On long κ-tuples with few prime factors

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ABSTRACT

We prove that there are infinitely many integers n such that the total number of prime factors of $(n + h_1) \ldots (n + h_\kappa)$ is exactly $(1 + o(1))\kappa \operatorname{Log} \kappa$. Our result even ensures us that these prime factors are fairly evenly distributed among and factor $n + h_i$.

1. Introduction and some results

The prime κ-tuple conjecture of Hardy and Littlewood in [**11**] asserts that, given any 'admissible' κ-tuple $(h_1 = 0, h_2, \ldots, h_\kappa)$, there are infinitely many integers such that each of $n, n + h_2, \ldots, n + h_\kappa$ is prime. By an *admissible* κ-tuple we mean a κ-tuple $(h_1 = 0, h_2, \ldots, h_\kappa)$ of distinct non-negative integers such that the set $\{h_1, \ldots, h_\kappa\}$ does *not* cover all of $\mathbb{Z}/p\mathbb{Z}$ for every prime p. For instance, $(0, 2)$ and $(0, 2, 6)$ are admissible tuples, while $(0, 2, 4)$ is not. This conjecture is, of course, out of reach of today's techniques, and the best one can do, since Brun in [**4**], is to produce tuples $(n, n + h_2, \ldots, n + h_\kappa)$ whose total number of prime factors is bounded in terms of κ. We are concerned in this paper with the case when κ is large. The best available result is still the one of Miech from [**17**], where he proves that there exist infinitely many κ-tuples having in all at most $\kappa \operatorname{Log} \kappa + \mathcal{O}(\kappa)$ prime factors. The best bound one can expect is κ by the Hardy–Littlewood conjecture, but, more realistically and taking the parity barrier into account, the best bound we hope to achieve here is 2κ. As a matter of fact, the parity principle is the only identified hurdle here, but even getting a bound at most $(1 - \delta)\kappa \operatorname{Log} \kappa$ for some positive δ has not been achieved. We will be confronted here by the same barrier, in a strong way (see the paragraph before Theorem 1.2) but will explain partly this phenomenon. The proof of the claim of [**21**, Proposition, p. 287] is faulty, as far as I can see. Different bounds appear in [**10**, Theorems 10.5, 10.7, Corollary 10.11.2], [**5**, Table 11.1]; all have the same asymptotic behaviour.

The main novelty in recent years is due to Heath-Brown in [**12**]. On developing an idea of Selberg on the twin prime conjecture, he investigated the problem of bounding the number of prime factors of each $n + h_i$ and has obtained that, given any admissible κ-tuple, there are infinitely many tuples $(n, n + h_2, \ldots, n + h_\kappa)$ such that each $n + h_i$ has at most $2(1 + o(1)) \operatorname{Log} \kappa / \operatorname{Log} 2$ prime factors.

Here is a typical corollary of the method we develop.

THEOREM 1.1. *Given any admissible κ-tuple $(h_1, \ldots, h_\kappa)$, there are infinitely many integers n such that*

$$(1/\kappa) \sum_{1 \leqslant i \leqslant \kappa} (\omega(n + h_i) - \operatorname{Log} \kappa)^2 \ll \operatorname{Log} \kappa,$$

where $\omega(m)$ denotes the number of prime factors of m counted without multiplicity.

In short, we are not only able to produce κ-tuples with *at most* $\kappa \operatorname{Log} \kappa + \mathcal{O}(\kappa)$, but with *exactly* $\kappa \operatorname{Log} \kappa + \mathcal{O}(\kappa \sqrt{\operatorname{Log} \kappa})$. If we seek only this total number, then we can also reach $\kappa \operatorname{Log} \kappa + \mathcal{O}(\kappa)$. Our result furthermore ensures that these primes factors are fairly evenly distributed among each factor $n + h_i$. As usual (with the exception of [7]), it is possible to ensure that there are more than a positive constant times $N/(\operatorname{Log} N)^{\kappa}$ such κ-tuples below N, and we can further guarantee them to be free of prime factors below N^{η} for a small but positive η. We concentrate on the simpler situation, and on the novel aspects of the method we propose.

It is striking that the limit $\kappa \operatorname{Log} \kappa$ should arise so vividly, and a partial explanation is this, anticipating somewhat on what follows: the host sequence $\beta(n)$ (see (1.2) and Sections 2 and 4) has in its support many integers that *indeed* are such that the product $(n + h_1) \ldots (n + h_{\kappa})$ has $(1 + o(1))\kappa \operatorname{Log} \kappa$ prime factors. Note that we are not able to produce integers in the support of $\beta(n)$ and such that $(n + h_1) \ldots (n + h_{\kappa})$ have, say, $(2 + o(1))\kappa \operatorname{Log} \kappa$ prime factors. But it does not rule out their existence either.

