-\tilde n))\qquad(60)$$

where $I(P, n, \tilde{n})$ is an interval of length $\leq P$. The inner sum is

$$\leq (2\|\alpha d(n-\tilde{n})\|)^{-1} \ll \|ad(n-\tilde{n})/q\|^{-1}$$

provided $qM2X|\beta|/P \leq \frac{1}{2}$. Since $\sum_{d\leq M} |\lambda_d^+| \leq M$, and $Mq \operatorname{Log} q \leq P/\operatorname{Log} P$ by hypothesis, the total contribution is

$$\ll_\varepsilon \rho^2 \frac{q \operatorname{Log} q}{\phi(q)} \frac{X^2 M}{P^2} \operatorname{Log}^{2\nu-2}\left(\frac{2X}{P}\right) \ll \rho^2 \frac{1}{\phi(q)} \cdot \frac{X^2}{P \operatorname{Log} P} \operatorname{Log}^{2\nu-2}\left(\frac{2X}{P}\right).$$

Finally we get

$$|\Sigma|^2 \ll_\varepsilon \rho^4 \operatorname{Log}^{2\nu-2}\left(\frac{2X}{P}\right) \operatorname{Log}^{2\nu-2}(2P) \frac{qX^2}{\phi(q)^2}.$$

Note that the constants do not depend on ν. Since $q^2 \leq X/P$, we surely have $q \leq X^{1/4}$. We take $M = \frac{1}{4}X^{1/20}$. Taking the squareroot yields the Lemma. $\diamond\diamond\diamond$

Using a diadic decomposition, we get

$$\sum_{\substack{\sqrt{X}<p\leq T\\ n<p, X<np\leq X'}} \Lambda^{(\nu)}(p)\ \Lambda^{(\nu)}(n) e(pn\alpha) \ll \frac{\sqrt{q}X}{\phi(q)} \frac{\overline{\Sigma}}{(\nu-1)!^2}$$

with

$$\overline{\Sigma} = \sum_{c=0}^{C} \Big\{ \operatorname{Log}\Big(\sqrt{X} 2^{1-c}\Big) \operatorname{Log}\Big(\sqrt{X} 2^{1+c}\Big) \Big\}^{\nu-1},$$

C being the integer part of $(\operatorname{Log}(2\sqrt{X})/\operatorname{Log} 2$. Putting $D = (\operatorname{Log} 2)/\operatorname{Log}(2\sqrt{X})$, our upper bound becomes

$$\frac{\sqrt{q}X}{\phi(q)} \frac{\operatorname{Log}^{2\nu-2}(2\sqrt{X})}{(\nu-1)!^2} \sum_{c=0}^{C} [(1-cD)(1+cD)]^{\nu-1}.$$

The summation over c is less than

$$1 + \int_0^1 (1-x^2)^{\nu-1} \frac{dx}{D} \ll 1 + \frac{\operatorname{Log} X}{\sqrt{\nu}} \ll \frac{\operatorname{Log} X}{\sqrt{\nu}}$$

under the assumption $\operatorname{Log} X \geq \nu$. Under this same assumption, we also have

$$\left(\frac{\operatorname{Log}(2\sqrt{X})}{\operatorname{Log}\sqrt{X}}\right)^{2\nu-2} = \left(1 + \frac{\operatorname{Log} 2}{\operatorname{Log} X}\right)^{2\nu-2} \ll 1.$$

Note also that thanks to Lemma 9, we have

$$\binom{2\nu-2}{\nu-1} \frac{2\nu-1}{2^{2\nu}\sqrt{\nu}} \ll 1.$$

Thus, for $\operatorname{Log} X \geq \nu$,

$$\sum_{\substack{\sqrt{X}<p\leq T\\ n<p, X<np\leq X'}} \Lambda^{(\nu)}(p)\ \Lambda^{(\nu)}(n)e(pn\alpha) \ll \frac{\sqrt{q}X}{\phi(q)}\frac{\operatorname{Log}^{2\nu-1} X}{(2\nu-1)!}. \tag{61}$$

