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NOTES ON RESTRICTION THEORY IN THE PRIMES

BY

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ABSTRACT

We study the mean $\sum_{x\in\mathcal{X}}\left|\sum_{p\le N} u_p e(xp)\right|^\ell$ when ℓ covers the full range $[2,\infty)$ and $\mathcal{X}\subset\mathbb{R}/\mathbb{Z}$ is a well-spaced set, providing a smooth transition from the case $\ell=2$ to the case $\ell>2$ and improving on the results of J. Bourgain and of B. Green and T. Tao. A uniform Hardy-Littlewood property for the set of primes is established as well as a sharp upper bound for $\sum_{x\in\mathcal{X}}\left|\sum_{p\le N} u_p e(xp)\right|^\ell$ when $\mathcal{X}$ is small. These results are extended to primes in *any* interval in a last section, provided the primes are numerous enough therein.

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1. Introduction and some results

Some historical background. During the proof of Theorem 3 of [2] and by using specific properties of the primes, J. Bourgain established (in Equation (4.39) therein) the estimate

$$\forall \ell > 2, \quad \biggl(\int_0^1 \biggl|\sum_{p\le N} u_p e(p\alpha)\biggr|^\ell d\alpha\biggr)^{1/\ell} \ll_\ell N^{-1/\ell}\biggl(\frac{N}{\log N}\sum_{p\le N}|u_p|^2\biggr)^{1/2}. \tag{1}$$

This proof was improved by B. Green in [5, Theorem 1.5] and in the paper [3], B. Green & T. Tao reduced the proof to using only sieve properties, enabling a wild generalization. A striking feature of (1) is that it is *not* valid when $\ell = 2$ as Parseval formula easily shows. Understanding the transition became then an open question, an answer to which is provided in Corollary 1.3 below.

Let us mention that, before the work of B.J. Green, it was customary in prime number theory to restrict our attention to the case $\ell = 2$, while Green used $\ell = 5/2$. This proved to be very valuable in applications.

The heart of the matter.

Theorem 1.1: *Let $\mathcal{X}$ be a δ-well spaced subset of $\mathbb{R}/\mathbb{Z}$ and $N \ge 1000$. Let $(u_p)_{p\le N}$ be a sequence of complex numbers. We have*

$$\sum_{x\in\mathcal{X}}\biggl|\sum_{p\le N} u_p e(xp)\biggr|^2 \le 280\frac{N+\delta^{-1}}{\log N}\log(2|\mathcal{X}|)\sum_{p\le N}|u_p|^2.$$

Let us recall that a set $\mathcal{X}\subset\mathbb{R}/\mathbb{Z}$ is said to be δ-well spaced when $\min_{x\neq x'\in\mathcal{X}}|x-x'|_{\mathbb{Z}}\ge\delta$, where $|y|_{\mathbb{Z}} = \min_{k\in\mathbb{Z}}|y-k|$ denotes in a rather unusual manner the distance to the nearest integer. In most applications, δ^{-1} is smaller than N.

B.J. Green & T. Tao's result in [3] relates to a similar inequality though with a larger dependence in $|\mathcal{X}|$ that the $\log(2|\mathcal{X}|)$ we have here. We shall prove this inequality in dual form in Theorem 5.1.

Though Theorem 1.1 is an L^2-estimate, a fundamental *maximal* character is hidden in the fact that the set $\mathcal{X}$ may be chosen freely.

Sieves and transference principle. The main ingredient to prove Theorem 1.1 is the large sieve inequality coupled with an *enveloping sieve*; our novelty with respect to [12] is to incorporate a preliminary *unsieving* into this sieving process. We shall spend some time to describe properly this enveloping sieve.

In some sense, *sieving*, and this is all the more true in the context of the large sieve, relies on describing a sequence through congruence properties. As a consequence, properties of arithmetical progressions may well be shared by sequences properly described by sieves. The terminology *transference principle* refers here to this idea. It leads in particular to some majorant properties as shown below.

Analytical usage. The maximal character of Theorem 1.1 may be used to control the size of level sets, and this is the path we follow in this subsection.

THEOREM 1.2: *Let $\mathcal{X}$ be a δ-well spaced subset of $\mathbb{R}/\mathbb{Z}$. Assume $N \geq 1000$ and let $h > 0$. We have*

$$\sum_{x\in\mathcal{X}}\biggl|\sum_{p\le N} u_p e(xp)\biggr|^{2+h} \le 7000\bigl((1+\tfrac{3}{2\log N})^h + 1/h\bigr)\biggl(\frac{N+\delta^{-1}}{\log N}\sum_{p\le N}|u_p|^2\biggr)^{1+h/2}.$$

On taking $\mathcal{X} = \{\beta + k/N, 0 \le k \le N-1\}$ and integrating over β in $[0, 1/N]$, we get the corollary we advertised above.

COROLLARY 1.3: *Assume $N \geq 1000$ and let $h > 0$. We have*

$$\int_0^1\biggl|\sum_{p\le N} u_p e(p\alpha)\biggr|^{2+h} d\alpha \le 7000\frac{(1+\frac{3}{2\log N})^h + 1/h}{N}\biggl(\frac{2N}{\log N}\sum_{p\le N}|u_p|^2\biggr)^{1+h/2}.$$

This result offers an optimal (save for the constant) transition to the case $h = 0$. Indeed, on selecting $h = 1/\log N$, this corollary implies that, when $|u_p| \le 1$, we have the best possible

$$\int_0^1\biggl|\sum_{p\le N} u_p e(p\alpha)\biggr|^{2+h} d\alpha \ll \sum_{p\le N}|u_p|^2.$$

A majorant property. In the same line, but maybe more strikingly, our result implies a *uniform* Hardy-Littlewood majorant property, in the sense of the paper [4] of B. Green & I. Ruzsa.

THEOREM 1.4: *Assume $N \geq 10^6$ and let $\ell \geq 2$. We have*

$$\biggl(\int_0^1\biggl|\sum_{p\le N} u_p e(p\alpha)\biggr|^{\ell} d\alpha\biggr)^{1/\ell} \le 10^5\biggl(\int_0^1\biggl|\sum_{p\le N} e(p\alpha)\biggr|^{\ell} d\alpha\biggr)^{1/\ell}$$

as soon as $\sum_{p\le N}|u_p|^2 \le \sum_{p\le N} 1$.

In other words, the constant $C(\ell)$ in Theorem 1.5 of [5] is *uniformly* bounded, and in fact by 10^5. Guessing and getting the optimal constant is open, whether under the L^∞-condition $|u_p| \le 1$ or under the L^2-condition we use.

Arithmetical usage. To better compare Theorem 1.1 with earlier results and to underline its maximal character, let us recall Theorem 5.3 of [13]: when $Q_0 \le \sqrt{N}$, we have

$$\sum_{q\le Q_0}\sum_{a\,mod^*\,q}\Bigl|\sum_n u_n e(na/q)\Bigr|^2 \le 7\frac{N\log Q_0}{\log N}\sum_n |u_n|^2 \tag{2}$$

valid provided u_n vanishes when n has a prime factor less than $\sqrt{N}$. Here is the estimate we can now get.

