Modified truncated Perron formulae

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Abstract

We prove two general and ready for use formulae relating variations of the summatory function $\sum_{\leq x} a_n$ together with $\frac{1}{2i\pi}\int_{\kappa-iT}^{\kappa+iT} F(z)x^z dz/z$, where $F(z)=\sum_{n\geq 1} a_n/n^z$ and κ is a parameter strictly larger than the abcissa of absolute convergence of F. File `TruncatedPerron-5.tex`.

1 Introduction and results

The Perron summation formula [14] gives a direct link between the summatory function $\sum_{n\leq x} a_n$ and the corresponding Dirichlet series $F(s)=\sum_{n\geq 1} a_n/n^s$, see Landau [9, section 86], or Montgomery & Vaughan [11, chapter 5] as well as the notes therein. The integral containing F extends over a full vertical line of the complex plane, and the need for truncated versions appeared very early. One of them can for instance be found in the classical book of Titchmarsh [17, Lemmas 3-12 and 3.19]. Here is the version proved and discussed in this paper.

Theorem 1.1 (The MT Perron summation formula)**.** *Let $F(s)=\sum_{n\geq 1} a_n/n^s$ be a Dirichlet series and let $\kappa>0$ be a real parameter chosen larger than the abscissa of absolute convergence of F. Let $T\geq 1$ be a real parameter. For every integer $n\in\left[x-\frac{x}{8T}, x+\frac{x}{8T}\right]$ we choose an arbitrary complex number θ_n of modulus bounded by 1. The following formula holds:*

$$\sum_{n<x-\frac{x}{8T}} a_n + \sum_{|n-x|\leq\frac{x}{8T}} \theta_n a_n = \frac{1}{2i\pi}\int_{\kappa-iT}^{\kappa+iT} F(z)\frac{x^z}{z}dz + \mathcal{O}\biggl(\frac{x^\kappa}{T}\sum_{n\geq 1}\frac{|a_n|}{n^\kappa} + e^\kappa\int_{1/T}^{1}\sum_{|n-x|\leq ux}|a_n|\frac{du}{Tu^2}\biggr).$$

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"MT" stands for "Modified Truncated". In particular, both $\sum_{n\le x} a_n$ and $\sum_{n<x} a_n$ are covered by this result. In [15], we shall produce similar formulae relying on the short sums of (a_n) and not of $|a_n|$ and replacing the abscissa of absolute convergence by the abscissa of convergence at the cost of a slightly more complicated kernel than x^z/z.

Another path is to use a smoothed version, i.e. to get a formula for the sum $\sum_{n\le x} a_n f(n/x)$, where f is a compactly supported and sufficiently differentiable function; it is commonly assumed to take the value 1 when $n/x \le 1$ and the value 0 when $n/x > 1+\delta$, where δ is some positive parameter. The smoothness of f ensures that its Mellin transform decreases fast enough in vertical trips. To recover $\sum_{n\le x} a_n$, we then need to evaluate $\sum_{x<n\le(1+\delta)x} a_n f(n/x)$ which relies on the behaviour of a_n on the "short" interval $[x,(1+\delta)x]$. The same kind of information is required to truncate the Perron formula; for instance, I proposed in [16, Theorem 7.1] a general version in which the error term is clearly dependent of the behaviour of short sums (a fact missed by Liu & Ye in [10, Theorem 2.1] – to be complete, one has to notice that the choice $H \ge T$ in this theorem leads to a useless result). The smoothed version offers flexibility but the truncated version offers simplicity. Theorem 1.1 shows that the truncated version can in fact handle any kind of bounded weight attached to a_n around the border $\{n = x\}$.

When using Theorem 1.1 for $F(s) = -\zeta'(s)/\zeta(s)$, or for $F(s) = 1/\zeta(s)$, we select $\kappa = 1+(\log(2x))^{-1}$. We then employ the inequalities $\sum_{n\ge1}\frac{|a_n|}{n^\kappa} \ll \log(2x)$ as well as $\sum_{|n-x|\le ux}|a_n| \ll ux$ when $1/\sqrt{x} \le u \le 1$ by the Brun-Titchmarsh Theorem (the harmless restriction $u \ge 1/\sqrt{x}$ comes from the handling of prime powers). On assuming $T \le \sqrt{x}$, this gives the error term $\mathcal{O}(x\log(xT)/T)$. Several authors set to improve this term, like Goldston in [3], Wolke in [18, Theorem 1] or Perelli & Puglisi in [13]. Our next result belongs to this vein.

Theorem 1.2 (The MT Perron summation formula, second form). *Let $F(s) = \sum_{n\ge1} a_n/n^s$ be a Dirichlet series and let $\kappa > 0$ be a real parameter chosen larger than the abscissa of absolute convergence of F. Let $T \ge 1$ be a real parameter. For every integer $n \in \left[x - \frac{20x}{T}, x + \frac{20x}{T}\right]$ we choose an arbitrary complex number θ_n of bounded modulus. Let δ and $\epsilon > 0$ be two real parameters in $(0,1]$. There exists a subset I^* of $[T,(1+\delta)T]$ of measure $\ge (1-\epsilon)\delta T$ such that for every $T^* \in I^*$, we have*

$$\sum_{n<x-\frac{20x}{T}} a_n + \sum_{|n-x|\le\frac{20x}{T}} \theta_n a_n = \frac{1}{2i\pi}\int_{\kappa-iT^*}^{\kappa+iT^*} F(z)\frac{x^z dz}{z} + \mathcal{O}\biggl(\frac{e^{1/\delta}x^\kappa}{T^2}\sum_{n\ge1}\frac{|a_n|}{n^\kappa} + e^{1/\delta}e^\kappa \sum_{|n-x|\le 20x/T}|a_n|\biggr).$$

We can for instance select $\delta = 1/\log\log T$. The fact that there is a abundant set of possible T^* is useful in practice; for instance, in the case of $-\zeta'/\zeta$, we want T^* to be at distance $\gg 1/\log T$ from the ordinates of the zeroes of ζ. Since there are $\mathcal{O}(T\log T)$ zeroes, the measure of the set of T's in $[T, 2T]$ that are at distance $\le c/\log T$ from the ordinate of (at least) one zero is $\ll cT$, and this is $< T$ if c is small enough.

When applied to $F(s) = -\zeta'(s)/\zeta(s)$, or for $F(s) = 1/\zeta(s)$, this formula leads to the remainder $\mathcal{O}(x(\log x)T^{-2} + xT^{-1})$. In case a remainder term with $\sum_{n\ge1}\frac{|a_n|}{n^\kappa}/T^k$ for a $k > 2$ is preferred, Theorem 5.3 is at the reader's disposal.