The traditional way to study this problem introduces the sequence of numbers

$$\Pi_{(h_1, \ldots, h_{\kappa})}(n) = \prod_{1 \leqslant i \leqslant \kappa} (n + h_i), \tag{1.1}$$

and looks for such numbers free of small prime factors. The so-called *linearized approach* restricts n to primes and we can thus remove the factor $n + h_1 = n$ from $\Pi_{(h_1, \ldots, h_{\kappa})}(n)$. This is efficient when κ is small, but we will neglect it since we are interested in what happens when κ goes to infinity. Our study relies on the quantity

$$S_i((a_{d^*})_{d^*}) = \sum_{n \leqslant N} \left(\sum_{d^* | n + h_i} a_{d^*} \right) \beta(n) \quad \text{with } \beta(n) = \left(\sum_{d | \Pi_{(h_1, \ldots, h_{\kappa})}(n)} \lambda_d \right)^2. \tag{1.2}$$

This is a simple version of a more elaborate sum we will present later on, at (2.5). The coefficient $\beta(n)$ arises in this form in the Selberg sieve. Miech in [17] chooses precisely the one that is chosen there, while Salerno in [21] and Greaves in [8, Section 7.3.2], following ideas of [2] and of [24], both take more elaborate versions, all explained thereafter.

The word 'weight' being overloaded in this theory, we are in need of a stricter terminology. We call $\beta(n)$ the *host sequence*, while referring to the coefficient $c(n) = \sum_{d^* | n + h_i} a_{d^*}$ as the *sieve coefficient*. We will try to add a* to what pertains to this coefficient. Two issues will make matters somewhat more intricate: the construction of the usual Selberg coefficients $\beta(n)$ depends on yet another sequence which, at this level, will be called (and be treated as) a host sequence. Secondly, we shall modify these Selberg coefficients by employing ... some weights! The word *weight* will be reserved for these, except in the expression *the weighted sieve*. The *weighted sieve approach* dealt with in this paper consists in showing that $\sum_i S_i((a_{d^*})_{d^*})$ is non-negative for some sieve coefficients $c(n)$ and a given host sequence β.

The choice of a proper family of host sequences will be fairly satisfactorily answered to in this paper, but a proper choice of sieve coefficients remains hazy, although we will make some progress. Concerning this coefficient, the traditional choice takes it in the form

$$c(n) = b - \sum_{\substack{p \leqslant P = N^{\theta}, \\ p | \Pi_{(h_1, \ldots, h_{\kappa})}(n)}} 1,$$

thus restricting the sum to d^* to being 1 or a prime number.

Historically speaking, the first weighted sieve had been used by Kuhn in 1941 (see [14]) to prove that every interval $[X, X + \sqrt{X}]$ contains numbers having at most four prime factors, provided that X be large enough.

At this level, we skip the choice of the host sequence. It is detailed in Sections 2 and 4. Each choice depends on a large parameter Q and on a non-negative non-increasing function

w on $[0, \infty[$, which furthermore is supposed to be continuous, piecewise differentiable with a bounded derivative and such that $w(1) = 0$. This choice, or rather the way to handle the resulting coefficient, is the first novelty of this paper. The setting we create for them leads to easy treatments, so easy that we are in a position to handle sieve coefficients $c(n)$ that do not only depend on prime divisors of $\Pi_{(h_1, \ldots, h_\kappa)}(n)$ but also on divisors that have a given number of prime factors. We examine the functions $w(t) = \max(0, 1 - t)^\nu$ for positive ν such that but $\mathrm{Log}\, \nu = o(\mathrm{Log}^2\, \kappa)$, but we will attain the same bound $\kappa \, \mathrm{Log}\, \kappa$, thus giving somewhat more consistency to this barrier. We further consider a sequence: the characteristic function $a_r[P]$ of those integers that have exactly r prime factors, all distinct and all at most P, and 0 at any other ones.

THEOREM 1.2.   *For any non-negative integer r, any integer parameter κ and any parameter $\tau > 0$, there exists a bounded continuous function $\mathfrak{G}_{\kappa,r}(t_1, t_2, \tau)$ with the following property. Let $(h_1, \ldots, h_\kappa)$ be an admissible κ-tuple, $Q \geqslant 1$ be a parameter and w be a function as above. We consider the sum $S_i(a_r[Q^\tau])$ from (1.2) when $\beta(n)$ is as above. We have*

$$\frac{S_i(a_r[Q^\tau])}{N/(\mathrm{Log}\, Q)^\kappa} = \mathscr{C}(h_1, \ldots, h_\kappa) \int_{0 \leqslant t_1, t_2 \leqslant 1} w'(t_1) w'(t_2) \mathfrak{G}_{\kappa,r}(t_1, t_2, \tau) \, dt_1 \, dt_2$$
$$+ \mathcal{O}(1/(\mathrm{Log}\, Q)^{1/9}) + \mathcal{O}(Q^{r\tau+2} N^{-1} (\mathrm{Log}\, Q)^\kappa),$$

where the constant $\mathscr{C}(h_1, \ldots, h_\kappa)$ is given by

$$\mathscr{C}(h_1, \ldots, h_\kappa) = \prod_{p \geqslant 2} \left(1 - \frac{\#\{h_1, \ldots, h_\kappa \mod p\}}{p} \right) \left(1 - \frac{1}{p} \right)^{-\kappa}.$$