Corollary 1.5: *Let $N \ge 1000$ and $Q_0 \in [2, \sqrt{N}]$. Let $(u_p)_{p\le N}$ be a sequence of complex numbers. We have*

$$\sum_{q\le Q_0}\sum_{a\,mod^*\,q}\max_{|\alpha-\frac{a}{q}|\le\frac{1}{qQ_0}}\Bigl|\sum_{p\le N} u_p e(\alpha p)\Bigr|^2 \le 1200\frac{N\log Q_0}{\log N}\sum_{p\le N}|u_p|^2.$$

Extensions. Rather than restricting their attention to prime numbers, B. Green & T. Tao considered a more general setting that could be encompassed in the framework of *sufficiently sifted sequences* of [14], and the same holds for our estimates. Indeed the methods used (essentially the large sieve inequality and enveloping sequences) remain general enough to warrant such an extension. We simply present an obvious generalization to the case of primes in some interval in the last section. The somewhat reverse situation of smooth numbers has been the subject of the work [7] by A.J. Harper, to which we borrow an idea (see around Equation (16) below).

Explicit values for the constants? Explicit values for the constants are provided for three reasons: it avoids us saying that these are independant of the involved parameters; it puts forward that our argument is elementary enough; and finally, it shows that some work is still required to improve on them and to determine the optimal ones. We did not work overmuch on these constants.

Notation. As this paper may be of interest to audiences having different background, let us review the notation we use throughout this paper.

- The number of prime factors of the ℓ is denoted by $\omega(\ell)$.

- The Möbius fuction is denoted by μ, so that $\mu(\ell)$ vanishes when ℓ is divisible by the square of a prime, and otherwise takes the value $(-1)^{\omega(\ell)}$. In particular, we have $\mu(1) = 1$.
- The gcd of the two integers a and b is often denoted by (a, b) while their lcm is denoted by $[a, b]$.
- The value of the Euler φ-function at the positive integer ℓ is the number of integers in $\{1, \cdots, \ell\}$ that are prime to ℓ. In particular $\varphi(1) = 1$.
- Notation "$a \ mod^* \ q$" denotes a variable a that ranges through the invertible (also called *reduced*) residue classes modulo q.
- Summations are usually over positive integers when the summation variable is not denoted by p, in which case the variable p runs through the primes. The stated conditions apply. In case more clarification seems necessary, we shall for instance write $\sum_{\substack{n:n\le N,\\ n|d}}$ to denote a sum over the positive integers $n \le N$ that divide the parameter d.
- The Ramanujan sum $c_q(n)$ is defined by

$$c_q(n) = \sum_{a\, mod^*\, q} e(na/q), \quad \Big(e(\alpha) = \exp(2i\pi\alpha)\Big) \tag{3}$$

and where $a \mod {}^*q$ is a shortcut for "$1 \le a \le q$ and $(a, q) = 1$". This quantity can also be computed via the von Sterneck expression

$$c_q(n) = \mu\biggl(\frac{q}{(q,n)}\biggr)\frac{\varphi(q)}{\varphi\Bigl(\frac{q}{(q,n)}\Bigr)}. \tag{4}$$

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2. Handling the G-functions

We define $P(z_0) = \prod_{p<z_0} p$ and

$$G_d(y; z_0) = \sum_{\substack{\ell\le y,\\ (\ell,dP(z_0))=1}} \frac{\mu^2(\ell)}{\varphi(\ell)}, \quad G(y; z_0) = G_1(y; z_0). \tag{5}$$

When $z_0 = 2$, these functions are classical in sieve theory (see for instance Equation (1.3) of [6, Chapter 3] by H. Halberstam and H.E. Richert)and we shall in fact reduce the analysis to this case. So we use the specific notation:

$$G(y) = G(y;2) = \sum_{\ell \le y} \frac{\mu^2(\ell)}{\varphi(\ell)}. \tag{6}$$

Here are the six lemmas we will combine for their evaluations.

Lemma 2.1: *We have* $G(yd; z_0) \ge \dfrac{d}{\varphi(d)} G_d(y; z_0) \ge G(y; z_0)$

Lemma 2.2: *We have* $\displaystyle\prod_{p<z_0} \frac{p}{p-1} G(z; z_0) \ge G(z)$.

Lemma 2.3: *When* $h \ge 0$, *we have* $\displaystyle\sum_{\ell \le y} \frac{\mu^2(\ell)}{\varphi(\ell)^{1+h}} \ge \sum_{q \le y} \frac{1}{q^{1+h}}$.

Lemma 2.4: *When* $z \ge 1$, *we have* $G(z) \ge \log z$.

Lemma 2.5: *When* $z_0 \ge 2$, *we have* $\displaystyle\prod_{p<z_0} \frac{p-1}{p} \ge \frac{e^{-\gamma}}{\log(2z_0)}$.

Lemma 2.6: *When* $z_0 \ge 2$, *we have* $G(z; z_0) \ge e^{-\gamma} \dfrac{\log z}{\log(2z_0)}$.

Proof of Lemma 2.1. This inequality has its origin in [18, Eq. (1.3)] by J. van Lint and H.E. Richert, but the argument is so simple that we reproduce it. We write

$$G(y; z_0) = \sum_{\delta | d} \sum_{\substack{\ell \le y, \\ (\ell, P(z_0))=1, \\ (\ell,d)=\delta}} \frac{\mu^2(\ell)}{\varphi(\ell)} = \sum_{\delta | d} \frac{\mu^2(\delta)}{\varphi(\delta)} \sum_{\substack{m \le y/\delta, \\ (m, dP(z_0))=1}} \frac{\mu^2(m)}{\varphi(m)}.$$

The last transformation is due to the fact that ℓ being squarefree, the condition $(\ell, d) = \delta$ implies that $\ell = \delta m$ with m squarefree and prime to d. The final step is to notice that

$$\sum_{\delta | d} \frac{\mu^2(\delta)}{\varphi(\delta)} = \frac{d}{\varphi(d)}$$

and that, when $\delta | d$, we have

$$\sum_{\substack{m \le y, \\ (m, dP(z_0))=1}} \frac{\mu^2(m)}{\varphi(m)} \ge \sum_{\substack{m \le y/\delta, \\ (m, dP(z_0))=1}} \frac{\mu^2(m)}{\varphi(m)} \ge \sum_{\substack{m \le y/d, \\ (m, dP(z_0))=1}} \frac{\mu^2(m)}{\varphi(m)}.$$

The lemma follows readily. ∎

Proof of Lemma 2.2. Let us change the notation in Lemma 2.1 and use z_1 rather than z_0. Then Lemma 2.2 follows from Lemma 2.1 with $z_1 = 2$ and $d = P(z_0)$, and on recalling definition (6). ∎