The localisation in T relies on an integral Gorny inequality for a special class of functions on which we now comment. Extending work of Hadamard [5], Hardy & Littlewood [6] (see also Cartan in [1] and Kolmogorov [8]), Gorny proved in [4] the following.

Theorem 1.3 (Gorny)**.** *Let f be a C^k-function on a finite interval. We have*

$$\|f^{(\ell)}\|_\infty \le 4e^{2\ell}(k/\ell)^\ell \|f\|_\infty^{1-\frac{\ell}{n}}\|f^{(k)}\|_\infty^{\frac{\ell}{n}}. \quad (0 \le \ell \le k)$$

This kind of result is often termed "Landau-Kolmogorov inequality" though these authors studied the somewhat different case of the segment $[0, \infty)$.

We consider here the class $\mathcal{C}_k(a, b)$ of functions f over an interval (a, b) (both a and b can be infinite) that are k-times differentiable, such that all $f^{(h)}$ when $h \in \{0, \cdots, k\}$ are in L^2 and such that, for all index $h \in \{0, \cdots, k-1\}$, we have $f^{(h)}(a) = f^{(h)}(b) = 0$. The following Theorem holds.

Theorem 1.4. *Let f be in class $\mathcal{C}_k(a, b)$. For any $h \in \{0, \cdots, k\}$, we have*

$$\int_a^b \left|f^{(h)}(v)\right|^2 dv \le \left(\int_a^b \left|f^{(k)}(v)\right|^2 dv\right)^{\frac{h}{k}} \left(\int_a^b \left|f(v)\right|^2 dv\right)^{1-\frac{h}{k}}.$$

The skeleton of the proof of Theorem 1.1 is the same as the proof of [16, Theorem 7.1], indeed confirming the fact that no new information is being incorporated: we extract *more* from the proof. The proof of Theorem 1.2 again starts from the same matrix.

We take the opportunity of this paper to point out what seems like a small mistake in [18, Theorem 1]: in inequality (2.5) therein, a factor $(x/n)^\sigma$ is missing as far as I can see. This has the consequence that [18, Theorem 2] is valid only for $T \ge \log x$, a restriction that is of no consequence for the applications. [7, Theorem 1] has thus the same restriction, as it relies on Wolke's paper.

Notation

Though the constants in the final results are not explicitely given to give as simple results as we could, they are computed for the most part of the paper. To do so we rely on the classical notation $f = \mathcal{O}^*(g)$ to mean that $|f| \le g$.

2 The MT Perron summation formula

Here is the more precise result we prove, from which deducing Theorem 1.1 is a matter of routine.

Theorem 2.1. *Let $v : \mathbb{R} \mapsto [0,1]$ be such that $v(y) = 1$ when $y \geq 1/4$ and $v(y) = 0$ when $y \leq -1/4$. Let $F(s) = \sum_{n\geq 1} a_n/n^s$ be a Dirichlet series and let $\kappa > 0$ be a real parameter chosen larger than the abscissa of absolute convergence of F. Let finally T and T' be two positive real parameters such that $T' \leq 4T$. We have*

$$\sum_{n\geq 1} a_n v(T\log(n/x)) = \frac{1}{2i\pi}\int_{\kappa-iT}^{\kappa+iT} F(z)\frac{x^z}{z}dz \\ + \mathcal{O}^*\biggl(\int_{1/T'}^{\infty} \sum_{n/|\log(x/n)|\leq u} \frac{|a_n|}{n^\kappa}\frac{7x^\kappa du}{5T'u^2}\biggr).$$

In this Theorem, the value of $v(T\log(n/x))$ can be chosen arbitrarily inside $[0,1]$ when $xe^{-1/(4T)} \leq n \leq xe^{1/(4T)}$. There is a very large freedom of choice for the function v, and, in fact, formula (3.6) below is valid in an even larger context. We denote by Y the (multiplicative) Heaviside function defined by

$$Y(x) = \begin{cases} 0 & 0 < x < 1, \\ 1/2 & x = 1, \\ 1 & 1 < x. \end{cases} \tag{2.1}$$

We consider also the function $a(y,\kappa')$ depending of the positive parameters κ' and defined for positive y by

$$a(y,\kappa') = \frac{e^{y\kappa'}}{\pi}\arctan(1/\kappa'). \tag{2.2}$$

Here is our main lemma.

Lemma 2.2. *Let v be a function of the positive variable y. For $\kappa' > 0$ and y, we have*

$$\left|v(y) - \frac{1}{2i\pi}\int_{\kappa'-i}^{\kappa'+i}\frac{e^{zy}dz}{z}\right| \leq \\ \min\left(|v(y) - Y(e^y)| + \frac{e^{y\kappa'}}{\pi|y|}, |v(y) - a(y,\kappa')| + \frac{e^{y\kappa'}}{\pi}|y|\right).$$

Proof. When $y < 0$, we write for $K > \kappa'$ going to infinity :

$$\left(\int_{\kappa'-i}^{\kappa'+i} + \int_{\kappa'+i}^{K+i} + \int_{K+i}^{K-i} + \int_{K-i}^{\kappa'-i}\right)\frac{e^{yz}dz}{z} = 0.$$

The third integral dwindles to zero when K increases. Both integrals on the horizontal segments are bounded by $e^{y\kappa'}/|y|$ (bound $1/|z|$ by 1 and integrate the $e^{y\sigma}$). This implies that

$$\left|v(y) - \frac{1}{2i\pi}\int_{\kappa'-i}^{\kappa'+i}\frac{e^{yz}dz}{z}\right| \leq |Y(e^y) - v(y)| + \frac{e^{y\kappa'}}{\pi|y|} \quad (y < 0).$$

The same bound holds for $y > 0$: the proof goes as above except that we shift the line of integration towards the left hand side. We get

$$\left| v(y) - \frac{1}{2i\pi}\int_{\kappa'-i}^{\kappa'+i} \frac{e^{yz}dz}{z} \right| \le |1 - v(y)| + \frac{e^{y\kappa'}}{\pi|y|} \quad (0 < y).$$

These bounds are efficients when $|y|$ is large enough; else we proceed more directly. Since we want this proof to holds also in the case of Theorem 3.1, we introduce a parameter $\tau \in [1, 2]$, which is thus equal to 1 in this very proof. We write

$$\int_{\kappa'-i\tau}^{\kappa'+i\tau} \frac{e^{yz}dz}{z} = e^{y\kappa'}\int_{\kappa'-i\tau}^{\kappa'+i\tau} \frac{dz}{z} + e^{y\kappa'}\int_{-\tau}^{\tau} \frac{(e^{ity}-1)idt}{\kappa'+it}.$$

