Addendum to Modified truncated Perron formulae

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Abstract

We add a corollary to the results from [1]. Furthermore D. Johnston spotted an error and a typo. The typo is what is below (3.1) and corrected therein. The same typo exists in the last displayed equation in page 125 of [1]. The error is more serious: the statement has a set $I^*$ of 'measure' verifying something. A closer look at the proof shows that this measure is taken with respect of a smooth multiple of the Lebesgue measure. When some parameters vary, this multiplier may have important consequences, see the present proof below (3.1) for a full description of the situation.

1 Introduction and results

Theorem 1.1 (The MT Perron summation formula, special form). Let $(a_n)$ be a sequence of complex numbers such that $|a_n| \leq 1$ and let $F(s) = \sum_{n \geq 1} a_n/n^s$ be the corresponding Dirichlet series. We select real parameters $\kappa \in (1,3/2]$, $\delta \in [0,1/2]$, $\epsilon \in (0,\delta]$ and $x \geq T \geq 2$. There exists a subset $I^*$ of $[T, (1 + \delta)T]$ of measure

$$\frac{|I^*|}{\delta T} \geq \max \left( \frac{1}{2}, \frac{\log \frac{\delta}{\epsilon}}{-4 \log(\kappa - 1)} + 6 \right)$$

such that for every $T^* \in I^*$, we have

$$\sum_{n \leq x} a_n = \frac{1}{2\pi i} \int_{\kappa-iT^*}^{\kappa+iT^*} F(z) \frac{x^z}{z} \, dz + \mathcal{O}^* \left( \frac{\delta 2x}{\epsilon T} e^{1/\delta} (23 + 2x^{\kappa-1}) \right).$$

Note 1: It is better to multiply the error terms of [1, Theorem 1.2] and [1, Theorem 5.3] by the factor $\delta/\epsilon$. As such, the results are correct since the $\mathcal{O}$ may depend on $\epsilon$ and $\delta$ is bounded. It would be however clearer.

Note 2: A typical situation is $\delta = 1/2 = 2\epsilon$ and $\kappa = 1 + \frac{1}{\log 2}$.

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2 Recall of previous result

We recall here part of [1, Section 5]. We define

\[ f_k(v) = \begin{cases} (v(1-v))^k & \text{when } v \in [0, 1], \\ 0 & \text{else} \end{cases} \]

and we further define \( w_{k, \xi} \) by

\[ w_{k, \xi}(u) = \frac{(2k + 1)!}{k! \xi^{k+1}} f_k \left( \frac{u}{\xi - 1} \right) \]

which satisfies \( \int_1^\xi w_{k, \xi}(u) du = 1 \).

**Corollary 2.1** (ex-Corollary 5.1). Let \( k \geq 1 \) be an integer and let \( \xi > 1 \) be a real number. Let \( F(z) = \sum_n a_n/n^2 \) be a Dirichlet series that converges absolutely for \( \Re z > \kappa_a \), and let \( \kappa > 0 \) be strictly larger than \( \kappa_a \). For \( x \geq 1 \) and \( T \geq 1 \), we have

\[ \sum_{n \leq x} a_n = \frac{1}{2\pi i} \int_T^{\xi T} \int_{\kappa - it}^{\kappa + it} F(z) \frac{x^z}{z} w_{k, \xi}(t/T) dt dz + O^* \left( \frac{7 \xi}{10} \sum_{\log(x/n) \leq u} \frac{|a_n|}{n^\kappa} \frac{(k + 1)x^u}{T^{k+1}u^{k+2}} \exp \frac{2/e}{\xi - 1} \right). \]

3 Proof of Theorem 1.1

Let us handle the remainder term in Corollary 2.1 under the assumption that \( |a_n| \leq 1 \) and \( \kappa \leq 3/2 \). We also set \( \xi = 1 + \delta \leq 3/2 \). We split the integral in \( u \) at 1. We get

- When \( u \geq U \), we use \( \sum_{\log(x/n) \leq u} \frac{|a_n|}{n^\kappa} \leq (\kappa) \), whence the contribution is bounded above by
  \[ x^\kappa \frac{7 \xi}{10} \frac{\zeta(\kappa)}{T^{k+1}} \exp \frac{2/e}{\xi - 1} \leq 1.6 \frac{x^\kappa}{T^{k+1}(\kappa - 1)} \exp \frac{1}{\delta} \]
  on recalling that \( \zeta(\kappa) \leq \kappa/|\kappa - 1| \).

- When \( 1/T \leq u \leq 1 \), we use \( \sum_{\log(x/n) \leq u} \frac{|a_n|}{n^\kappa} \leq x^{1-\kappa} e^{\kappa u} (e^u - e^{-u} + 1/x) \).

Next some easy manipulation shows that \( e^u - e^{-u} = 2 \sinh u \leq 2u \sinh 1 \leq 2.36 u \). Since we assumed that \( T \leq x \), we get

\[ \sum_{\log(x/n) \leq u} \frac{|a_n|}{n^\kappa} \leq 3.36 x^{1-\kappa} e^{\kappa u}. \]

The corresponding contribution is thus bounded above by

\[ 3.36 x^\kappa \frac{7 \xi}{10} \frac{k + 1}{kT} \exp \frac{2/e}{\xi - 1} \leq \frac{46 x}{T} \exp \frac{1}{\delta} . \]
We thus get
\[
\int_T^{\xi T} \left| \sum_{n \leq x} a_n - \frac{1}{2i\pi} \int_T^{\xi T} \int_{-iT}^{iT} F(z) \frac{x^2 dz}{z} \right| \frac{w_{k,\xi}(t/T) dt}{T} \leq \frac{46 \pi}{T} e^{1/\delta} + \frac{2\kappa \pi}{T^{k+1}(\kappa - 1)} e^{1/\delta}.
\]