Proof of Lemma 2.3. We simply notice that

$$\sum_{\ell \le y} \frac{\mu^2(\ell)}{\varphi(\ell)^{1+h}} \ge \sum_{\ell \le y} \frac{\mu^2(\ell)}{\ell^{1+h}} \prod_{p|\ell} \Big(\sum_{k \ge 0} \frac{1}{p^k} \Big)^{1+h}$$

$$\ge \sum_{\ell \le y} \frac{\mu^2(\ell)}{\ell^{1+h}} \prod_{p|\ell} \Big(\sum_{k \ge 0} \frac{1}{p^{k(1+h)}} \Big) = \sum_{\substack{q \ge 1, \\ k(q) \le y}} \frac{1}{q^{1+h}} \ge \sum_{q \le y} \frac{1}{q^{1+h}}$$

where $k(d) = \prod_{p|d} p$ is the so-called *squarefree kernel* of d. The lemma is proved. ∎

Proof of Lemma 2.4. This lemma is as classical as the lemmas in this section. It can for instance be found page 24 of the book [1] by E. Bombieri. An upper bound of similar strength can be found in [12, Lemma 3.5]. For a proof, use Lemma 2.3 with $h = 0$ and recall that $\sum_{q \le y} 1/q \ge \log y$. ∎

Proof of Lemma 2.5. In [15, Theorem 8] by J.B. Rosser & L. Schoenfeld, we find the estimate $\displaystyle\prod_{p < z_0} \frac{p}{p-1} < e^\gamma \log z_0 \Big(1 + \frac{1}{2\log^2 z_0} \Big)$ which is valid when $z_0 > 286$. Hence, when $z_0 > 286$ we find that

$$e^\gamma \log(2z_0) \prod_{p < z_0} \frac{p-1}{p} \ge \Big(1 + \frac{\log 2}{\log z_0} \Big) \Big(1 + \frac{1}{2\log^2 z_0} \Big)^{-1} \ge 1$$

as $\log 2 \ge 1/2$. A direct inspection using Pari-GP [11] establishes the stated inequality for the remaining values of z_0. ∎

Proof of Lemma 2.6. Though the proof that follows does not require it, let us notice that the lemma is obvious when $\log z_0 \ge e^{-\gamma} \log z$, as $G(z; z_0) \ge 1$ (consider the contribution of the summand $\ell = 1$ in (5)). For the proof, simply combine Lemma 2.2 together with Lemma 2.5. ∎

3. Auxiliary lemmas

Lemma 3.1: *Let $h > 0$. We have* $\sum_{d\le D}\frac{\mu^2(d)}{\varphi(d)^{1+h}} \ge \frac{1-D^{-h}}{h}$.

Proof. We first appeal to Lemma 2.3 and then further simplify the lower bound as follows:

$$\begin{aligned}\sum_{q\le D}\frac{1}{q^{1+h}} &= \int_1^D \sum_{q\le t} 1\frac{(1+h)dt}{t^{2+h}} + \frac{[D]}{D^{1+h}}\\ &\ge \int_1^D (t-1)\frac{(1+h)dt}{t^{2+h}} + \frac{D-1}{D^{1+h}} = \frac{h+1}{h}\left(1-\frac{1}{D^h}\right) + \frac{D-1}{D^{1+h}}\\ &\ge \frac{1-D^{-h}}{h} + 1 - \frac{1}{D^{1+h}} \ge \frac{1-D^{-h}}{h}\end{aligned}$$

as required. ∎

Lemma 3.2: *We have* $\pi(x) = \sum_{p\le x} 1 \le \frac{x}{\log x}(1+\frac{3}{2\log x})$ *and* $\pi(x) \le \frac{5x}{4\log x}$, *both valid when* $x \ge 114$. *Finally,* $\pi(x) \ge x/(\log x)$ *when* $x \ge 17$.

This can be found in [15, Theorem 1, Corollary 2] by J.B. Rosser & L. Schoenfeld.

Lemma 3.3: *Let* $M \in \mathbb{R}$, *and* N *and* δ *be positive real number. There exists a smooth function* ψ *on* $\mathbb{R}$ *such that*

- *The function* ψ *is non-negative.*
- *When* $t \in [M, M+N]$, *we have* $\psi(t) \ge 1$.
- $\psi(0) = N + \delta^{-1}$.
- *When* $|\alpha| > \delta$, *we have* $\hat{\psi}(\alpha) = 0$.
- *We have* $\psi(t) = \mathcal{O}_{M,N,\delta}(1/(1+|t|^2))$.

This lemma is due to A. Selberg, see [16, Section 20]. See also [9] by H.L. Montgomery and [17] by J.D. Vaaler. A similar construction but with a stronger decreasing condition can be found in [13, Chapter 15], based on the paper [8] by J.J. Holt and J.D. Vaaler.

4. An enveloping sieve

We fix two real parameters $z_0 \le z$ and consider the sole case of prime numbers. It is easy to reproduce the analysis of [14, Section 3] as far as exact formulae are concerned, but one gets easily sidetracked towards slightly different formulae. The reader may for instance compare [12, Lemma 4.2] and [14, (4.1.14)]. Similar material is also the topic of [13, Chapter 12]. So we present a complete analysis in our special case. Here is the main end-product we shall use.

THEOREM 4.1: *Let $z_0 \le z$ be two parameters. There exists an upper bound $\beta_{z_0,z}$ of the characteristic function of those integers that do not have any prime factor in the interval $[z_0, z)$. The function $\beta_{z_0,z}$ admits the expansion:*

$$\beta_{z_0,z}(n) = \sum_{\substack{q \le z^2,\\ q|P(z)/P(z_0)}} w_q(z;z_0)c_q(n)$$

where $c_q(n)$ is the Ramanujan sum and where

$$w_q(z;z_0) = \frac{\mu(q)}{\varphi(q)G(z;z_0)}\frac{G_{[q]}(z;z_0)}{G(z;z_0)},$$

with the definition

$$G_{[q]}(z;z_0) = \sum_{\substack{\ell \le z/\sqrt{q},\\ (\ell,qP(z_0))=1}} \frac{\mu^2(\ell)}{\varphi(\ell)}\xi_q(z/\ell) \tag{7}$$

and

$$\xi_q(y) = \sum_{\substack{q_1q_2q_3=q,\\ q_1q_3\le y,\\ q_2q_3\le y}} \frac{\mu(q_3)\varphi_2(q_3)}{\varphi(q_3)} \quad \text{where} \quad \varphi_2(q_3) = \prod_{p|q}(p-2).$$

Notice that $\xi_q(y) = q/\varphi(q)$ when $y \ge q$ and that $|\xi_q(y)| \le 3^{\omega(q)}$ always.

Remark 4.1: The factor $G_{[q]}(z;z_0)/G(z;z_0)$ should be looked upon as a mild perturbation. It can be shown to equivalent to 1 when q goes to infinity, and, in general, it only introduces technicalities that can be handled.