The first integral is $2\arctan(1/\kappa') \le \pi$ while we deal with the second one by using

$$\left|\frac{e^{ity}-1}{ity}\right| = \left|\int_0^1 e^{iuty}du\right| \le 1.$$

This leads to the upper bound $2\tau|y|$ (even if $y = 0$), and thus

$$\left| v(y) - \frac{1}{2i\pi}\int_{\kappa'-i\tau}^{\kappa'+i\tau} \frac{e^{yz}dz}{z} \right| \le \left| v(y) - \frac{e^{y\kappa'}}{\pi}\arctan(1/\kappa') \right| + \frac{e^{y\kappa'}\tau}{\pi}|y|. \tag{2.3}$$

The lemma follows readily. □

Let us continue the path of generality. The parameters κ' from Lemma 2.2 and the parameter κ from Theorem 2.1 are linked by $\kappa' = \kappa/T$. We suppose given a function v and the existence of three parameters c_1, c_2 and θ such that

$$\begin{cases} \max\limits_{\substack{y/|y|\ge\theta,\\ 0<\kappa'\le\kappa}} |v(y) - Y(e^y)|\dfrac{|y|}{e^{y\kappa'}} \le c_1, \\ \max\limits_{\substack{y/|y|\le\theta,\\ 0<\kappa'\le\kappa}} \min\left(\dfrac{|v(y)-Y(e^y)|}{e^{y\kappa'}} + \dfrac{1}{\pi|y|}, \dfrac{|v(y)-a(y,\kappa')|}{e^{y\kappa'}} + \dfrac{|y|}{\pi}\right) \le c_2. \end{cases} \tag{2.4}$$

We write

$$\begin{aligned} &\left| \sum_{n\ge1} a_n v(T\log(n/x)) - \frac{1}{2i\pi}\int_{\kappa-iT}^{\kappa+iT} F(z)\frac{x^z}{z}dz \right| x^{-\kappa} \\ &\qquad \le c_2 \sum_{T|\log(x/n)|<\theta} \frac{|a_n|}{n^\kappa} + \frac{c_1+\pi^{-1}}{T}\sum_{T|\log(x/n)|\ge\theta} \frac{|a_n|}{n^\kappa|\log(x/n)|}. \end{aligned}$$

We continue via

$$
\begin{aligned}
\sum_{T|\log(x/n)|\geq\theta}\frac{|a_n|}{n^\kappa|\log(x/n)|}
&=\sum_{T|\log(x/n)|\geq\theta}\frac{|a_n|}{n^\kappa}\int_{|\log(x/n)|}^\infty\frac{du}{u^2}\\
&=\int_{\theta/T}^\infty\sum_{\theta/T\leq|\log(x/n)|\leq u}\frac{|a_n|}{n^\kappa}\frac{du}{u^2}\\
&=\int_{\theta/T}^\infty\sum_{|\log(x/n)|\leq u}\frac{|a_n|}{n^\kappa}\frac{du}{u^2}-\theta^{-1}\sum_{T|\log(x/n)|<\theta}\frac{|a_n|}{n^\kappa}
\end{aligned}
$$

and thus

$$
\begin{aligned}
&\left|\sum_{n\geq1}a_nv(T\log(n/x))-\frac{1}{2i\pi}\int_{\kappa-iT}^{\kappa+iT}F(z)\frac{x^z}{z}dz\right|x^{-\kappa}\\
&\leq\frac{c_1+\pi^{-1}}{T}\int_{\theta/T}^\infty\sum_{|\log(x/n)|\leq u}\frac{|a_n|}{n^\kappa}\frac{du}{u^2}+\Bigl(c_2-\frac{c_1+\pi^{-1}}{\theta}\Bigr)\sum_{T|\log(x/n)|<\theta}\frac{|a_n|}{n^\kappa}.
\end{aligned}
$$

We define

$$
c_3=\max\biggl(c_2,\frac{c_1+\pi^{-1}}{\theta}\biggr). \tag{2.5}
$$

If $(c_1+\pi^{-1})/\theta<c_2$, we may replace c_1 by the larger value $c_2\theta-\pi^{-1}$. This yields:

$$
\begin{aligned}
&\left|\sum_{n\geq1}a_nv(T\log(n/x))-\frac{1}{2i\pi}\int_{\kappa-iT}^{\kappa+iT}F(z)\frac{x^z}{z}dz\right|x^{-\kappa}\\
&\qquad\qquad\leq\frac{c_3}{T/\theta}\int_{\theta/T}^\infty\sum_{|\log(x/n)|\leq u}\frac{|a_n|}{n^\kappa}\frac{du}{u^2}.
\end{aligned} \tag{2.6}
$$

Proof of Theorem 2.1

We assume that $v(y)=Y(e^y)$ when $y\geq\theta$ for some $\theta\geq1/4$ and thus $c_1\geq0$ is enough. We further assume that $0\leq v(y)\leq1$ in general, and since

$$
\max_{0\leq u\leq\theta}\left(1+\frac{1}{\pi u},1+\frac{u}{\pi}\right)\leq1+\pi^{-1}, \tag{2.7}
$$

we can chose $c_2=1+\pi^{-1}$. In this context $c_3=1+\pi^{-1}\leq7/5$. Finally, on setting $T'=T/\theta$, we get

$$
\left|\sum_{n\geq1}a_nv(T\log(n/x))-\frac{1}{2i\pi}\int_{\kappa-iT}^{\kappa+iT}F(z)\frac{x^z}{z}dz\right|\leq\int_{1/T'}^\infty\sum_{n/|\log(x/n)|\leq u}\frac{|a_n|}{n^\kappa}\frac{7x^\kappa du}{5T'u^2}
$$

for *any* function v satisfying the above hypotheses.