A more general theorem is proved in Theorem 14.1. The expression we find for $\mathfrak{G}_{\kappa,r}(t_1, t_2, \tau)$ is fairly explicit but too complicated to get even the asymptotic dependence in κ. See (17.1) and (18.1) in case $r = 0$ and $r = 1$. Let us summarize here the properties we prove:

(1) $\mathfrak{G}_{\kappa,r}$ is symmetrical in t_1 and t_2, that is, $\mathfrak{G}_{\kappa,r}(t_1, t_2, \tau) = \mathfrak{G}_{\kappa,r}(t_2, t_1, \tau)$;

(2) for any $\lambda > 0$, we have $\mathfrak{G}_{\kappa,r}(t_1, t_2, \tau) = \lambda^\kappa \mathfrak{G}_{\kappa,r}(t_1/\lambda, t_2/\lambda, \tau/\lambda)$;

(3) $\mathfrak{G}_{\kappa,r}(t_1, 0, \tau) = \mathfrak{G}_{\kappa,r}(0, t_2, \tau) = \mathfrak{G}_{\kappa,r}(t_1, t_2, 0) = 0$ and we extend $\mathfrak{G}_{\kappa,r}(t_1, t_2, \tau)$ to negative values of t_1 and/or t_2 by attributing it the value 0;

(4) $\mathfrak{G}_{\kappa,r}$ is a bounded continuous function;

(5) when $\tau > 0$, the following recursion formula holds:

$$\tau \frac{d\mathfrak{G}_{\kappa,r}(t_1, t_2, \tau)}{d\tau} = \mathfrak{G}_{\kappa,r-1}(t_1, t_2, \tau) - \mathfrak{G}_{\kappa,r-1}(t_1 - \tau, t_2, \tau)$$
$$- \mathfrak{G}_{\kappa,r-1}(t_1, t_2 - \tau, \tau) + \mathfrak{G}_{\kappa,r-1}(t_1 - \tau, t_2 - \tau, \tau); \qquad (1.3)$$

(6) when $\tau > 0$, we have $\mathfrak{G}_{\kappa,0}(t_1, t_2, \tau) = \min(t_1, t_2)^\kappa$.

The formula (14.2) offers a distinct approach, as it avoids any notion of derivative, but we need to have access to the second derivative of $\mathfrak{G}_{\kappa,r}$: we do not know how to prove that it exists in general. See (17.2) and (18.2) in case $r = 0$ and $r = 1$.

We use sieve coefficients supported by products of two prime factors, in a general context; prove a precise formula for a specific host sequence for products of two primes, and prove a general but very cumbersome formula for general sieve coefficients supported by products of r primes.

Final note: we restricted our attention in this work to prime κ-tuples and did not consider the case of a general κ-dimensional sieve setting, but similar results are available, at least when κ is an integer. We also did not try to rule out primes appearing with multiplicities. In fact, we tried to stick to the simplest problem: the technique is still so heavy that we are not yet able to do all what one would wish, even in this special situation.

Notation. We assume that $Q \geqslant 2$ and we set $\alpha = 1/\operatorname{Log} Q$, $\tau = (\operatorname{Log} P)/(\operatorname{Log} Q)$ as well as $R = \operatorname{Log} P_0$, where $P_0 \leqslant P$ will be introduced later. We also set $\varepsilon = \alpha R$ and $Z = (\operatorname{Log} Q)^\kappa$. We use

$$\mathcal{H}_m = \sum_{1 \leqslant k \leqslant m} 1/k. \tag{1.4}$$

The symbol $+$ as an exponent will be used to avoid negative values: $x^+ = \max(0, x)$ and $x^{+m} = \max(0, x)^m$. We close this part by a remark concerning one-dimensional integrals. We use most often $\int_{a \leqslant t \leqslant b} f(t)\, dt$ and not $\int_a^b f(t)\, dt$: the first one vanishes when $b < a$, while, in such a situation, the second one is commonly understood as $-\int_b^a f(t)\, dt$.

2. A general framework

For an admissible shift $(h_1, \ldots, h_\kappa)$, we define

$$\mathcal{K}(h_1, \ldots, h_\kappa)_d = \bigcap_{1 \leqslant i \leqslant \kappa} (\mathcal{U}_d - h_i) \quad (\text{with } \mathcal{U}_d = (\mathbb{Z}/d\mathbb{Z})^*). \tag{2.1}$$

When n lies in $\mathcal{K}(h_1, \ldots, h_\kappa)_d$, then $n + h_i$ falls in $\mathcal{U}_d$, that is, is prime to d, for all i from 1 to κ.

We proposed some time ago (see [19, 20]) a geometrical approach that dispenses with building an auxiliary polynomial, like $\Pi_{(h_1, \ldots, h_\kappa)}$ above. The exposition will be made easier by the following definition.

DEFINITION 2.1. A collection $(\mathcal{K}_d)_{d \geqslant 1}$ is said to be a *multiplicatively split compact set* when the following conditions are satisfied.

