Addendum to Modified truncated Perron formulae

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October 3, 2021

Abstract

We add a corollary to the results from [1].

Furthermore D. Johnston spotted an error and a typo. The typo is what is below (3.1) and corrected therein. The same typo exists in the last displayed equation in page 125 of [1]. The error is more serious: the statement has a set I^* of 'measure' verifying something. A closer look at the proof shows that this measure is taken with respect of a smooth multiple of the Lebesgue measure. When some parameters vary, this multiplier may have important consequences, see the present proof below (3.1) for a full description of the situation.

1 Introduction and results

Theorem 1.1 (The MT Perron summation formula, special form)**.** *Let (a_n) be a sequence of complex numbers such that $|a_n| \le 1$ and let $F(s) = \sum_{n\ge1} a_n/n^s$ be the corresponding Dirichlet series. We select real parameters $\kappa \in (1, 3/2]$, $\delta \in [0, 1/2]$, $\epsilon \in (0, \delta]$ and $x \ge T \ge 2$. There exists a subset I^* of $[T, (1+\delta)T]$ of measure*

$$\frac{|I^*|}{\delta T} \ge \max\left(\tfrac{1}{2}, \frac{\log\frac{\delta}{\epsilon}}{\frac{-4\log(\kappa-1)}{\log T} + 6}\right)$$

such that for every $T^* \in I^*$, we have

$$\sum_{n\le x} a_n = \frac{1}{2i\pi}\int_{\kappa-iT^*}^{\kappa+iT^*} F(z)\frac{x^z dz}{z} + \mathcal{O}^*\left(\frac{\delta}{\epsilon}\frac{2x}{T}e^{1/\delta}\left(23 + 2x^{\kappa-1}\right)\right).$$

Note 1: *It is better to multiply the error terms of [1, Theorem 1.2] and [1, Theorem 5.3] by the factor δ/ϵ. As such, the results are correct since the $\mathcal{O}$ may depend on ϵ and δ is bounded. It would be however clearer.*
Note 2: *A typical situation is $\delta = 1/2 = 2\epsilon$ and $\kappa = 1 + \frac{1}{\log x}$.*

2010 *Mathematics Subject Classification*: Primary , ; Secondary .
Key words and phrases:

2 Recall of previous result

We recall here part of [1, Section 5]. We define

$$f_k(v) = \begin{cases} (v(1-v))^k & \text{when } v \in [0,1], \\ 0 & \text{else} \end{cases} \tag{2.1}$$

and we further define $w_{k,\xi}$ by

$$w_{k,\xi}(u) = \frac{(2k+1)!}{k!^2(\xi-1)} f_k\Big(\frac{u-1}{\xi-1}\Big) \tag{2.2}$$

which satisfies $\int_1^\xi w_{k,\xi}(u)du = 1$.

Corollary 2.1 (ex-Corollary 5.1)**.** *Let $k \ge 1$ be an integer and let $\xi > 1$ be a real number. Let $F(z) = \sum_n a_n/n^z$ be a Dirichlet series that converges absolutely for $\Re z > \kappa_a$, and let $\kappa > 0$ be strictly larger than κ_a. For $x \ge 1$ and $T \ge 1$, we have*

$$\sum_{n \le x} a_n = \frac{1}{2i\pi} \int_T^{\xi T} \int_{\kappa - it}^{\kappa + it} F(z) \frac{x^z dz}{z} \frac{w_{k,\xi}(t/T)dt}{T}$$
$$+ \mathcal{O}^*\left(\frac{7\xi}{10} \int_{1/T}^{\infty} \sum_{|\log(x/n)| \le u} \frac{|a_n|}{n^\kappa} \frac{(k+1)x^\kappa du}{T^{k+1}u^{k+2}} \exp\frac{2/e}{\xi - 1} \right).$$