Proof of Theorem 1.1

We recall the simple inequalities $e^y \geq 1+y$ valid for $y \geq 0$ as well as $e^{-y} \leq 1-y/2$ valid for $y \in [0, 3/2]$ to replace $v(T\log(n/x))$ by θ_n. We select $T' = 2T$ and we use furthermore

$$\int_{1/T'}^{\infty} \sum_{n/|\log(x/n)| \leq u} \frac{|a_n|}{n^\kappa} \frac{7x^\kappa du}{10Tu^2} \leq \frac{7x^\kappa}{5T} \sum_{n\geq 1} \frac{|a_n|}{n^\kappa} + \int_{1/(2T)}^{1/2} \sum_{n/|\log(x/n)| \leq u} |a_n| \frac{7e^\kappa du}{5Tu^2}.$$

We use $e^y \leq 1+2y$ when $y \in [0,1]$ and $e^{-y} \geq 1-2y$ to simplify the second term via

$$\int_{1/(2T)}^{1/2} \sum_{n/|\log(x/n)| \leq u} |a_n| \frac{du}{u^2} \leq \int_{1/2T}^{1/2} \sum_{|n-x| \leq 2ux} |a_n| \frac{du}{u^2}.$$

The change of variable $2u \mapsto u$ concludes the proof.

3 The WMT Perron summation formula

We start here the proof of Theorem 1.2, but we first prove a general and more precise result, namely Theorem 3.1 below, which we will then specialize. The idea is inherited from [18, Theorem 1]. It is expedient to state it in a general form. Let $\xi > 1$ be a some fixed real parameter. A function w over $[1, \xi]$ with value in $\mathbb{C}$ is said to be (k, ξ)-*admissible* for some non-negative integer k when

1. w is k-times differentiable and $w^{(k)}$ is in L^1.
2. We have $\int_1^\xi w(t)dt = 1$.
3. We have $w^{(\ell)}(1) = w^{(\ell)}(\xi) = 0$ for $0 \leq \ell \leq k-2$. This condition is empty when $k = 1$.

For such a function, we define $\mathrm{N}_{k,\xi}(w)$, $\mathscr{L}_\xi(w)$ and $\mathrm{M}_{k,\xi}(w)$ by

$$2\pi\, \mathrm{N}_{k,\xi}(w) = \tfrac{1}{\xi}|w^{(k-1)}(\xi)| + |w^{(k-1)}(1)| + k! \sum_{0\leq h\leq k} \int_1^\xi \frac{|w^{(h)}(u)|}{h!} du, \tag{3.1}$$

$\mathscr{L}_\xi(w) = \int_1^\xi uw(u)du/\pi$ and

$$\mathrm{M}_{k,\xi}(w) = 1 + \big((k+1)\,\mathrm{N}_{k,\xi}(w)\big)^{1/(k+2)} \mathscr{L}_\xi(w)^{(k+1)/(k+2)}. \tag{3.2}$$

In our usual application, we select $w = 1$ with $k = 1$ and $\xi = 2$. Note that, since w belongs to the class $\mathcal{C}_{k-1}(1, \xi)$ defined in the introduction, we can use Theorem 1.4 to bound $\int_1^\xi |w^{(h)}(u)|du$ when $h \leq k-1$ in the sole terms of $\int_1^\xi |w(u)|^2 du$ and $\int_1^\xi |w^{(k-1)}(u)|^2 du$. We will do so only in an application we have in mind.

Theorem 3.1 (The WMT Perron formula)**.** *Let $k \geq 1$ be an integer and let $\xi > 1$ be a real number. Let w be a (k,ξ)-admissible function. Let $v : \mathbb{R} \mapsto [0,1]$ be such that $v(y) = 1$ when $y \geq \mathrm{M}_{k,\xi}(w)^{-1/(k+1)}$ and $v(y) = 0$ when $y \leq -\mathrm{M}_{k,\xi}(w)^{-1/(k+1)}$. Let $F(z) = \sum_n a_n/n^z$ be a Dirichlet series that converges absolutely for $\Re z > \kappa_a$, and let $\kappa > 0$ be strictly larger than κ_a. For $x \geq 1$, $T \geq 1$ and $T' \leq \mathrm{M}_{k,\xi}(w)^{1/(k+1)}T$, we have*

$$\sum_{n\geq 1} a_n v(T\log(n/x)) = \frac{1}{2i\pi}\int_T^{\xi T}\int_{\kappa-it}^{\kappa+it} F(z)\frac{x^z dz}{z}\frac{w(t/T)dt}{T} + \mathcal{O}^*\left(\mathrm{M}_{k,\xi}(w)\int_{1/T'}^{\infty}\sum_{|\log(x/n)|\leq u}\frac{|a_n|}{n^\kappa}\frac{x^\kappa du}{T'^{(k+1)}u^{k+2}}\right).$$

Here WMT is for "Weighted Modified Truncated". This term is so long it is better to use an acronym! Note that $\mathrm{M}_{k,\xi}(w) \geq 1$, so the choice $T' = T$ is always possible. Notice further that for the choices $w_{0,\xi} = 1/(\xi-1)$ and $k = 1$, we have $\int_1^\xi w_{0,\xi} = 1$, then $\mathrm{N}_{1,\xi}(w_{0,\xi}) \leq 3/(2(\xi-1)\pi)$ and $\mathscr{L}_\xi(w_{0,\xi}) \leq \xi/(2\pi)$. On assuming that $\xi \leq 2$, we find that $\mathrm{M}_{1,\xi}(w_{0,\xi}) \leq 1 + (3\xi^2/(\xi-1))^{1/3}/(2\pi)$. In case $\xi = 2$, we more precisely find that $M_{1,2}(1) \leq 1 + \frac{5}{3}(\xi-1)^{1/3}$, and thus, in particular, we find that, for any $\xi > 1$,

$$\sum_{n\leq x} a_n = \frac{1}{2i\pi}\int_T^{\xi T}\int_{\kappa-it}^{\kappa+it} F(z)\frac{x^z dz}{z}\frac{dt}{(\xi-1)T} + \mathcal{O}^*\left(\int_{1/T}^{\infty}\sum_{|\log(x/n)|\leq u}\frac{|a_n|}{n^\kappa}\frac{5x^\kappa du/3}{(\xi-1)^{1/3}T^2u^3}\right).$$

We have thus handled the question of the localization of t. Concerning the main contribution in the error term, the convergence in $1/u^3$ is usually what is required for applications. The proof of Theorem 3.1 relies on the next lemma.