(i) For each $d \geqslant 1$, $\mathcal{K}_d$ is a subset of $\mathbb{Z}/d\mathbb{Z}$.

(ii) When d_1 and d_2 are co-prime, $\mathcal{K}_{d_1 d_2}$ is in one to one correspondence, via the Chinese remainder map, with $\mathcal{K}_{d_1} \times \mathcal{K}_{d_2}$.

We further say that it is *square-free* when the following condition is satisfied.

(iii) If $d_1 | d_2$, but both have the same prime factors, then $\mathcal{K}_{d_2}$ is the reverse image of $\mathcal{K}_{d_1}$ through the canonical surjection from $\mathbb{Z}/d_2\mathbb{Z}$ to $\mathbb{Z}/d_1\mathbb{Z}$.

The main compact set we consider is $\mathcal{K}(h_1, \ldots, h_\kappa)$. Its definition (2.1) shows clearly that it is multiplicatively split and square-free. We need another one, $\mathcal{K}^*$, that will be successively $\mathcal{U}_d - h_1, \mathcal{U}_d - h_2, \ldots, \mathcal{U}_d - h_\kappa$. In general, we simply assume that

(H_1) $\mathcal{K}$ is a multiplicatively split compact set;

(H_2) $\mathcal{K}^*$ is a square-free multiplicatively split compact set that contains $\mathcal{K}$.

We associate to $\mathcal{K}^*$ (and to any multiplicatively split compact set) its bordering system $(\mathcal{L}_d^*)_{d \geqslant 1}$. Each $\mathcal{L}_d^*$ is the subset of $\mathbb{Z}/d\mathbb{Z}$ defined by the following conditions:

(1) $\mathcal{L}_1^* = \{1\}$ and $\mathcal{L}_d^* = \emptyset$ when d is not square-free;

(2) when d_1 and d_2 are co-prime, $\mathcal{L}_{d_1 d_2}^*$ is in bijection via the Chinese remainder map to $\mathcal{L}_{d_1}^* \times \mathcal{L}_{d_2}^*$;

(3) $\mathcal{L}_p^* = \mathbb{Z}/p\mathbb{Z} \setminus \mathcal{K}_p^*$.

This may look complicated, but the situation clears when one looks at characteristic functions (the expressions are valid whether the compact set $\mathcal{K}^*$ be square-free or not):

$$\mathbb{1}_{\mathcal{L}_d^*} = \prod_{p^\nu \| d} \left(\mathbb{1}_{\mathcal{K}_{p^{\nu-1}}} - \mathbb{1}_{\mathcal{K}_{p^\nu}} \right) = (-1)^{\omega(d)} \sum_{\delta | d} \mu(d/\delta) \mathbb{1}_{\mathcal{K}_\delta} \tag{2.2}$$

and

$$1_{\mathcal{K}_d} = \prod_{p^\nu \| d} (1 - 1_{\mathcal{L}_p^*} - \ldots - 1_{\mathcal{L}_{p^\nu}^*}) = \sum_{\delta | d} (-1)^{\omega(\delta)} 1_{\mathcal{L}_\delta^*}. \tag{2.3}$$

We shall also use the bordering system $(\mathcal{L}_d)_{d \geqslant 1}$ associated with $\mathcal{K}$. Note that the condition $\mathcal{K}^* \supset \mathcal{K}$ has the following consequence: when p is a prime number and a is a positive integer, we have

$$1_{\mathcal{K}_{p^a}} \cdot 1_{\mathcal{L}_p^*} = 1_{\mathcal{K}_{p^a}} \cdot (1 - 1_{\mathcal{K}_p^*}) = 0. \tag{2.4}$$

Having this preparation at hand, we can present the main actor of this paper, namely, the sum

$$S((a_{d^*})_{d^*}; \mathcal{K}, \mathcal{K}^*) = \sum_{n \leqslant N} \left(\sum_{d^*/n \in \mathcal{L}_{d^*}^*} a_{d^*} \right) \left(\sum_{d/n \in \mathcal{K}_d} \lambda_d^\sharp \right)^2. \tag{2.5}$$

We assume that a_{d^*} vanishes when $d^* > D^*$. The coefficients $(\lambda_d^\sharp)_d$ are completely free for us to choose. We simply assume that they vanish when $d > Q$, for some parameter Q. We need some more material from sieve theory. We define the coefficients $(\lambda_d)_d$ by

$$\lambda_d = (-1)^{\omega(d)} \sum_{d | \ell} \lambda_\ell^\sharp. \tag{2.6}$$

We have (see [**19**, (11.5)])

$$\lambda_d^\sharp = \sum_{d | \ell} \mu(\ell/d)(-1)^{\omega(\ell)} \lambda_\ell \quad \text{and} \quad \sum_{d/n \in \mathcal{K}_d} \lambda_d^\sharp = \sum_{d/n \in \mathcal{L}_d} \lambda_d. \tag{2.7}$$

In practice, the condition $n \in \mathcal{K}_d$ leads to easier treatment of the main term, while the λ_d's will be smaller, leading to a better treatment of the error term. We finally introduce the multiplicative function

$$h(d) = \prod_{p^\nu \| d} \left(\frac{p^\nu}{|\mathcal{K}_{p^\nu}|} - \frac{p^{\nu-1}}{|\mathcal{K}_{p^{\nu-1}}|} \right).$$

