Addendum to Modified truncated Perron formulae

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Abstract

We add a corollary to the results from [1].
File `TruncatedPerron-5-Addendum-01.tex`.

1 Introduction and results

Theorem 1.1 (The MT Perron summation formula, special form). *Let (a_n) be a sequence of complex numbers such that $|a_n| \le 1$ and let $F(s) = \sum_{n\ge1} a_n/n^s$ be the corresponding Dirichlet series. We select real paramaters $\kappa \in (1, 3/2]$, $\delta \in [0, 1/2]$, $\epsilon > 0$ and $x \ge T \ge 2$. There exists a subset I^* of $[T, (1+\delta)T]$ of measure $\ge (1-\epsilon)\delta T$ such that for every $T^* \in I^*$, we have*

$$\sum_{n\le x} a_n = \frac{1}{2i\pi}\int_{\kappa-iT^*}^{\kappa+iT^*} F(z)\frac{x^z dz}{z} + \mathcal{O}^*\biggl(\frac{\delta}{\epsilon}\frac{2x}{T}e^{1/\delta}\bigl(23 + 2x^{\kappa-1}\bigr)\biggr).$$

Note: *It is better to multiply the error terms of [1, Theorem 1.2] and [1, Theorem 5.3] by the factor δ/ϵ. As such, the results are correct since the $\mathcal{O}$ may depend on ϵ and δ is bounded. It would be however clearer.*

2 Recall of previous result

We recall here part of [1, Section 5]. We define

$$f_k(v) = \begin{cases} (v(1-v))^k & \text{when } v \in [0,1], \\ 0 & \text{else} \end{cases} \tag{2.1}$$

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and we further define $w_{k,\xi}$ by

$$w_{k,\xi}(u) = \frac{(2k+1)!}{k!^2(\xi-1)} f_k\Big(\frac{u-1}{\xi-1}\Big) \tag{2.2}$$

which satisfies $\int_1^\xi w_{k,\xi}(u)du = 1$.

Corollary 2.1 (ex-Corollary 5.1)**.** *Let $k \geq 1$ be an integer and let $\xi > 1$ be a real number. Let $F(z) = \sum_n a_n/n^z$ be a Dirichlet series that converges absolutely for $\Re z > \kappa_a$, and let $\kappa > 0$ be strictly larger than κ_a. For $x \geq 1$ and $T \geq 1$, we have*

$$\sum_{n \leq x} a_n = \frac{1}{2i\pi} \int_T^{\xi T} \int_{\kappa - it}^{\kappa + it} F(z) \frac{x^z dz}{z} \frac{w_{k,\xi}(t/T)dt}{T}$$
$$+ \mathcal{O}^*\left(\frac{7\xi}{10} \int_{1/T}^{\infty} \sum_{|\log(x/n)| \leq u} \frac{|a_n|}{n^\kappa} \frac{(k+1)x^\kappa du}{T^{k+1}u^{k+2}} \exp \frac{2/e}{\xi - 1} \right).$$

3 Proof of Theorem 1.1

Let us handle the remainder term in Corollary 2.1 under the assumption that $|a_n| \leq 1$ and $\kappa \leq 3/2$. We also set $\xi = 1 + \delta \leq 3/2$. We split the integral in u at 1. We get

- When $u \geq U$, we use $\sum_{|\log(x/n)| \leq u} \frac{|a_n|}{n^\kappa} \leq \zeta(\kappa)$, whence the contribution is bounded above by

$$x^\kappa \frac{7\xi}{10} \frac{\zeta(\kappa)}{T^{k+1}} \exp \frac{2/e}{\xi - 1} = \frac{7\xi}{10} \frac{x^\kappa \zeta(\kappa)}{T^{k+1}} \exp \frac{2/e}{\xi - 1} \leq 1.6 \frac{x^\kappa}{T^{k+1}(\kappa - 1)} \exp \frac{1}{\delta}$$

  on recalling that $\zeta(\kappa) \leq \kappa/(\kappa - 1)$.

- When $1/T \leq u \leq 1$, we use $\sum_{|\log(x/n)| \leq u} \frac{|a_n|}{n^\kappa} \leq x^{1-\kappa} e^{\kappa u}(e^u - e^{-u} + 1/x)$. Next some easy manipulation shows that $e^u - e^{-u} = 2\sinh u \leq 2u \sinh 1 \leq 2.36\, u$. Since we assumed that $T \leq x$, we get

$$\sum_{|\log(x/n)| \leq u} \frac{|a_n|}{n^\kappa} \leq 3.36\, x^{1-\kappa} e^\kappa u.$$

  The corresponding contribution is thus bounded above by

$$3.36\, x e^\kappa \frac{7\xi}{10} \frac{k+1}{kT} \exp \frac{2/e}{\xi - 1} \leq \frac{46\, x}{T} \exp \frac{1}{\delta}.$$

We thus get

$$\int_T^{\xi T} \left| \sum_{n\le x} a_n - \frac{1}{2i\pi}\int_T^{\xi T}\int_{\kappa-it}^{\kappa+it} F(z)\frac{x^z dz}{z} \right| \frac{w_{k,\xi}(t/T)dt}{T} \le \frac{46x}{T}e^{1/\delta} + \frac{2x^\kappa}{T^{k+1}(\kappa-1)}e^{1/\delta}.$$

Let us select $k = [-\log(\kappa-1)/\log T]+1$, so that $T^k(\kappa-1)\ge 1$. We call $\mathfrak{R}$ the right-hand side of this inequality. For such a k, we find that

$$\int_T^{\xi T} \left| \sum_{n\le x} a_n - \frac{1}{2i\pi}\int_T^{\xi T}\int_{\kappa-it}^{\kappa+it} F(z)\frac{x^z dz}{z} \right| \frac{w_{k,\xi}(t/T)dt}{T} \le \frac{2x}{T}e^{1/\delta}\bigl(23+2x^{\kappa-1}\bigr).$$

For any parameter $\epsilon>0$, the set $I(x,k)$ of $t\in[T,\xi T]$ for which

$$\left| \sum_{n\le x} a_n - \frac{1}{2i\pi}\int_{\kappa-it}^{\kappa+it} F(z)\frac{x^z dz}{z} \right| \ge \epsilon^{-1}(\xi-1)T\cdot\mathfrak{R}$$

verifies $|I|\le\epsilon(\xi-1)T$.

References

[1] O. Ramaré. Modified truncated Perron formulae. *Ann. Blaise Pascal*, 23(1):109–128, 2016.