Lemma 3.2. *Let v be a function of the positive variable y. Let w be (k,ξ)-admissible function. For $\kappa' > 0$ and y, we have*

$$\left|v(y) - \frac{1}{2i\pi}\int_1^\xi\int_{\kappa'-i\tau}^{\kappa'+i\tau}\frac{e^{zy}dz}{z}w(\tau)d\tau\right| \leq \min\left(|Y(e^y) - v(y)| + \frac{\mathrm{N}_{k,\xi}(w)e^{\kappa' y}}{|y|^{k+1}}, |v(y) - a(y,\kappa')| + \frac{|y|e^{\kappa' y}}{\pi}\mathscr{L}_\xi(w)\right).$$

Proof. The proof starts like the one of Lemma 2.2. When $y < 0$, we consider the equality

$$\left(\int_{\kappa'-i\tau}^{\kappa'+i\tau} + \int_{\kappa'+i}^{K+i\tau} + \int_{K+i\tau}^{K-i\tau} + \int_{K-i\tau}^{\kappa'-i\tau}\right)\frac{e^{yz}dz}{z} = 0.$$

The third integral dwindles to zero when K increases. We integrate these four integrals with respect to $\tau \in [1,\xi]$, after multiplication by $w(\tau)$. We can take avantage of this integral sign, by writing

$$\int_1^\xi \int_{\kappa'+i\tau}^{K+i\tau} \frac{e^{zy}dz}{z} w(\tau)d\tau = \int_{\kappa'}^{K} e^{uy} \int_1^\xi \frac{e^{i\tau y} w(\tau)d\tau}{u+it} du. \tag{3.3}$$

Concerning the inner integral, we momentarily set $f(\tau) = w(\tau)/(u+i\tau)$; we check, by using Leibnitz formula for the ℓ-th derivative of a product for instance, that $f^{(\ell)}(\xi) = f^{(\ell)}(1) = 0$ when $0 \le \ell \le k-1$. Since we need to bound the k-derivative, let us recall this formula in our context:

$$f^{(m)}(\tau) = \sum_{0 \le h \le m} \binom{m}{h} \frac{i^{m-h}(m-h)!w^{(h)}(\tau)}{(u+i\tau)^{m-h+1}}.$$

With $m = k-1$, this implies that

$$f^{(k-1)}(1) = \frac{w^{(k-1)}(1)}{u+i}, \quad f^{(k-1)}(\xi) = \frac{w^{(k-1)}(\xi)}{u+i\xi},$$

while with $m = k$, the above formula gives

$$f^{(k)}(\tau) = \sum_{0 \le h \le m} \binom{k}{h} \frac{i^{k-h}(k-h)!w^{(h)}(\tau)x^{it}}{(u+i\tau)^{k-h+1}}.$$

We employ k integrations by parts to reach

$$\begin{aligned}\int_1^\xi \frac{e^{ity}w(\tau)d\tau}{u+i\tau} &= \frac{(-1)^{k-1}}{(iy)^{k-1}} \int_1^\xi e^{i\tau y} f^{(k-1)}(\tau)d\tau \\ &= \frac{(-1)^{k-1} f^{(k-1)}(\xi)e^{2iy}}{(iy)^k} - \frac{(-1)^{k-1} f^{(k-1)}(1)e^{iy}}{(iy)^k} \\ &\quad - \frac{(-1)^{k-1}}{(iy)^k} \int_1^\xi e^{i\tau y} f^{(k)}(\tau)d\tau.\end{aligned}$$

We return to (3.3) and integrate with respect to u:

$$\begin{aligned}&\int_1^\xi \int_{\kappa'+i\tau}^{K+i\tau} \frac{e^{zy}dz}{z} w(\tau)d\tau = \\ &\quad \frac{(-1)^{k-1}w^{(k-1)}(\xi)}{(iy)^k} \int_{\kappa'}^{K} \frac{e^{(u+2i)y}}{u+i\xi} du - \frac{(-1)^{k-1}w^{(k-1)}(1)}{(iy)^k} \int_{\kappa'}^{K} \frac{e^{(u+i)y}}{u+i} du \\ &\quad + \sum_{0 \le h \le K} \binom{k}{h} \int_1^\xi \frac{(-1)^k i^{k-h}(k-h)!w^{(h)}(\tau)}{(iy)^k} \int_{\kappa'}^{K} \frac{e^{(u+i\tau)y}}{(u+i\tau)^{k-h+1}} du d\tau.\end{aligned}$$

The integrals over u are bounded respectively by $\frac{1}{\xi} e^{\kappa' y}/|y|$, $e^{\kappa' y}/|y|$ and $e^{\kappa' y}/|y|$ (In the first one, bound $1/(u+i\xi)$ by $1/\xi$, and integrate e^{uy}, and proceed similarly for the next two). We thus have reached

$$\left|\int_1^2\int_{\kappa'+it}^{K+it}\frac{e^{zy}dz}{z}w(\tau)d\tau\right|\le$$
$$\frac{\frac{1}{\xi}|w^{(k-1)}(\xi)|+|w^{(k-1)}(1)|}{|y|^{k+1}}+\int_1^\xi\sum_{0\le h\le k}\binom{k}{h}\frac{(k-h)!|w^{(h)}(\tau)|}{|y|^{k+1}}d\tau$$

which is thus bounded by $2\pi e^{\kappa' y}\,\mathrm{N}_{k,\xi}(w)/|y|^{k+1}$. On gathering our results, we conclude that we have proved, for $y>0$, that

$$\left|v(y)-\frac{1}{2i\pi}\int_1^\xi\int_{\kappa'-i\tau}^{\kappa'+i\tau}\frac{e^{yz}dz}{z}w(\tau)d\tau\right|\le|Y(e^y)-v(y)|+\mathrm{N}_{k,\xi}(w)\frac{e^{y\kappa'}}{|y|^{k+1}}. \tag{3.4}$$

This is the counterpart of (2). The same bound holds for $y>0$: the proof goes as above except that we shift the line of integration towards the left hand side. These bounds are efficients when $|y|$ is large enough; else we again resort to (2.3). The lemma follows readily. □

We continue to follow the previous section. The parameter κ' from Lemma 3.2 and the parameter κ from Theorem 3.1 are again linked by $\kappa'=\kappa/T$. We now suppose given a function v and the existence of three parameters c_1, c_2' and θ such that

$$\begin{cases}\displaystyle\max_{\substack{y/|y|\ge\theta,\\0<\kappa'\le\kappa}}|\frac{\mathrm{N}_{k,\xi}(w)|y|^{k+1}}{e^{y\kappa'}}\le c_1,\\[2ex] \displaystyle\max_{\substack{y/|y|\le\theta,\\0<\kappa'\le\kappa}}\min\left(\frac{|v(y)-Y(e^y)|}{e^{y\kappa'}}+\frac{\mathrm{N}_{k,\xi}(w)}{|y|^{k+1}},\frac{|v(y)-a(y,\kappa')|}{e^{y\kappa'}}+|y|\,\mathscr{L}(w)\right)\le c_2'.\end{cases}$$