We need to handle averages of this function and we follow [**10**, Chapter 5] (see also [**9**]). A condition labelled (Ω_1) is introduced therein, in the fourth part of the first chapter, p. 49, but it is expedient to assume a much stronger hypothesis, namely

$$\sum_{p \leqslant x} \frac{h(p) \operatorname{Log} p}{p} = \kappa \operatorname{Log} x + \mathcal{O}(1). \tag{H_3}$$

Our main hypothesis on h is a slight simplification of what is called $(\Omega_2(\kappa, L))$ at the beginning of [**10**, Chapter 5]. It reads

$$h(p) \ll \kappa/p. \tag{H_4}$$

Note that we use the fact that κ is an integer in Section 12. This implies classically that

$$\sum_{\delta \leqslant x} h(\delta) = A(\operatorname{Log} x)^\kappa + \mathcal{O}((\operatorname{Log}(2x))^{\kappa-1})$$

when $x \leqslant Q$, by using [**10**, Lemmas 5.3 and 5.4] and Merten's Theorem. We deduce from (H_3) and (H_4) the following weaker form which will be easier to use:

$$\sum_{\delta \leqslant x} h(\delta) = A(\operatorname{Log} x)^\kappa + \mathcal{O}(\alpha Z) \quad (1 \leqslant x \leqslant Q). \tag{2.8}$$

When $\mathcal{K} = \mathcal{K}(h_1, \ldots, h_\kappa)$, the constant A is equal to the constant $\mathscr{C}$ of Theorem 1.2. It is positive when $(h_1, \ldots, h_\kappa)$ is admissible, or, and this is an equivalent statement, when $\mathcal{K}(h_1, \ldots, h_\kappa)$ is non-empty.

3. Generalization of a formula of Bombieri

The quantity (2.5) has a summation over four variables (n, d^*, d and d'). We take care here of the summation over n. We set

$$\gamma(d^*) = |\mathcal{L}_{d^*}^*|/d^* \tag{3.1}$$

as well as

$$S_0((a_{d^*})_{d^*}; \mathcal{K}, \mathcal{K}^*) = \sum_{d^*, \delta} \gamma(d^*) a_{d^*} h(\delta) \left(\sum_{\substack{\delta|d, \\ (d,d^*)=1}} |\mathcal{K}_d| \lambda_d^\sharp / d \right)^2 . \tag{3.2}$$

LEMMA 3.1.   *We have*

$$S((a_{d^*})_{d^*}; \mathcal{K}, \mathcal{K}^*) = N S_0((a_{d^*})_{d^*}; \mathcal{K}, \mathcal{K}^*) + \mathcal{O}\left(\sum_{d_1^*, d_2, d_3} |a_{d_1^*}| |\lambda_{d_2}| |\lambda_{d_3}| |\mathcal{L}_{d_1}^*| |\mathcal{L}_{d_2}| |\mathcal{L}_{d_3}| \right) .$$

Proof.   We first revert to $(\mathcal{L}_d)$ on invoking (2.7) and get

$$S((a_{d^*})_{d^*}; \mathcal{K}, \mathcal{K}^*) = \sum_{d_1^*, d_2, d_3} a_{d_1^*} \lambda_{d_2} \lambda_{d_3} \sum_{\substack{n \leqslant N, \\ n \in \mathcal{L}_{d_1^*}^* \cap \mathcal{L}_{d_2} \cap \mathcal{L}_{d_3}}} 1.$$

Note that $\mathcal{L}_{d_1^*}^* \cap \mathcal{L}_{d_2} \cap \mathcal{L}_{d_3}$ vanishes when d_1^* is not square-free, or when there is a prime p and two distinct powers $a \geqslant 1$ and $b \geqslant 1$ that divides, respectively, d_2 and d_3. The reader will conclude that this set defines modulo $[d_1^*, d_2, d_3]$ a subset of cardinality at most $|\mathcal{L}_{d_1}^*| |\mathcal{L}_{d_2}| |\mathcal{L}_{d_3}|$. Concerning the main term, we divide it by N and write it as

$$\text{M} = \sum_{d_1^*, d_2, d_3} a_{d_1^*} \lambda_{d_2} \lambda_{d_3} \frac{|\mathcal{L}_{d_1^*}^* \cap \mathcal{L}_{d_2} \cap \mathcal{L}_{d_3}|}{[d_1^*, d_2, d_3]}.$$

It can be defined as the limit when N goes to infinity of $S((a_{d^*})_{d^*}; \mathcal{K}, \mathcal{K}^*)/N$. To compute this limit, we use (2.7) and switch to the $\lambda_d^\sharp$. This gives us

$$\text{M} = \sum_{d_1^*, d_2, d_3} a_{d_1^*} \lambda_{d_2}^\sharp \lambda_{d_3}^\sharp \frac{|\mathcal{L}_{d_1^*}^* \cap \mathcal{K}_{d_2} \cap \mathcal{K}_{d_3}|}{[d_1^*, d_2, d_3]}.$$