Remark 4.2: The coefficient $w_q(z;z_0)$ is the main actor here. It saves the density $1/G(z;z_0)$ of the sequence $(\beta_{z_0,z}(n))$. The further saving introduced by the factor $1/\varphi(q)$ is essential, though a milder decreasing rate is enough (see Lemma 4.2). It comes from an equidistribution of the sequence $(\beta_{z_0,z}(n))$ in invertible arithmetical progressions modulo q. It can easily be shown that

$$w_q(z;z_0) = \lim_{N\to\infty}\frac{1}{N}\sum_{n\le N}\beta_{z_0,z}(n)e(na/q)$$

for every a prime to q.

Remark 4.3: The parameter z_0 will be essential: the coefficients $w_q(z;z_0)$ for $q \in (1, z_0)$ vanish. See around (13).

Proof of Theorem 4.1. We split the proof in three steps. We follow closely Section 3 of [14]. See also [13, Chapter 11].

Building the upper bound. The initial idea of the Selberg sieve is to consider the family

$$\beta_{z_0,z}(n) = \Bigl(\sum_{\substack{d:d|n,\\ (d,P(z_0))=1,\\ d\le z}} \lambda_d \Bigr)^2 \tag{8}$$

for arbitrary real coefficients λ_d that are only constrained by the condition $\lambda_1 = 1$. Indeed, for any such set of of coefficients, the resulting function is non-negative and takes the value 1 at integers n that have no prime factors dividing $P(z)/P(z_0)$. After an optimization step that we skip, one reaches the choice

$$\lambda_d = 1\!\!1_{(d,P(z_0))=1} \frac{\mu(d) d G_d(z/d;z_0)}{\varphi(d) G(z;z_0)} \tag{9}$$

where G_d is given by (5) (notice that, indeed, $\lambda_1 = 1$). From now onward, we reserve the notation λ_d for this special choice. Though we shall not use it, notice that Lemma 2.1 implies the bound $|\lambda_d| \le 1$.

We develop the square above and get

$$\begin{aligned}\beta_{z_0,z}(n) &= \sum_{\substack{d_1,d_2,\\ [d_1,d_2]|n}} \lambda_{d_1}\lambda_{d_2} = \sum_{d_1,d_2} \frac{\lambda_{d_1}\lambda_{d_2}}{[d_1,d_2]} \sum_{q|[d_1,d_2]} \sum_{a \,mod^*\, q} e(na/q) \\ &= \sum_{\substack{q\le z^2,\\ (q,P(z_0))=1}} w_q(z;z_0) c_q(n)\end{aligned}$$

where

$$w_q(z;z_0) = \sum_{q|[d_1,d_2]} \frac{\lambda_{d_1}\lambda_{d_2}}{[d_1,d_2]}. \tag{10}$$

Note that $w_q(z;z_0) = 0$ when q does not divide $P(z)/P(z_0)$, and in particular when it is not squarefree. Let us assume now that $q|P(z)/P(z_0)$.

Expliciting $w_q(z;z_0)$. We introduce the definition (9) of the λ_d's and obtain

$$G(z;z_0)^2 w_q(z;z_0) = \sum_{\substack{\ell_1,\ell_2\le z,\\ (\ell_1\ell_2,P(z_0))=1}} \frac{\mu^2(\ell_1)}{\varphi(\ell_1)}\frac{\mu^2(\ell_2)}{\varphi(\ell_2)} \sum_{\substack{q|[d_1,d_2],\\ d_1|\ell_1,d_2|\ell_2}} \frac{d_1\mu(d_1)d_2\mu(d_2)}{[d_1,d_2]}.$$

The inner sum vanishes is ℓ_1 has a prime factor prime to $q\ell_2$, and similarly for ℓ_2. Furthermore, we need to have $q|[\ell_1,\ell_2]$ for the inner sum not to be empty. Whence we may write $\ell_1 = q_1q_3\ell$ and $\ell_2 = q_2q_3\ell$ where $(\ell,q)=1$ and $q=q_1q_2q_3$. The part of the inner sum corresponding to ℓ has value $\prod_{p|\ell}(p-2+1)=\varphi(\ell)$. We have reached

$$G(z;z_0)^2 w_q(z;z_0) = \sum_{\substack{\ell\le z,\\ (\ell,qP(z_0))=1}} \frac{\mu^2(\ell)}{\varphi(\ell)} \sum_{\substack{q_1q_2q_3=q,\\ q_1q_3\ell\le z,\\ q_2q_3\ell\le z}} \frac{1}{\varphi(q)\varphi(q_3)} \sum_{\substack{q|[d_1,d_2],\\ d_1|q_1q_3,\\ d_2|q_2q_3}} \frac{d_1\mu(d_1)d_2\mu(d_2)}{[d_1,d_2]}.$$

In this last inner sum, we have necessarily $d_1 = q_1d_1'$ and $d_2 = q_2d_2'$, so $q_3 = [d_1',d_2']$. We may thus write

$$G(z;z_0)^2 w_q(z;z_0) = \sum_{\substack{\ell\le z,\\ (\ell,qP(z_0))=1}} \frac{\mu^2(\ell)}{\varphi(\ell)} \sum_{\substack{q_1q_2q_3=q,\\ q_1q_3\ell\le z,\\ q_2q_3\ell\le z}} \frac{\mu(q)\mu(q_3)}{\varphi(q)\varphi(q_3)} \sum_{\substack{d_1',d_2':\\ q_3=[d_1',d_2']}} \frac{d_1'\mu(d_1')d_2'\mu(d_2')}{[d_1',d_2']}.$$

This last inner sum has value $\varphi_2(q_3)$, whence

$$G(z;z_0)^2 w_q(z;z_0) = \frac{\mu(q)}{\varphi(q)} \sum_{\substack{\ell\le z,\\ (\ell,qP(z_0))=1}} \frac{\mu^2(\ell)}{\varphi(\ell)} \sum_{\substack{q_1q_2q_3=q,\\ q_1q_3\ell\le z,\\ q_2q_3\ell\le z}} \frac{\mu(q_3)\varphi_2(q_3)}{\varphi(q_3)}$$

as announced. The size conditions are readily seen to imply that $\ell \le z/\sqrt{q}$.
∎

LEMMA 4.2: *When $4\le z_0\le z$, we have $|w_q(z;z_0)| \le 6\,\dfrac{\log z_0}{\sqrt{q}\log z}$.*

Proof. We deduce from the definition the estimate $|\xi_q(y)|\le 3^{\omega(q)}$, and thus

$$|G(z;z_0)w_q(z;z_0)| \le 3^{\omega(q)}/\varphi(q). \tag{11}$$

As $z_0 > 3$, we may assume that q is prime to 6, since otherwise $w_q(z; z_0) = 0$. We use Lemma 2.6 to get

$$|w_q(z;z_0)| \leq \prod_{p\geq 5} \max\left(\frac{3\sqrt{p}}{p-1}, 1\right)\frac{1}{G(z;z_0)\sqrt{q}} \leq \frac{2.23\times e^{\gamma}\ \log 2z_0}{\sqrt{q}\log z}$$
$$\leq \frac{2.23\times e^{\gamma}\ \log z_0}{\sqrt{q}\log z}\left(1+\frac{\log 2}{\log 4}\right) \leq 6\,\frac{\log z_0}{\sqrt{q}\log z}$$