Let us select \(k = \lceil -\log(\kappa - 1)/\log T \rceil + 1\), so that \(T^k(\kappa - 1) \geq 1\). For such a \(k\), we find that
\[
\int_T^{\xi T} \left| \sum_{n \leq x} a_n - \frac{1}{2i\pi} \int_T^{\xi T} \int_{-iT}^{iT} F(z) \frac{x^2 dz}{z} \right| \frac{w_{k,\xi}(t/T) dt}{T} \leq \frac{2\pi}{T} e^{1/\delta} (23 + 2\kappa) = \mathfrak{R}.
\]

For any parameter \(\epsilon > 0\), the set \(I(x,k)\) of \(t \in [T,\xi T]\) for which
\[
(3.1) \quad \left| \sum_{n \leq x} a_n - \frac{1}{2i\pi} \int_T^{\xi T} \int_{-iT}^{iT} F(z) \frac{x^2 dz}{z} \right| \geq \epsilon^{-1}(\xi - 1) \mathfrak{R}
\]

verifies
\[
\epsilon^{-1}(\xi - 1) \mathfrak{R} \int_{I(x,k)} \frac{w_{k,\xi}(t/T) dt}{T} \leq \int_{I(x,k)} \left| \sum_{n \leq x} a_n - \frac{1}{2i\pi} \int_T^{\xi T} \int_{-iT}^{iT} F(z) \frac{x^2 dz}{z} \right| \frac{w_{k,\xi}(t/T) dt}{T} \leq \int_T^{\xi T} \left| \sum_{n \leq x} a_n - \frac{1}{2i\pi} \int_T^{\xi T} \int_{-iT}^{iT} F(z) \frac{x^2 dz}{z} \right| \frac{w_{k,\xi}(t/T) dt}{T} \leq \mathfrak{R}.
\]

We now have to switch from the ‘\(w_{k,\xi}\)-measure’ of \(I(k, x)\) to its usual Lebesgue measure, which we denote by \(2(\xi - 1)\nu \cdot T\) and which we assume to be less than \((1 - \xi)T = \delta T\). The shape of \(w_{k,\xi}\) implies that, when \(\ell \in \{0, \cdots, k - 1\}\), we have
\[
2 \int_T^{(1+(1-\xi)\nu)T} \frac{w_{k,\xi}(t/T) dt}{T} \leq \frac{\epsilon}{\xi - 1},
\]
i.e., on recalling the definition (2.2), that
\[
\frac{(2k + 1)!}{k!^2} \int_0^{\nu} f_k(v) dv \leq \frac{\epsilon}{2(\xi - 1)}.
\]

We readily check, by repeated integration by parts, that
\[
\int_0^{\nu} f_k(v) dv = \int_0^{\nu} v^k(1 - v)^k dv = \frac{\nu^{k+1}(1 - \nu)^k}{k + 1} + \frac{k \nu^{k+1}(1 - \nu)^{k-1}}{(k + 1)(k + 2)} + \cdots + \frac{k \cdots (k - \ell + 1) \nu^{k+\ell+1}(1 - \nu)^{k-\ell}}{(k + 1) \cdots (k + \ell + 1)}
\]

\[
+ \frac{k \cdots (k - \ell)}{(k + 1) \cdots (k + \ell + 1)} \int_0^{\nu} v^{k+\ell+1}(1 - v)^{k-\ell-1} dv.
\]
from which we infer, with the choice \( \ell = k - 1 \) and on setting \( y = \nu/(1 - \nu) \), that

\[
\int_0^\nu f_k(v)dv = \nu^{k+1}(1 - \nu)^k \left( \frac{1}{k+1} + \frac{ky}{(k+1)(k+2)} + \cdots + \frac{k \cdots 1 y^{k+1}}{(k+1) \cdots (2k+1)} \right)
\]

\[
= \nu^{k+1}(1 - \nu)^k \sum_{\ell=0}^k \frac{k!^2}{(k-\ell)!(k+1+\ell)!} y^\ell.
\]

When \( \nu = 1/2 \), we get \( y = 1 \) and this formula yields

\[
\frac{1}{2} \frac{k!^2}{(2k+1)!} = \frac{1}{2^{2k+1}} \sum_{\ell=0}^k \frac{k!^2}{(k-\ell)!(k+1+\ell)!}.
\]

Hence

\[
\int_0^\nu f_k(v)dv \geq \nu^{k+1}(1 - \nu)^k 2^{2k} \frac{(2k+1)!}{k!^2} y^k = \frac{1}{2} \nu^{2k+1} \frac{(2k+1)!}{k!^2}.
\]

We thus infer from this argument that

\[
(2\nu)^{2k+1} \leq \frac{\epsilon}{\xi - 1}.
\]

Recall that we have assumed that \( \epsilon \leq \xi - 1 = \delta \). The above inequality gives us

\[
2\nu \leq \exp \left( -\frac{\log \frac{\epsilon}{\xi}}{2k + 1} \right).
\]

Because the conclusion of our Theorem reads \(|I^*|/(\delta T) \geq \max(\frac{1}{2}, \cdots)\), we may assume that

\[
\log \frac{\delta}{\epsilon} \leq \frac{-2 \log(\kappa - 1)}{\log T} + 3.
\]

This enables us to use the elementary inequality \( e^{-u} \leq 1 - u/2 \) when \( u \in [0, 1] \) and to get

\[
1 - 2u \geq \frac{\log \frac{\delta}{\epsilon}}{-4 \log(\kappa - 1) + \log T} + 6.
\]

Our Theorem follows readily from this last inequality.

References