Note that $\mathcal{K}_{d_2} \cap \mathcal{K}_{d_3} = \mathcal{K}_{[d_2, d_3]}$. We use equation (2.4) to introduce the condition $(d_1^*, d_2 d_3) = 1$. This gives us

$$\frac{|\mathcal{L}_{d_1^*}^* \cap \mathcal{K}_{d_2} \cap \mathcal{K}_{d_3}|}{[d_1^*, d_2, d_3]} = \frac{|\mathcal{L}_{d_1^*}^*|}{d_1^*} \frac{|\mathcal{K}_{d_2} \cap \mathcal{K}_{d_3}|}{[d_2, d_3]}.$$

We complete the separation of d_2 and d_3 via the diagonalization process of Selberg, that is, we write

$$\frac{|\mathcal{K}_{d_2} \cap \mathcal{K}_{d_3}|}{[d_2, d_3]} = \frac{|\mathcal{K}_{d_2}| |\mathcal{K}_{d_3}|}{d_2 d_3} \sum_{\substack{\delta|d_1, \\ \delta|d_2}} h(\delta).$$

The lemma follows readily.   □

This lemma generalizes [2, Theorem 18]. This same formula occurs as [8, Section 7.3.1, Lemma 1]. This is also [24, (5.6')]. Our proof is much shorter than the initial one of Bombieri. Greaves' proof is also remarkably short and shares with the above one the fact of treating the

variable d_1^* in a distinct manner. Our switching between $\mathcal{L}$ and $\mathcal{K}$ as usual enables us to extend the proof to the case when $\mathcal{K}$ is not assumed to be square-free.

4. A family of host sequences

It is time to narrow our family of host sequences. But to do so, we first develop some material to motivate our choice. Let $M = \mathrm{lcm}(d \leqslant Q)$ and let us look at $\mathcal{K}_M \subset \mathbb{Z}/M\mathbb{Z}$. We assume momentarily that the compact set satisfies the Johnsen–Gallagher condition (see [6, 19, 20, 23]), that is,

$$\forall d \leqslant Q, \forall d | q, \forall a \in \mathcal{K}_d, \text{ the number } |\{b \in \mathcal{K}_q / b \equiv a[d]\}| \text{ is independent of } a. \qquad \text{(JG)}$$

Let us expand the characteristic function of $\mathcal{K}_M$ in Fourier series:

$$\begin{aligned} 1_{\mathcal{K}_M}(n) &= \sum_{b \bmod M} \left(\frac{1}{M} \sum_{c \in \mathcal{K}_M} e(-bc/M) \right) e(bn/M) \\ &= \sum_{d | M} \sum_{a \bmod {}^* d} \left(\frac{1}{M} \sum_{c \in \mathcal{K}_M} e(-ac/d) \right) e(an/d) \\ &= \sum_{d | M} \sum_{a \bmod {}^* d} \left(\frac{|\mathcal{K}_M|}{M |\mathcal{K}_d|} \sum_{c \in \mathcal{K}_d} e(-ac/d) \right) e(an/d), \end{aligned}$$

where we have used the Johnsen–Gallagher condition (see (JG)). We define

$$\psi_d^*(n) = \sum_{a \bmod {}^* d} \left(\frac{1}{|\mathcal{K}_d|} \sum_{c \in \mathcal{K}_d} e(-ac/d) \right) e(an/d). \qquad (4.1)$$

These functions are, up to a multiplicative factor, the pseudo-characters introduced by Selberg in 1973 (see [2, 18] as well as [13]). An L^2 approximation of $1_{\mathcal{K}_M}$ is thus given by

$$\frac{|\mathcal{K}_M|}{M} \sum_{d \leqslant Q} \psi_d^*(n). \qquad (4.2)$$

It is also possible to define $\psi_d^*(n)$ by Moebius inversion since we readily verify that

$$\sum_{d | q} \psi_d^* = \frac{q}{|\mathcal{K}_q|} 1_{\mathcal{K}_q}. \qquad (4.3)$$

The coefficient $q/|\mathcal{K}_q|$ is somewhat mysterious and explained in [19, Section 9.4]. Inverting the above equation leads to the definition

$$\psi_d^* = \sum_{q | d} \mu(d/q) \frac{q}{|\mathcal{K}_q|} 1_{\mathcal{K}_q}. \qquad (4.4)$$