3 Proof of Theorem 1.1

Let us handle the remainder term in Corollary 2.1 under the assumption that $|a_n| \le 1$ and $\kappa \le 3/2$. We also set $\xi = 1 + \delta \le 3/2$. We split the integral in u at 1. We get

- When $u \ge U$, we use $\sum_{|\log(x/n)| \le u} \frac{|a_n|}{n^\kappa} \le \zeta(\kappa)$, whence the contribution is bounded above by

  $$x^\kappa \frac{7\xi}{10} \frac{\zeta(\kappa)}{T^{k+1}} \exp\frac{2/e}{\xi - 1} = \frac{7\xi}{10} \frac{x^\kappa \zeta(\kappa)}{T^{k+1}} \exp\frac{2/e}{\xi - 1} \le 1.6 \frac{x^\kappa}{T^{k+1}(\kappa - 1)} \exp\frac{1}{\delta}$$

  on recalling that $\zeta(\kappa) \le \kappa/(\kappa - 1)$.

- When $1/T \le u \le 1$, we use $\sum_{|\log(x/n)| \le u} \frac{|a_n|}{n^\kappa} \le x^{1-\kappa} e^{\kappa u}(e^u - e^{-u} + 1/x)$. Next some easy manipulation shows that $e^u - e^{-u} = 2\sinh u \le 2u \sinh 1 \le 2.36\, u$. Since we assumed that $T \le x$, we get

  $$\sum_{|\log(x/n)| \le u} \frac{|a_n|}{n^\kappa} \le 3.36\, x^{1-\kappa} e^\kappa u.$$

  The corresponding contribution is thus bounded above by

  $$3.36\, x e^\kappa \frac{7\xi}{10} \frac{k+1}{kT} \exp\frac{2/e}{\xi - 1} \le \frac{46\, x}{T} \exp\frac{1}{\delta}.$$

We thus get

$$\int_T^{\xi T}\left|\sum_{n\le x}a_n-\frac{1}{2i\pi}\int_T^{\xi T}\int_{\kappa-it}^{\kappa+it}F(z)\frac{x^z dz}{z}\right|\frac{w_{k,\xi}(t/T)dt}{T} \le \frac{46x}{T}e^{1/\delta}+\frac{2x^\kappa}{T^{k+1}(\kappa-1)}e^{1/\delta}.$$

Let us select $k=[-\log(\kappa-1)/\log T]+1$, so that $T^k(\kappa-1)\ge 1$. For such a k, we find that

$$\int_T^{\xi T}\left|\sum_{n\le x}a_n-\frac{1}{2i\pi}\int_T^{\xi T}\int_{\kappa-it}^{\kappa+it}F(z)\frac{x^z dz}{z}\right|\frac{w_{k,\xi}(t/T)dt}{T}\le\frac{2x}{T}e^{1/\delta}\big(23+2x^{\kappa-1}\big)=\mathfrak{R}.$$

For any parameter $\epsilon>0$, the set $I(x,k)$ of $t\in[T,\xi T]$ for which

$$\left|\sum_{n\le x}a_n-\frac{1}{2i\pi}\int_{\kappa-it}^{\kappa+it}F(z)\frac{x^z dz}{z}\right|\ge\epsilon^{-1}(\xi-1)\mathfrak{R} \tag{3.1}$$

verifies

$$\epsilon^{-1}(\xi-1)\mathfrak{R}\int_{I(x,k)}\frac{w_{k,\xi}(t/T)dt}{T}\le\int_{I(x,k)}\left|\sum_{n\le x}a_n-\frac{1}{2i\pi}\int_{\kappa-it}^{\kappa+it}F(z)\frac{x^z dz}{z}\right|\frac{w_{k,\xi}(t/T)dt}{T}$$
$$\le\int_T^{\xi T}\left|\sum_{n\le x}a_n-\frac{1}{2i\pi}\int_{\kappa-it}^{\kappa+it}F(z)\frac{x^z dz}{z}\right|\frac{w_{k,\xi}(t/T)dt}{T}\le\mathfrak{R}.$$