It has been enough, in the Euler product, to consider the primes $p = 5$ and $p = 7$. The lemma follows swiftly. ∎

5. The fundamental estimate

Theorem 5.1: *Let $N \geq 1000$. Let $\mathcal{B}$ be a δ-well spaced subset of $\mathbb{R}/\mathbb{Z}$. For any function f on $\mathcal{B}$, we have*

$$\sum_{p\leq N}\left|\sum_{b\in\mathcal{B}} f(b)e(bp)\right|^2 \leq 280(N+\delta^{-1})\|f\|_2^2 \frac{\log(2\|f\|_1^2/\|f\|_2^2)}{\log N}.$$

where $\|f\|_q^q = \sum_{b\in\mathcal{B}} |f(b)|^q$ for any positive q.

Proof. Let us first notice that $\|f\|_1^2 \geq \|f\|_2^2$. Let $z = N^{1/4}$ and

$$z_0 = \left(2\frac{\|f\|_1^2}{\|f\|_2^2}\right)^2 \geq 4.$$

We have $z_0 \leq z$ when $\|f\|_1^2/\|f\|_2^2 \leq N^{1/8}/2$. When this condition is not met, we use the dual of the usual large sieve inequality (see [9] by H.L. Montgomery) to infer that

$$\sum_{p\leq N}\left|\sum_{b\in\mathcal{B}} f(b)e(bp)\right|^2 \leq (N+\delta^{-1})\|f\|_2^2$$
$$\leq (N+\delta^{-1})\|f\|_2^2 \frac{\log(2\|f\|_1^2/\|f\|_2^2)}{\log(N^{1/8})}.$$

This establishes our inequality in this case. Henceforth, we assume that $z_0 \le z$. We discard the small primes trivially:

$$\begin{aligned}\sum_{p\le z}\Big|\sum_{b\in\mathcal{B}} f(b)e(bp)\Big|^2 &\le z\|f\|_1^2 \le N^{3/8}\|f\|_2^2/\sqrt{8}\\ &\le N\frac{\|f\|_2^2\log(2\|f\|_1^2/\|f\|_2^2)}{\log N}\frac{\log N}{\sqrt{8}N^{5/8}\log 2}\\ &\le \frac{N}{2880}\frac{\|f\|_2^2\log(2\|f\|_1^2/\|f\|_2^2)}{\log N}.\end{aligned}$$

Let us now define

$$W = \sum_{z<p\le N}\Big|\sum_{b\in\mathcal{B}} f(b)e(bp)\Big|^2. \tag{12}$$

We bound above the characteristic function of those primes by our enveloping sieve and further majorize the characteristic function of the interval $[1,N]$ by a function ψ (see Lemma 3.3) of Fourier transform supported by $[-\delta_1,\delta_1]$ where $\delta_1 = \min(\delta, 1/(2z^4))$, and which is such that $\hat{\psi}(0) = N + \delta_1^{-1}$. This leads to

$$W \le \sum_{\substack{q\le z^2,\\ (q,P(z_0))=1}} w_q(z;z_0)\sum_{a\,mod^*\,q}\sum_{b_1,b_2} f(b_1)\overline{f(b_2)}\sum_{n\in\mathbb{Z}} e((b_1-b_2)n)e(an/q)\psi(n).$$

We split this quantity according to whether $q < z_0$ or not:

$$W = W(q<z_0) + W(q\ge z_0).$$

When $q \ge z_0$, Poisson summation formula tells us that the inner sum is also $\sum_{m\in\mathbb{Z}}\hat{\psi}(b_1-b_2-(a/q)+m)$. The sum over b_1, b_2 and n is thus

$$\le (N+\delta_1^{-1})\sum_{b_1,b_2} f(b_1)\overline{f(b_2)}\#\big\{(a/q)/\|b_1-b_2+a/q\| < \delta_1\big\}.$$

Given (b_1,b_2), at most one a/q may work, since $1/z^4 > 2\delta_1$. By bounding above $w_q(z;z_0)$ by Lemma 4.2, we see that

$$\begin{aligned}W(q\ge z_0) &\le 6(N+\delta_1^{-1})\frac{\|f\|_1^2\log z_0}{\sqrt{z_0}\log z}\\ &\le \frac{6}{\sqrt{2}}(N+\delta_1^{-1})\frac{\|f\|_2^2\log z_0}{\log z}.\end{aligned} \tag{13}$$

When $w_q(z;z_0)\neq 0$, we have $q|P(z)/P(z_0)$; on adding the condition $q<z_0$, only $q=1$ remains. Since $\mathcal{B}$ is δ-well-spaced and $w_1(z;z_0)=1/G(z;z_0)$, Lemma 2.6 leads to

$$W(q<z_0)\leq (N+\delta_1^{-1})\frac{e^\gamma\|f\|_2^2\log 2z_0}{\log z}.$$

We check that $(N+\delta_1^{-1})\leq\frac{N+4N}{N}(N+\delta^{-1})$. We finally get

$$\sum_{p\leq N}\Bigl|\sum_{b\in\mathcal{B}}f(b)e(bn)\Bigr|^2\leq\left(\frac{1}{2880}+5\times 2\times 4\times\left(\frac{6}{\sqrt{2}}+e^\gamma\left(1+\frac{\log 2}{\log z_0}\right)\right)\right)$$
$$\times(N+\delta^{-1})\|f\|_2^2\frac{\log(2\|f\|_1^2/\|f\|_2^2)}{\log N}.$$

The proof of the theorem follows readily. ∎

6. On moments. Proof of Theorem 1.2

Lemma 6.1: *Assume $y/\log y\leq t$ with $y\geq 2$ and $t\geq e$. Then $y\leq 2t\log t$.*

Proof. Our property is trivial when $y\leq 2e$. Notice that the function $f:y\mapsto y/\log y$ is non-increasing when $y\geq e$. We find that $f(2t\log t)\geq t\geq f(y)$, whence $2t\log t\geq y$ as sought. ∎

Proof of Theorem 1.2. For typographical simplification, we define

$$B=\left(\frac{N+\delta^{-1}}{\log N}\sum_{p\leq N}|u_p|^2\right)^{1/2}. \tag{14}$$

We also set $\ell=2+h$. For any $\xi>0$, we examine the set

$$\mathcal{X}_\xi=\Bigl\{x\in\mathcal{X}/\Bigl|\sum_{p\leq N}u_pe(xp)\Bigr|\geq\xi B\Bigr\}. \tag{15}$$