We write

$$\left|\sum_{n\ge1}a_nv(T\log(n/x))-\frac{1}{2i\pi T}\int_T^{\xi T}\int_{\kappa-it}^{\kappa+it}F(z)\frac{x^z}{z}dzw(t/T)dt\right|x^{-\kappa}$$
$$\le c_2'\sum_{T|\log(x/n)|<\theta}\frac{|a_n|}{n^\kappa}+\frac{c_1+1}{T^{k+1}}\sum_{T|\log(x/n)|\ge\theta}\frac{|a_n|}{n^\kappa|\log(x/n)|^{k+1}}.$$

We continue via

$$\begin{aligned}\sum_{T|\log(x/n)|\ge\theta}\frac{|a_n|}{n^\kappa|\log(x/n)|^{k+1}}&=\sum_{T|\log(x/n)|\ge\theta}\frac{|a_n|}{n^\kappa}\int_{|\log(x/n)|}^\infty\frac{(k+1)du}{u^{k+2}}\\&=(k+1)\int_{\theta/T}^\infty\sum_{\theta/T\le|\log(x/n)|\le u}\frac{|a_n|}{n^\kappa}\frac{du}{u^{k+2}}\\&=(k+2)\int_{\theta/T}^\infty\sum_{|\log(x/n)|\le u}\frac{|a_n|}{n^\kappa}\frac{du}{u^{k+2}}-\frac{T^{k+1}}{\theta^{k+1}}\sum_{T|\log(x/n)|<\theta}\frac{|a_n|}{n^\kappa}\end{aligned}$$

and thus

$$\left|\sum_{n\ge1} a_n v(T\log(n/x)) - \frac{1}{2i\pi T}\int_T^{\xi T}\int_{\kappa-it}^{\kappa+it} F(z)\frac{x^z}{z}dz w(t/T)dt\right| x^{-\kappa}$$
$$\le \frac{c_1+1}{T^{k+1}}\int_{\theta/T}^{\infty}\sum_{|\log(x/n)|\le u}\frac{|a_n|}{n^\kappa}\frac{du}{u^{k+2}} + \Big(c_2' - \frac{c_1+1}{\theta^{k+1}}\Big)\sum_{T|\log(x/n)|<\theta}\frac{|a_n|}{n^\kappa}.$$

We define

$$c_3' = \max\bigg(c_2', \frac{c_1+1}{\theta^{k+1}}\bigg). \tag{3.5}$$

If $(c_1+1)/\theta^{k+1} < c_2'$, we may replace c_1 by the larger value $c_2'\theta^{k+1}-1$. This yields:

$$\left|\sum_{n\ge1} a_n v(T\log(n/x)) - \frac{1}{2i\pi T}\int_T^{\xi T}\int_{\kappa-it}^{\kappa+it} F(z)\frac{x^z}{z}dz w(t/T)dt\right| x^{-\kappa} \le \frac{c_3'}{(T/\theta)^{k+1}}\int_{\theta/T}^{\infty}\sum_{|\log(x/n)|\le u}\frac{|a_n|}{n^\kappa}\frac{du}{u^{k+2}}. \tag{3.6}$$

Proof of Theorem 3.1

We assume that $v(y) = Y(e^y)$ when $y \ge \theta$ for some $\theta \ge \mathrm{M}_{k,\xi}(w)^{-1/(k+1)}$ and thus $c_1 = 0$ is enough. We further assume that $0 \le v(y) \le 1$ in general, and since

$$\max_{u\ge0}\bigg(1 + \frac{\mathrm{N}_{k,\xi}(w)}{u^{k+1}}, 1 + u\,\mathscr{L}_\xi(w)\bigg) \le \mathrm{M}_{k,\xi}(w), \tag{3.7}$$

so we can chose $c_2' = \mathrm{M}_{k,\xi}(w)$. In this context $c_3' = c_2'$ (this is where the bound on θ is required). Finally, on setting $T' = T/\theta$, we get

$$\left|\sum_{n\ge1} a_n v(T\log(n/x)) - \frac{1}{2i\pi T}\int_T^{\xi T}\int_{\kappa-it}^{\kappa+it} F(z)\frac{x^z}{z}dz w(t/T)dt\right| \le M_{k,\xi}(w)\int_{1/T'}^{\infty}\sum_{n/|\log(x/n)|\le u}\frac{|a_n|}{n^\kappa}\frac{x^\kappa du}{T'^{(k+1)}u^{k+2}}$$

for *any* function v satisfying the above hypotheses.

4 An integral Gorny inequality for a restricted class

Proof of Theorem 1.4. We use the notation $\mathfrak{m}_h = \int_a^b |f_k^{(h)}(v)|^2 dv$. We only consider the case $0 \le h < k$. Repeated integrations by parts followed by Cauchy's inequality give the recursion

$$\mathfrak{m}_h \le \mathfrak{m}_{h+t}^{1/2}\,\mathfrak{m}_{h-t}^{1/2}, \quad (0 \le h-t \le h+t \le k)$$

from which we infer that

$$\mathfrak{m}_h \le \begin{cases} \mathfrak{m}_k^{1/2}\,\mathfrak{m}_{2h-k}^{1/2} & \text{when } h > k/2, \\ \mathfrak{m}_0^{1/2}\,\mathfrak{m}_{2h}^{1/2} & \text{when } h \le k/2. \end{cases} \tag{4.1}$$

We will use this rule recursively. Let us write

$$\frac{h}{k} = \sum_{i \ge 1} \frac{a_i}{2^i}$$

with $a_i \in \{0, 1\}$. Let $I \ge 1$ be some fixed index. We write

$$\sum_{1 \le i \le I} \frac{a_i}{2^i} = \frac{b_I}{2^I}.$$

We prove by recursion on $I \ge 1$ that

$$\mathfrak{m}_h \le \mathfrak{m}_k^{\frac{b_I}{2^I}}\,\mathfrak{m}_0^{1-\frac{1}{2^I}-\frac{b_I}{2^I}}\,\mathfrak{m}_{2^I h - b_I k}^{\frac{1}{2^I}} \tag{4.2}$$