We now have to switch from the '$w_{k,\xi}$-measure' of $I(k,x)$ to its usual Lebesgue measure, which we denote by $2(\xi-1)\nu\cdot T$ and which we assume to be less than $(1-\xi)T=\delta T$. The shape of $w_{k,\xi}$ implies that, when $\ell\in\{0,\cdots,k-1\}$, we have

$$2\int_T^{(1+(1-\xi)\nu)T}\frac{w_{k,\xi}(t/T)dt}{T}\le\frac{\epsilon}{\xi-1},$$

i.e., on recalling the definition (2.2), that

$$\frac{(2k+1)!}{k!^2}\int_0^\nu f_k(v)dv\le\frac{\epsilon}{2(\xi-1)}.$$

We readily check, by repeated integration by parts, that

$$\begin{aligned}\int_0^\nu f_k(v)dv&=\int_0^\nu v^k(1-v)^k dv\\&=\frac{\nu^{k+1}(1-\nu)^k}{k+1}+\frac{k\nu^{k+2}(1-\nu)^{k-1}}{(k+1)(k+2)}+\cdots+\frac{k\cdots(k-\ell+1)\nu^{k+\ell+1}(1-\nu)^{k-\ell}}{(k+1)\cdots(k+\ell+1)}\\&\quad+\frac{k\cdots(k-\ell)}{(k+1)\cdots(k+\ell+1)}\int_0^\nu v^{k+\ell+1}(1-v)^{k-\ell-1}dv\end{aligned}$$

from which we infer, with the choice $\ell = k-1$ and on setting $y = \nu/(1-\nu)$, that

$$\begin{aligned}\int_0^\nu f_k(v)dv &= \nu^{k+1}(1-\nu)^k\left(\frac{1}{k+1} + \frac{ky}{(k+1)(k+2)} + \cdots + \frac{k\cdots 1 y^{k+1}}{(k+1)\cdots(2k+1)}\right)\\ &= \nu^{k+1}(1-\nu)^k\sum_{\ell=0}^{k}\frac{k!^2}{(k-\ell)!(k+1+\ell)!}y^\ell.\end{aligned}$$

When $\nu = 1/2$, we get $y = 1$ and this formula yields

$$\frac{1}{2}\frac{k!^2}{(2k+1)!} = \frac{1}{2^{2k+1}}\sum_{\ell=0}^{k}\frac{k!^2}{(k-\ell)!(k+1+\ell)!}.$$

Hence

$$\int_0^\nu f_k(v)dv \geq \nu^{k+1}(1-\nu)^k 2^{2k}\frac{(2k+1)!}{k!^2}y^k = \frac{1}{2}(2\nu)^{2k+1}\frac{(2k+1)!}{k!^2}.$$

We thus infer from this argument that

$$(2\nu)^{2k+1} \leq \frac{\epsilon}{\xi - 1}.$$

Recall that we have assumed that $\epsilon \leq \xi - 1 = \delta$. The above inequality gives us

$$2\nu \leq \exp\frac{-\log\frac{\delta}{\epsilon}}{2k+1}.$$

Because the conclusion of our Theorem reads '$|I^*|/(\delta T) \geq \max(\frac{1}{2}, \cdots)$', we may assume that

$$\log\frac{\delta}{\epsilon} \leq \frac{-2\log(\kappa-1)}{\log T} + 3.$$

This enables us to use the elementary inequality $e^{-u} \leq 1 - u/2$ when $u \in [0,1]$ and to get

$$1 - 2u \geq \frac{\log\frac{\delta}{\epsilon}}{\frac{-4\log(\kappa-1)}{\log T} + 6}.$$

Our Theorem follows readily from this last inequality.

References

[1] O. Ramaré. Modified truncated Perron formulae. *Ann. Blaise Pascal*, 23(1):109–128, 2016.