By the Cauchy-Schwartz inequality and Lemma 3.2, we see that $\xi\leq c_1=\min(5/4,1+\frac{3}{2\log N})$ or else, the set $\mathcal{X}_\xi$ is empty. We consider (as in [7], bottom of page 1141, by A.J. Harper)

$$\Gamma(\xi)=\sum_{x\in\mathcal{X}_\xi}\Bigl|\sum_{p\leq N}u_pe(xp)\Bigr|. \tag{16}$$

We write it as $\Gamma(\xi)=\sum_{x\in\mathcal{X}_\xi}c(x)\sum_{p\leq N}u_pe(xp)$ for some $c(x)$ of modulus 1 and develop it in

$$\Gamma(\xi)=\sum_{p\leq N}u_p\sum_{x\in\mathcal{X}_\xi}c(x)e(xp).$$

We apply Cauchy's inequality to this expression to get

$$\Gamma(\xi)^2 \le \sum_{p\le N}|u_p|^2 \sum_{p\le N}\Bigl|\sum_{x\in\mathcal{X}_\xi} c(x)e(xp)\Bigr|^2 \le 280\,B^2|\mathcal{X}_\xi|\log(2|\mathcal{X}_\xi|)$$

by Theorem 5.1. It follows from this upper bound that

$$\xi^2|\mathcal{X}_\xi|^2B^2 \le \Gamma(\xi)^2 \le 280\,B^2|\mathcal{X}_\xi|\log(2|\mathcal{X}_\xi|)$$

whence

$$2|\mathcal{X}_\xi|/\log(2|\mathcal{X}_\xi|) \le 560\,/\xi^2.$$

We convert this inequality via Lemma 6.1 in $2|\mathcal{X}_\xi| \le 1120\xi^{-2}\log(560/\xi^2)$.

We can now turn towards the proof of the stated inequality and select $\xi_j = c_1/c^j$ for some $c>1$ that we will select later. We get

$$\begin{aligned}\sum_{x\in\mathcal{X}}\Bigl|\sum_{p\le N} u_p e(xp)\Bigr|^\ell/B^\ell &\le c_1^\ell|\mathcal{X}_{\xi_0}| + \sum_{j\ge1}\frac{c_1^\ell}{c^{\ell j}}(|\mathcal{X}_{\xi_j}|-|\mathcal{X}_{\xi_{j-1}}|)\\ &\le 560(1-c^{-\ell})\sum_{j\ge0}\frac{c_1^{\ell-2}(\log(560)-2\log c_1+2j\log c)}{c^{(\ell-2)j}}.\\ &\le 560\sum_{j\ge0}\frac{c_1^{\ell-2}(7+2j\log c)}{c^{(\ell-2)j}}.\end{aligned}$$

We check that

$$560\sum_{j\ge0}\frac{c_1^{\ell-2}\times 7}{c^{(\ell-2)j}} = \frac{560\times 7\times c_1^{\ell-2}}{1-c^{2-\ell}}$$

and that

$$\begin{aligned}560\sum_{j\ge1}\frac{c_1^{\ell-2}j\,2\log c}{c^{(\ell-2)j}} &\le 1120\frac{(\log c)}{c^{\ell-2}}c_1^{\ell-2}\sum_{j\ge1}\frac{j}{c^{(\ell-2)(j-1)}}\\ &\le \frac{1120\times(c_1/c)^{\ell-2}\log c}{(1-c^{2-\ell})^2}.\end{aligned}$$

When $\ell\ge3$, we select $c=2$, getting after some numerical work

$$\sum_{x\in\mathcal{X}}\Bigl|\sum_{p\le N}u_pe(xp)\Bigr|^{2+h} \le (3920(1+\tfrac{3}{2\log N})^h+2000)\biggl(\frac{N+\delta^{-1}}{\log N}\sum_{p\le N}|u_p|^2\biggr)^{1+h/2}.$$

When $\ell\in(2,3)$, we select $c=\exp(1/h)$, getting similarly

$$\sum_{x\in\mathcal{X}}\Bigl|\sum_{p\le N}u_pe(xp)\Bigr|^{2+h} \le \biggl(6300(1+\tfrac{3}{2\log N})^h+\frac{2900}{h}\biggr)\biggl(\frac{N+\delta^{-1}}{\log N}\sum_{p\le N}|u_p|^2\biggr)^{1+h/2}.$$

Our theorem follows readily. ∎

7. Large sieve bound on small sets. Proof of Theorem 1.1

Proof of Theorem 1.1. This is a classical argument of duality. We write

$$V=\sum_{x\in\mathcal{X}}\biggl|\sum_{p\le N}u_pe(xp)\biggr|^2=\sum_{x\in\mathcal{X}}\sum_{p\le N}u_p\overline{S(x)}e(xp)$$

where $S(x)=\sum_{1\le p\le N}u_pe(xp)$. On using the Cauchy-Schwarz inequality, we get

$$V^2\le\sum_{p\le N}|u_p|^2\sum_{p\le N}\biggl|\sum_{x\in\mathcal{X}}\overline{S(x)}e(xp)\biggr|^2.$$

We invoke Theorem 5.1 and notice to control $\|S\|_1^2/\|S\|_2^2$ that

$$\Bigl(\sum_{x\in\mathcal{X}}|\overline{S(x)}|\Bigr)^2\le|\mathcal{X}|\sum_{x\in\mathcal{X}}|\overline{S(x)}|^2.$$

This leads to

$$V^2\le 280\,\frac{N+\delta^{-1}}{\log N}\sum_{p\le N}|u_p|^2\sum_{x\in\mathcal{X}}|S(x)|^2\log(2|\mathcal{X}|).$$

On simplifying by $\sum_{x\in\mathcal{X}}|S(x)|^2$ (after discussing whether it vanishes or not), we get our estimate. ∎

8. Optimality and uniform boundedness. Proof of Theorem 1.4

Proof of Theorem 1.4. We assume that $N\ge 10^6$ and set

$$S(\alpha)=\sum_{p\le N}e(p\alpha). \tag{17}$$

The argument employed at the bottom of page 1626 of [5] by B. Green is not enough for us. Instead, we got our inspiration from the argument developped by R.C. Vaughan in [19]. It runs as follows. We first notice that

$$\biggl|\sum_{a\,mod^*q}S\Bigl(\frac{a}{q}+\beta\Bigr)\biggr|\le\biggl(\sum_{a\,mod^*q}\Bigl|S\Bigl(\frac{a}{q}+\beta\Bigr)\Bigr|^\ell\biggr)^{1/\ell}\biggl(\sum_{a\,mod^*q}1\biggr)^{(\ell-1)/\ell}.$$