We first notice that $b_I/2^I \le h/k$ hence $2^I h - b_I k \ge 0$, while $(b_I + 1)/2^I > h/k$ and thus $k > 2^I h - b_I k$. Let us first consider the case $I = 1$: when $b_1 = a_1 = 1$ or when $b_1 = a_1 = 0$, this is what we have just proved in (4.1). Let us now assume the formula proved for index I and let us consider index $I+1$. If $2^I h - b_I k > k/2$, then $a_{I+1} = 1$ and we can use the first rule in (4.1), getting

$$\mathfrak{m}_h \le \mathfrak{m}_k^{\frac{b_I}{2^I}+\frac{1}{2^{I+1}}}\,\mathfrak{m}_0^{1-\frac{1}{2^I}-\frac{b_I}{2^I}}\,\mathfrak{m}_{2(2^I h - b_I k)-k}^{\frac{1}{2^{I+1}}} = \mathfrak{m}_k^{\frac{b_{I+1}}{2^{I+1}}}\,\mathfrak{m}_0^{1-\frac{1}{2^I}-\frac{b_I}{2^I}}\,\mathfrak{m}_{2^{I+1}h - b_{I+1}k}^{\frac{1}{2^{I+1}}}$$

and we further notice that, in this case,

$$1 - \frac{1}{2^I} - \frac{b_I}{2^I} = 1 - \frac{1}{2^{I+1}} - \frac{1}{2^{I+1}} - \frac{b_I}{2^I} = 1 - \frac{1}{2^{I+1}} - \frac{b_{I+1}}{2^{I+1}}.$$

This concludes the proof in this case and the case $a_{I+1} = 0$ is similarly handled. Once (4.2) is established, we only need to let I go to infinity, since the values of $\mathfrak{m}_h$ are bounded (they belong to a finite set). The Theorem follows readily. $\square$

5 Proof of Theorem 1.2

We define

$$f_k(v) = \begin{cases} (v(1-v))^k & \text{when } v \in [0, 1], \\ 0 & \text{else} \end{cases} \tag{5.1}$$

and we select the (k,ξ)-admissible function $w_{k,\xi}$ defined by

$$w_{k,\xi}(u)=\frac{(2k+1)!}{k!^2(\xi-1)}f_k\Bigl(\frac{u-1}{\xi-1}\Bigr) \tag{5.2}$$

which indeed satisfies $\int_1^\xi w_{k,\xi}(u)du=1$. Here is a first corollary to Theorem 3.1, from which we will deduce Theorem 1.2.

Corollary 5.1. *Let $k\ge 1$ be an integer and let $\xi>1$ be a real number. Let $F(z)=\sum_n a_n/n^z$ be a Dirichlet series that converges absolutely for $\Re z>\kappa_a$, and let $\kappa>0$ be strictly larger than κ_a. For $x\ge1$ and $T\ge1$, we have*

$$\sum_{n\le x}a_n=\frac{1}{2i\pi}\int_T^{\xi T}\int_{\kappa-it}^{\kappa+it}F(z)\frac{x^z dz}{z}\frac{w_{k,\xi}(t/T)dt}{T}$$
$$+\mathcal{O}^*\left(\frac{7\xi}{10}\int_{1/T}^{\infty}\sum_{|\log(x/n)|\le u}\frac{|a_n|}{n^\kappa}\frac{(k+1)x^\kappa du}{T^{k+1}u^{k+2}}\exp\frac{2/e}{\xi-1}\right).$$

We start with a classical lemma, see for instance [2, (2.9)].

Lemma 5.2. *We have $n!=(2\pi n)^{1/2}(n/e)^n e^{\theta_+/(12n)}$ for $n\ge1$ and some $\theta_+\in\,]0,1[$.*

Proof of Corollary 5.1. Recall (5.1) and (5.2). We have

$$\int_1^\xi w_{k,\xi}(u)du=\frac{(2k+1)!}{k!^2}\int_0^1 f_k(v)dv=1$$

by the value of the Euler beta-function. This function is (k,ξ)-admissible and even better: we have $w_{k,\xi}^{(k-1)}(1)=w_{k,\xi}^{(k-1)}(\xi)=0$. We need to evaluate the L^1-norm of its h-th derivative, when $h\le k$, and we consider the L^2-norm instead. First note that

$$\int_1^\xi|w_{k,\xi}^{(h)}(u)|du=\frac{(2k+1)!}{k!^2(\xi-1)^h}\int_0^1|f_k^{(h)}(v)|dv.$$

We again put $\mathfrak{m}_h=\int_0^1|f_k^{(h)}(v)|^2dv$ and use Theorem 1.4. First note that k integrations by parts yields

$$\int_0^1|f_k^{(k)}(v)|^2dv=f_k^{(2k)}(1)\int_0^1 f_k(v)dv=\frac{(2k)!k!^2}{(2k+1)!} \tag{5.3}$$

and thus

$$\int_1^\xi|w_{k,\xi}^{(h)}(u)|du\le\frac{(2k+1)!}{k!^2(\xi-1)^h}((2k)!)^{\frac{h}{2k}}\sqrt{\frac{k!^2}{(2k+1)!}}=\frac{((2k)!)^{\frac{h}{2k}}}{(\xi-1)^h}\sqrt{\frac{(2k+1)!}{k!^2}}.$$

On using Lemma 5.2, we readily get that, for $k \geq 1$,

$$\int_1^\xi |w_{k,\xi}^{(h)}(u)|du \leq \frac{(4\pi k)^{1/4} e^{1/24}}{(\xi-1)^h} \left(\frac{2k}{e}\right)^{2k\frac{h}{2k}} \sqrt{\frac{(2k+1)\sqrt{4\pi k}(2k/e)^{2k} e^{1/24}}{2\pi k (k/e)^{2k}}},$$
$$\leq 3\sqrt{k}\left(\frac{2k}{e(\xi-1)}\right)^h.$$

As a consequence, we find that

$$2\pi\,\mathrm{N}_{k,\xi}(w_{k,\xi}) \leq 3\sqrt{k}\cdot k!\exp\frac{2k}{e(\xi-1)}$$

and, consequently,

$$\mathrm{M}_{k,\xi}(w_{k,\xi}) \leq 1 + \frac{\xi}{\pi}\left(\tfrac{3}{2}\sqrt{k}\cdot k!\exp\frac{2k}{e(\xi-1)}\right)^{1/(k+2)}.$$

We check with Pari/GP [12] and Lemma 5.2 above that this is not more than

$$1 + \frac{\xi}{\pi}\cdot 0.6\cdot(k+1)\exp\frac{2}{e(\xi-1)} \leq \frac{7\xi}{10}(k+1)\exp\frac{2}{e(\xi-1)}.$$

The proof of Corollary 5.1 is complete. □

A first step towards Theorem 1.2

Theorem 5.3. *Hypotheses and datas being the same as in Theorem 1.1, let further δ and $\epsilon > 0$ be two real parameters in $(0,1]$ and k be a positive integer. There exists a subset I^* of $[T,(1+\delta)T]$ of measure $\geq (1-\epsilon)\delta T$ such that for every $T^* \in I^*$, we have*

$$\sum_{n<x-\frac{x}{8T}} a_n + \sum_{|n-x|\leq\frac{x}{8T}} \theta_n a_n = \frac{1}{2i\pi}\int_{\kappa-iT^*}^{\kappa+iT^*} F(z)\frac{x^z dz}{z}$$
$$+\mathcal{O}\left(\frac{e^{1/\delta}x^\kappa}{T^k}\sum_{n\geq 1}\frac{|a_n|}{n^\kappa} + e^{\kappa+\delta^{-1}}\int_{1/T}^1 \sum_{|n-x|\leq 2ux}|a_n|\frac{k\,du}{T^k u^{k+1}}\right).$$

It may be worth mentionning that the choice of T^* depends on T and on k but that the constant implied in the $\mathcal{O}$-symbol does not.