This definition is valid whether $\mathcal{K}$ verifies condition (JG) or not, and is thus the one we take in general. Note that $\psi_d^*(n) = h(d)$ as soon as n belongs to $\mathcal{K}_d$. As a consequence, the function

$$\sum_{d \leqslant Q} \psi_d^* \qquad (4.5)$$

is constant over $\mathcal{K}_M$. Squared, this is the usual Selberg coefficient up to a normalizing coefficient. The expression above calls immediately for a modification, namely,

$$\sum_{d \leqslant Q} \zeta_d \psi_d^*, \qquad (4.6)$$

for some arbitrary coefficients $(\zeta_d)_{d \leqslant Q}$. We readily find that

$$
\sum_{d \leqslant Q} \zeta_d \psi_d^*(n) = \sum_{d \leqslant Q} \zeta_d \sum_{\substack{q \mid d, \\ n \in \mathcal{K}_q}} \mu(d/q) \frac{q}{|\mathcal{K}_q|}
$$

$$
= \sum_{q/n \in \mathcal{K}_q} \frac{q}{|\mathcal{K}_q|} \sum_{q \mid d \leqslant Q} \zeta_d \mu(d/q)
$$

so that we take

$$
\lambda_q^\sharp = \frac{q}{|\mathcal{K}_q|} \sum_{q \mid d \leqslant Q} \zeta_d \mu(d/q)/Z, \tag{4.7}
$$

where Z is a size parameter. Note that, when d^* is square-free and co-prime with δ,

$$
Z \sum_{\substack{\delta \mid d, \\ (d,d^*)=1}} |\mathcal{K}_d| \lambda_d^\sharp / d = \sum_{\substack{\delta \mid d, \\ (d,d^*)=1}} \sum_{d \mid q \leqslant Q} \zeta_q \mu(q/d) = \sum_{\delta \mid q} \zeta_q \sum_{\substack{\delta \mid d \mid q, \\ (d,d^*)=1}} \mu(q/d).
$$

Let us write $q = \delta \ell$. It is obvious that ℓ is prime to d^*. Furthermore,

$$
\sum_{\substack{\delta \mid d \mid \delta \ell, \\ (d,d^*)=1}} \mu(\delta \ell / d) = \mu(\ell)
$$

(since only $d = \delta$ appears in this sum), which does not vanish only when $\mu^2(\ell) \neq 0$, that is, $\ell \mid d^*$. We define

$$
G_d(Q) = \sum_{\substack{f \leqslant Q, \\ [f,d] \leqslant Q}} h(f) \zeta_{[d,f]}. \tag{4.8}
$$

This leads to

$$
\lambda_d = (-1)^{\omega(d)} G_d(Q)/Z. \tag{4.9}
$$

The usual normalization $Z = G_1(Q)$ is not required here because all our quantities are homogeneous with respect to Z. The next lemma in a generalization of [15].

LEMMA 4.1.   *When $\zeta \geqslant 0$ decreases on chains of multiples, the inequalities $G_\ell(Q\ell/d) \leqslant G_d(Q) \leqslant G_\ell(Q)$ hold whenever $\ell \mid d$.*

This has the nice consequence that $|\lambda_d| \leqslant G_1(Q)/Z$, while $\lambda_d^\sharp$ can be much bigger.

Proof.   The condition $[f,d] \leqslant Q$ implies that $[f,\ell] \leqslant Q$, which proves the first claim (note that $h \geqslant 0$). In the other direction, let f be such that $[f,\ell] \leqslant Q$. We have $[f,d] \leqslant [f,\ell](d/\ell)$ and the lemma follows readily. $\square$

With such a choice of $(\lambda_d)_d$, $S_0((a_{d^*})_{d^*}; \mathcal{K}, \mathcal{K}^*)$ becomes

$$
Z^2 S_0((a_{d^*})_{d^*}; \mathcal{K}, \mathcal{K}^*) = \sum_{\substack{d^*, \delta, \\ (d^*, \delta)=1}} \gamma(d^*) a_{d^*} h(\delta) \left(\sum_{\substack{\ell \leqslant Q/\delta, \\ \ell \mid d^*}} \mu(\ell) \zeta_{\delta \ell} \right)^2 . \tag{4.10}
$$

To compare with earlier work, our family of parameters (ζ_d) has this name in [2, 21]. Salerno in [21] chooses for ζ_d a step function with only two steps. Greaves in [8, Section 7.3.2, (2.4)] calls this parameter $y(d)$ and chooses a logarithmic smoothing of $1_{d \leqslant Q}$. Selberg in [24, (7.6),

(7.9), (7.11)] uses weights that are similar to (4.11). The paper [**7**, (3.14)] uses also a similar shape although the sum they study is somewhat different. Heath-Brown in [**12**, (4)] determines λ_d directly, but the sum he studies differs notably from ours.

We shall further restrict our attention to weights of shape

$$\zeta_d = w\left(\frac{\mathrm{Log}\, d}{\mathrm{Log}\, Q}\right) = w(\alpha\,\mathrm{Log}\, d), \tag{4.11}$$

for some non-negative non-increasing function w on $(0,1]$. We further assume that w is continuous with $w(1) = 0$, prolonged to $(0,\infty)$ by setting $w(t) = 0$ when $t \geqslant 1$, piecewise differentiable and such that w' is bounded. These hypotheses ensure that $w(t) = -\int_t^1 w'(u)du$, which is what we need. We thus have

$$\zeta_d = -\int_{\alpha\,\mathrm{Log}\, d}^1 w'(u)\, du. \tag{4.12}$$

5. *Two reduction steps*

In this section, we first restrict d^* to integers without any small prime factors. This step may seem harmless and usual, but is in fact crucial; it will rid us of many constant terms in asymptotic expressions and will enable us to disregard most of the co-primality conditions. This introduces a parameter P_0 (and later $\varepsilon = (\mathrm{Log}\, P_0)/\mathrm{Log}\, Q$) which should disappear from the main term. However, showing that the limit of this main term as ε goes to zero exists is more difficult than expected and will be the subject of the technical Sections 12 and 13.