A direct inspection shows that

$$\sum_{a\,mod^*q}S\Bigl(\frac{a}{q}+\beta\Bigr)=\mu(q)S(\beta)+T(q,\beta)$$

where

$$T(q,\beta) = \sum_{p:p|q} e(p\beta)(c_q(p) - \mu(q)). \tag{18}$$

The bound $|c_q(n)| \le \varphi((n,q))$ for the Ramanujan sum $c_q(n)$ (use for instance the von Sterneck expression (4)) gives us

$$|T(q,\beta)| \le \sum_{p:p|q} (p-1+1) \le q. \tag{19}$$

The last inequality follows from the trivial property that a sum of positive integers is certainly not more than its product. We next get a lower bound for $S(\beta)$ by writing

$$1 - e(\beta p) = 2i\pi \int_0^{\beta p} e(t)dt$$

whence

$$|S(\beta)| \ge S(0) - 2\pi\beta N S(0) \ge (1-2\pi\beta N)S(0) \ge (1-2\pi\beta N)\frac{N}{\log N} \tag{20}$$

by Lemma 3.2. When $|\beta| \le 1/(7N)$, this leads to $|S(\beta)| \ge c_2 N/\log N$ with $c_2 = 1 - 2\pi/7$, and, when q is squarefree and not more than $\sqrt{N}$, to

$$\big|\mu(q)S(\beta) + T(q,\beta)\big| \ge c_2 \frac{N}{\log N} - \sqrt{N} \ge \frac{N}{12\log N}. \tag{21}$$

We thus get, when $|\beta| \le 1/(7N)$,

$$\sum_{a \,mod^*\, q} \left|S\Big(\frac{a}{q} + \beta\Big)\right|^\ell \ge \frac{\mu^2(q)}{\varphi(q)^{\ell-1}} |S(\beta) + T(q,\beta)|^\ell \ge \frac{\mu^2(q)}{\varphi(q)^{\ell-1}} \left(\frac{N}{12\log N}\right)^\ell.$$

Thus

$$\begin{aligned}\int_0^1 |S(\alpha)|^\ell d\alpha &\ge \sum_{q \le \sqrt{N}} \sum_{a \,mod^*\, q} \mu^2(q) \int_{\frac{a}{q} - \frac{1}{7N}}^{\frac{a}{q} + \frac{1}{7N}} \left|S\Big(\frac{a}{q} + \beta\Big)\right|^\ell d\beta \\ &\ge \frac{2}{7N} \sum_{q \le \sqrt{N}} \frac{\mu^2(q)}{\varphi(q)^{\ell-1}} \left(\frac{N}{12\log N}\right)^\ell.\end{aligned}$$

By Lemma 3.1, we conclude that

$$\int_0^1 |S(\alpha)|^\ell d\alpha \ge \frac{1 - \sqrt{N}^{2-\ell}}{\ell - 2} \frac{2}{7N} \left(\frac{N}{12\log N}\right)^\ell.$$

We thus find that

$$\int_0^1 \Big|\sum_{p\le N} u_p e(p\alpha)\Big|^\ell d\alpha \le K(\ell)\Big(\frac{\log N}{N}\sum_{p\le N}|u_p|^2\Big)^{\ell/2}\int_0^1\Big|\sum_{p\le N} e(p\alpha)\Big|^\ell d\alpha \tag{22}$$

where $\ell = 2+h$ and

$$K(2+h) = 7000\frac{(1+\frac{3}{2\log N})^h + 1/h}{N}\Big(\frac{2N^2}{(\log N)^2}\Big)^{\ell/2}\frac{h}{1-\sqrt{N}^{-h}}\frac{7N}{2}\Big(\frac{N}{12\log N}\Big)^{-\ell}$$
$$= 7000\Big((1+\tfrac{3}{2\log N})^h + 1/h\Big)2^{\ell/2}\frac{h}{1-\sqrt{N}^{-h}}\frac{7}{2}12^\ell.$$

When $h\ge 1$, we use

$$K(2+h)\le 24500\big(1.11^h h+1\big)\frac{1}{0.999}(12\sqrt{2})^\ell \le 10^7\cdot 20^\ell$$

where the worst case is reached next to $\ell = 18.19\cdots$. When $h<1$, the quantity $K(2+h)$ is bounded above by

$$\frac{3\cdot 10^8}{h}\frac{h}{1-\sqrt{N}^{-h}} = \frac{3\cdot 10^8}{1-\sqrt{N}^{-h}}.$$

This is bounded above by $8\cdot 10^8$ when $h\ge 1/\log N$. When $0\le h\le 1/\log N$, we use

$$\int_0^1\Big|\sum_{p\le N} u_p e(p\alpha)\Big|^\ell d\alpha \le \Bigg(\pi(N)\sum_{p\le N}|u_p|^2\Bigg)^{h/2}\int_0^1\Big|\sum_{p\le N}u_p e(p\alpha)\Big|^2 d\alpha$$
$$\le \sqrt{5/4}\Big(\frac{\log N}{N}\sum_{p\le N}|u_p|^2\Big)^{\ell/2}\Big(\frac{N}{\log N}\Big)^\ell N^{-1}\log N$$

which leads to (22) with

$$K(2+h) = \frac{\sqrt{5/4}}{N}\Big(\frac{N}{\log N}\Big)^\ell\frac{h\log N}{1-\sqrt{N}^{-h}}\frac{7N}{2}\Big(\frac{N}{12\log N}\Big)^{-\ell}$$
$$\le \frac{7\sqrt{5/4}\cdot 12^3}{2(1-\exp(-1/2))}\le 8\cdot 10^8.$$

Theorem 1.4 follows readily. ∎

9. Small sets large sieve estimates. Proof of Corollary 1.5

Proof of Corollary 1.5. The split the Farey sequence

$$F(Q_0) = \left\{\frac{a}{q}, 1 \le a \le q \le Q_0, (a,q)=1\right\} = \left\{0 < x_1 < x_2 < \ldots < x_K = 1\right\} \tag{23}$$

in $F_1(Q_0) = \{x_{2i}, 1 \le i \le K/2\}$ union $F_2(Q_0) = \{x_{2i+1}, 1 \le 0 \le (K-1)/2\}$. We recall that the distance between two consecutive points a/q and a'/q' in $F(Q_0)$ is $1/(qq')$; this is at least as large as $\frac{1}{qQ_0} + \frac{1}{q'Q_0}$ by the known property $q+q' \ge Q_0$. Hence two intervals $[\frac{a_1}{q_1} - \frac{1}{q_1 Q_0}, \frac{a_1}{q_1} + \frac{1}{q_1 Q_0}]$ and $[\frac{a_2}{q_2} - \frac{1}{q_2 Q_0}, \frac{a_2}{q_2} + \frac{1}{q_2 Q_0}]$ with $\frac{a_1}{q_1}, \frac{a_2}{q_2} \in F_1(Q_0)$ are separated by at least $1/Q_0^2$. We check this is also true when seen on the unit circle: the largest point of $F(Q_0)$ is 1 and its smallest is $\frac{1}{[Q_0]}$. The same applies to $F_2(Q_0)$. We finally notice that $|F(Q_0)| \le Q_0(Q_0+1)/2 \le Q_0^2$.