Proof. We take $\xi = 1+\delta$ in Corollary 5.1. Note that k in Corollary 5.1 is $k+1$ in Theorem 5.3. Let us set

$$\mathfrak{R} = \frac{7\xi}{10}\int_{1/T}^\infty \sum_{|\log(x/n)|\leq u}\frac{|a_n|}{n^\kappa}\frac{(k+1)x^\kappa du}{T^{k+1}u^{k+2}}\exp\frac{2/e}{\xi-1}. \tag{5.4}$$

For any parameter $\epsilon > 0$, the set I of $t \in [T, \xi T]$ for which

$$\left|\sum_{n\leq x} a_n - \frac{1}{2i\pi}\int_{\kappa-it}^{\kappa+it} F(z)\frac{x^z dz}{z}\right| \geq \epsilon^{-1}(\xi-1)T\cdot\mathfrak{R}$$

verifies $|I| \leq \epsilon(\xi-1)T$.

We finally treat $\mathfrak{R}$ in a similar way as for Theorem 1.1, though we cannot use $T' = 2T$ but stick to the simpler choice $T' = T$. This forbids the simplification of the constants that arose from the final change of variable $2u \mapsto u$. Note that we split the integral at $u = 1$. The conclusion is easy. □

Proof of Theorem 1.2

A treatment of the error term slightly different from the one perforemd in the proof of Theorem 5.3 leads to Theorem 1.2. We split the integral at $u = 1/T^{(k-2)/k}$ with $k = [\log(3T)]$ when T is larger than 10 say. One readily checks that $T^{2/k} \leq 10$ and, on recalling the definition of $\mathfrak{R}$ in (5.4), we obtain

$$\mathfrak{R} \leq \frac{7e^{1/\delta}(1+\delta)x^\kappa}{10T^2}\sum_{n\geq 1}\frac{|a_n|}{n^\kappa} + \int_{1/T}^{10/T}\sum_{n/|\log(x/n)|\leq u}|a_n|\frac{7(1+\delta)e^{1/\delta}ke^\kappa du}{10T^k u^{k+1}}.$$

For the u's considered, we have

$$\sum_{n/|\log(x/n)|\leq u}|a_n| \leq \sum_{|n-x|\leq 20x/T}|a_n|.$$

When we are at this level, we see that this error term is anyway larger than

$$\sum_{|n-x|\leq x/(8T)}\theta_n a_n.$$

and larger than the same quantity with $20x/T$ instead of $x/(8T)$. This is also the reason why we can relax the condition $|\theta_n| \leq 1$ in "θ_n bounded". The Theorem 1.2 follows readily.

References

[1] H. Cartan. Sur les inégalités entre les maxima des dérivées successives d'une fonction. *C. R. Acad. Sci., Paris*, 208:414–416, 1939.

[2] H.G. Diamond and J. Steinig. An Elementary Proof of the Prime Number Theorem with a Remainder Term. *Inventiones math.*, 11:199–258, 1970.

[3] D. Goldston. On a result of Littlewood concerning prime numbers, ii. *Acta Arith.*, 43:49–51, 1983.

[4] A. Gorny. Contribution à l'étude des fonctions dérivables d'une variable réelle. *Acta Math., Uppsala*, 71:317–358, 1939. `http://www.numdam.org/item?id=THESE_1940__228__315_0`.

[5] J. Hadamard. Sur le module maximum d'une fonction et de ses dérivées. *Bull. Soc. Math. France*, 42:68–72, 1914. (Œuvres 1, 379-382).

[6] G.H. Hardy and J.E. Littlewood. Contributions to the Arithmetic Theory of Series. *Proc. London Math. Soc.*, S2-11(1):411, 1912.

[7] J. Kaczorowski and A. Perelli. A new form of the Riemann-van Mangold explicit formula. *Boll. Un. Mat. Ital. B (7)*, 10(1):51–66, 1996.

[8] A.N. Kolmogorov. Une généralisation de l'inégalité de M.J. Hadamard entre les bornes supérieures des dérivées. *C. R. Acad. Sci. Paris*, pages 764–765, 1938.

[9] E. Landau. Handbuch der Lehre von der Verteilung der Primzahlen. Erster Band. Leipzig u. Berlin: B. G. Teubner. X + 564 S (1909)., 1909.

[10] Jianya Liu and Yangbo Ye. Perron's formula and the prime number theorem for automorphic L-functions. *Pure Appl. Math. Q.*, 3(2, Special Issue: In honor of Leon Simon. Part 1):481–497, 2007.

[11] H.L. Montgomery and R.C. Vaughan. *Multiplicative Number Theory: I. Classical Theory*, volume 97 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, 2006.

[12] The PARI Group, Bordeaux. *PARI/GP, version* `2.5.2`, 2011. `http://pari.math.u-bordeaux.fr/`.

[13] A. Perelli and G. Puglisi. On some classical explicit formulae. *Bolletino U.M.I.*, 6, 4-A:269–278, 1985.

[14] O. Perron. Zur Theorie der *Dirichlet*schen Reihen. *J. Reine Angew. Math.*, 134:95–143, 1908.

[15] D.S. Ramana and O. Ramaré. An Exact Truncated Perron Formula. *In preparation*, 2014.

[16] O. Ramaré. Eigenvalues in the large sieve inequality. *Funct. Approximatio, Comment. Math.*, 37:7–35, 2007.

[17] E.C. Titchmarsh. *The Theory of Riemann Zeta Function.* Oxford Univ. Press, Oxford 1951, 1951.

[18] D. Wolke. On the explicit formula of Riemann-von Mangoldt, II. *J. London Math. Soc.*, 2(28), 1983.