To complete the proof of Theorem 3, we write

$$\begin{aligned} S(\Lambda^{(2\nu)};\alpha) &= \Sigma_\nu(e(n\alpha), X') - \Sigma_\nu(e(n\alpha), X) \\ &\quad + \mathcal{O}\left(\frac{\sqrt{q}X \operatorname{Log}^{2\nu-1} X}{\phi(q)(2\nu-1)!}\left[1+\sqrt{q}\nu^{\frac{1}{2}}\left(\frac{4\operatorname{Log}(X/T)}{\operatorname{Log} X}\right)^{\nu-1}\right]\right), \end{aligned} \tag{62}$$

provided $(\operatorname{Log}(X/T)/\operatorname{Log} X)^{\nu-1} \geq X^{-1/4}$, $X/T \geq q^2$ and $q \leq X^{1/24}$. We take $X/T = q^2$ and use (58) and Corollary 31 to get

$$|S(\Lambda,\alpha)|\frac{\phi(q)}{\sqrt{q}X} \ll 1 + \sqrt{q\nu}\left(\frac{8\operatorname{Log} q}{\operatorname{Log} X}\right)^{\nu-1} + q^{3/2}\exp\{-\tfrac{1}{2}\delta \operatorname{Log} X\} + \sqrt{q}\delta^{\nu/2}$$

provided that

$$|X^{1-\delta}q\beta| \leq \tfrac{1}{4}, \quad 11\nu \leq \delta\frac{\operatorname{Log} X}{\operatorname{Log}\operatorname{Log} X}, \quad \nu^2\delta^{3/2}\operatorname{Log}(1/\delta) \leq 1/200$$

(the LHS of first condition should have X' instead of X which is why we have divided the RHS by 2). We take $\delta = (\operatorname{Log} X)^{-2/3}$ and assume $\operatorname{Log} q \leq \frac{1}{6}(\operatorname{Log} X)^{1/3}$ as well as $\operatorname{Log} q \leq \frac{1}{4}\nu \operatorname{Log}\operatorname{Log} X$ and X large enough. Our bound simplifies into

$$|S(\Lambda,\alpha)|\frac{\phi(q)}{\sqrt{q}X} \ll 1$$

as required. We finally take ν to be the integer part of $(\operatorname{Log} X)^{1/3}/(11 \operatorname{Log}\operatorname{Log} X)$ and assume that $\operatorname{Log} q \leq \frac{1}{50}(\operatorname{Log} X)^{1/3}$.

References

[1] E. Bombieri. On twin almost primes. *Acta Arith.*, 28(2):177–193, 457–461, 1975/76.

[2] Jing-run Chen. On large odd numbers as sum of three almost equal primes. *Sci. Sin.*, 14:1113–1117, 1965.

[3] H. Daboussi. Brun's fundamental lemma and exponential sums over primes. *J. Number Theory*, 90:1–18, 2001.

[4] H. Daboussi and J. Rivat. Explicit upper bounds for exponential sums over primes. *Math. Comp.*, 70(233):431–447, 2001.

[5] H.G. Diamond and J. Steinig. An Elementary Proof of the Prime Number Theorem with a Remainder Term. *Inventiones math.*, 11:199–258, 1970.

[6] E. Fouvry and P. Michel. Crible asymptotique et sommes de Kloosterman. In *Proceedings of the Session in Analytic Number Theory and Diophantine Equations*, volume 360 of *Bonner Math. Schriften*, page 27, Bonn, 2003. Univ. Bonn.

[7] J. Friedlander and H. Iwaniec. On Bombieri's asymptotic sieve. *Ann. Sc. Norm. Sup. (Pisa)*, 5:719–756, 1978.

[8] J. Friedlander and H. Iwaniec. Bombieri's sieve. In Bruce C. (ed.) et al. Berndt, editor, *Analytic number theory. Vol. 1. Proceedings of a conference in honor of Heini Halberstam, May 16-20, 1995, Urbana, IL, USA. Boston, MA*, volume 138 of *Birkhäuser. Prog. Math.*, pages 411–430, 1996.

[9] R. Hall and G. Tenenbaum. *Divisors*, volume 90 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1988.

[10] H. Iwaniec and E. Kowalski. *Analytic number theory.* American Mathematical Society Colloquium Publications. American Mathematical Society, Providence, RI, 2004. xii+615 pp.

[11] H.L. Montgomery and R.C. Vaughan. The large sieve. *Mathematika*, 20(2):119–133, 1973.

[12] J.B. Rosser and L. Schoenfeld. Approximate formulas for some functions of prime numbers. *Illinois J. Math.*, 6:64–94, 1962.

[13] I. M. Vinogradov. *The method of trigonometrical sums in the theory of numbers.* Dover Publications Inc., Mineola, NY, 2004. Translated from the Russian, revised and annotated by K. F. Roth and Anne Davenport, Reprint of the 1954 translation.

[14] Jie Wu. Chen's double sieve, Goldbach's conjecture and the twin prime problem. *Acta Arith.*, 114(3):215–273, 2008.