To prove our corollary, for every $x_{2i} \in F_1(Q_0)$, we select a point $\tilde{x}_{2i}$ such that

$$\Big|\sum_{p \le N} u_p e(p\tilde{x_{2i}})\Big| = \max_{|x - x_{2i}| \le \frac{1}{qQ_0}} \Big|\sum_{p \le N} u_p e(px)\Big| \tag{24}$$

and apply Theorem 1.1 to the set $\tilde{X}_1 = \{\tilde{x}_{2i}\}$. We proceed similarly with $F_2(Q_0)$. The last details are left to the readers. ∎

10. Extension to primes in intervals

We discuss here how our results extend from the case of primes in the initial interval to primes in $[M+1, M+N]$ for some non-negative M. During the proof of Theorem 5.1, we used the property that our sequence has at most $N^{1/4}$ elements below $N^{1/4}$, and that the remaining ones are prime to any integer below $N^{1/4}$. This is certainly still true when looking at intervals.

Theorem 10.1: *Let $N \ge 1000$. Let B be a δ-well spaced subset of $\mathbb{R}/\mathbb{Z}$. For any function f on $\mathcal{B}$, we have*

$$\sum_{M+1 \le p \le M+N} \Big|\sum_{b \in \mathcal{B}} f(b) e(bp)\Big|^2 \le 280(N + \delta^{-1}) \|f\|_2^2 \frac{\log(2\|f\|_1^2/\|f\|_2^2)}{\log N}.$$

When defining c_1 in the proof of Theorem 1.2, we used an upper bound for the number of elements in our set. The version of the Brun-Titchmarsh inequality proved by H. Montgomery & R.C. Vaughan in [10] enables us to use $c_1 = 2$. After some trivial modifications, we reach the following.

THEOREM 10.2: *Let $\mathcal{X}$ be a δ-well spaced subset of $\mathbb{R}/\mathbb{Z}$. Assume $N \geq 1000$ and let $h > 0$. We have*

$$\sum_{x\in\mathcal{X}}\Big|\sum_{M+1\leq p\leq M+N} u_p e(xp)\Big|^{2+h} \leq 14000\big(2^h+1/h\big)\biggl(\frac{N+\delta^{-1}}{\log N}\sum_{M+1\leq p\leq M+N}|u_p|^2\biggr)^{1+h/2}.$$

COROLLARY 10.3: *Assume $N \geq 1000$ and let $h > 0$. We have*

$$\int_0^1\Big|\sum_{M+1\leq p\leq M+N} u_p e(p\alpha)\Big|^{2+h} d\alpha \leq 14000\frac{2^h+1/h}{N}\biggl(\frac{2N}{\log N}\sum_{M+1\leq p\leq M+N}|u_p|^2\biggr)^{1+h/2}.$$

Modifying the proof of Theorem 1.4 is more delicate as it requires bounding the trigonometric polynomial S from below in (20) to discard the contribution of $T(q,\beta)$. A simple solution is to assume that all the elements of our sequence are further larger than $\sqrt{N}$, which is readily granted by assuming that $M \geq \sqrt{N}$.

THEOREM 10.4: *There exists a constant $C > 0$ such that the following holds. Assume $N \geq 10^6$, $M \geq \sqrt{N}$ and let $\ell \geq 2$. We have*

$$\biggl(\int_0^1\Big|\sum_{M+1\leq p\leq M+N} u_p e(p\alpha)\Big|^{\ell} d\alpha\biggr)^{1/\ell} \leq C\sqrt{\frac{N/\log N}{1+R}}\biggl(\int_0^1\Big|\sum_{M+1\leq p\leq M+N} e(p\alpha)\Big|^{\ell} d\alpha\biggr)^{1/\ell}$$

as soon as $\sum_{M+1\leq p\leq M+N}|u_p|^2 \leq \sum_{M+1\leq p\leq M+N} 1 = R$.

So the uniform Hardy-Littlewood majorant property holds for primes in the interval $[M+1, M+N]$ provided the number of such primes is $\gg N/\log N$.

Proof of Theorem 10.4. We set

$$S(\alpha) = \sum_{M+1\leq p\leq M+N} e(p\alpha). \tag{25}$$

We can assume that $S(0) \geq 1$. On following the proof of Theorem 1.4, we readily reach, when $|\beta| \leq 1/(7N)$,

$$\sum_{a \bmod^* q}\Big|S\Big(\frac{a}{q}+\beta\Big)\Big|^{\ell} \geq \frac{\mu^2(q)}{\varphi(q)^{\ell-1}}S(0)^{\ell}. \tag{26}$$

This again leads to

$$\int_0^1 |S(\alpha)|^{\ell} d\alpha \geq \frac{1-\sqrt{N}^{2-\ell}}{\ell-2}\frac{2}{7N}S(0)^{\ell}.$$

Corollary 10.3 gives us with $\ell = 2 + h$

$$\int_0^1 \Big| \sum_{M+1 \le p \le M+N} u_p e(p\alpha) \Big|^\ell d\alpha \le 14000 \frac{2^h + 1/h}{N} \left(\frac{2N}{\log N} S(0) \right)^{\ell/2}$$
$$\le 10^5 \frac{h2^h + 1}{1 - \sqrt{N}^{-h}} \left(\frac{2N}{S(0) \log N} \right)^{\ell/2} \int_0^1 |S(\alpha)|^\ell d\alpha.$$

The factor $(\frac{h2^h+1}{1-\sqrt{N}^{-h}})^{1/\ell}$ is bounded when $h \in [1/\log N, \infty)$. We treat separately the case $h \in [0, 1/\log N]$. ∎

Theorem 1.1 and Corollary 1.5 go through with no modifications, and are in this manner closer to (2).

Theorem 10.5: *Let $\mathcal{X}$ be a δ-well spaced subset of $\mathbb{R}/\mathbb{Z}$ and $N \ge 1000$. Let $(u_p)_{M+1 \le p \le M+N}$ be a sequence of complex numbers. We have*

$$\sum_{x \in \mathcal{X}} \Big| \sum_{M+1 \le p \le M+N} u_p e(xp) \Big|^2 \le 280 \frac{N + \delta^{-1}}{\log N} \sum_{M+1 \le p \le M+N} |u_p|^2 \log(2|\mathcal{X}|).$$

Corollary 10.6: *Let $N \ge 1000$ and $Q_0 \in [2, \sqrt{N}]$. Let $(u_p)_{M+1 \le p \le M+N}$ be a sequence of complex numbers. We have*

$$\sum_{q \le Q_0} \sum_{a \, mod^* q} \max_{|\alpha - \frac{a}{q}| \le \frac{1}{qQ_0}} \Big| \sum_{M+1 \le p \le M+N} u_p e(\alpha p) \Big|^2 \le 1200 \frac{N \log Q_0}{\log N} \sum_{M+1 \le p \le M+N} |u_p|^2.$$

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