To restrict d^*, we limit our investigation to integers with fairly few divisors, as quantified by (H_5) below.

Our second step here will be to remove the coprimality condition $(d^*, \delta) = 1$ from (4.10).

LEMMA 5.1. *Let $d^* > 1$ be an integer and p be its smallest prime factor. We have, with the choice given by* (4.12),

$$\left|\sum_{\substack{\ell' \leqslant Q/\delta, \\ \ell' | d^*}} \mu(\ell')\zeta_{\delta\ell'}\right| \leqslant \tau(d^*)\|w'\|_\infty \alpha\,\mathrm{Log}\, p.$$

Proof.   Indeed, it is enough to consider the case when d^* is square-free. Let us set $d^* = pd_0^*$. We can dispense with the condition $\ell' \leqslant Q/\delta$ since it is included in w (for $w(1) = 0$). We thus find that

$$\sum_{\substack{\ell' \leqslant Q/\delta, \\ \ell' | d^*}} \mu(\ell')\zeta_{\delta\ell'} = \sum_{\ell' | d_0^*} \mu(\ell')(\zeta_{\delta\ell'} - \zeta_{\delta p\ell'}),$$

which gets majorized as announced.   □

We assume that

$$\sum_{d^*} \tau(d^*)^2 \gamma(d^*)|a_{d^*}| = o(\mathrm{Log}\, Q). \tag{H_5}$$

This hypothesis follows from (H_7) below for the sequence $a_r[P]$, the upper bound being $\mathcal{O}((\mathrm{Log}\,\mathrm{Log}\, P)^r)$.

Removing the small prime factors of d^*. Getting rid of the small prime factors of d^* will simplify the computation of the main term, essentially by removing constant terms. Let P_0 be a parameter to be chosen later. We set

$$\mathfrak{f}_0 = \prod_{p \leqslant P_0} p. \tag{5.1}$$

We find that, on invoking (H_5),

$$\left| \sum_{\substack{\delta, \\ (d^*, \mathfrak{f}_0) \neq 1}} \gamma(d^*) h(\delta) a_{d^*} \left(\sum_{\substack{\ell' \leqslant Q/\delta, \\ \ell' | d^*}} \mu(\ell') \zeta_{\delta \ell'} \right)^2 \right|$$

$$\leqslant \alpha^2 \|w'\|_\infty^2 Z (\operatorname{Log} P_0)^2 \sum_{d^*} \tau(d^*)^2 \gamma(d^*) |a_{d^*}| \ll \|w'\|_\infty^2 \alpha Z (\operatorname{Log} P_0)^2.$$

This is more than enough. It is also $\mathcal{O}(\|w'\|_\infty^2 Z (\operatorname{Log} \operatorname{Log} P)^r (\operatorname{Log} P_0)^2)$ for the sequence $a_r[P]$ under (H_7).

Removing the co-primality condition. We now remove the condition $(d^*, \delta) = 1$ in (4.10). Indeed on using (H_3), we find that

$$\left| \sum_{\substack{d^*, \delta, \\ (d^*, \mathfrak{f}_0) = 1, \\ (d^*, \delta) \neq 1}} \gamma(d^*) h(\delta) a_{d^*} \left(\sum_{\substack{\ell' \leqslant Q/\delta, \\ (d^*, \mathfrak{f}_0) = 1, \\ \ell' | d^*}} \mu(\ell') \zeta_{\delta \ell'} \right)^2 \right|$$

$$\leqslant \alpha^2 \|w'\|_\infty^2 \sum_{P_0 < p \leqslant P} \operatorname{Log}^2 p \sum_{\substack{d^*, \delta, \\ (d^*, \mathfrak{f}_0) = 1, \\ p | d^*, p | \delta}} \gamma(d^*) h(\delta) |a_{d^*}| \tau(d^*)^2$$

$$\ll \alpha^2 \|w'\|_\infty^2 \sum_{P_0 < p \leqslant P} \frac{\operatorname{Log}^2 p}{p} \sum_{\substack{d^*, \delta, \\ (d^*, \mathfrak{f}_0) = 1, \\ p | d^*, p | \delta}} \gamma(d^*) h(\delta/p) |a_{d^*}| \tau(d^*)^2$$

$$\ll \alpha^2 \|w'\|_\infty^2 Z \sum_{\substack{d^* \\ (d^*, \mathfrak{f}_0) = 1}} \gamma(d^*) |a_{d^*}| \tau(d^*)^2 \sum_{p | d^*} \frac{\operatorname{Log}^2 p}{p}.$$

Note that d^* has at most $(\operatorname{Log} D^*)/\operatorname{Log} P_0$ prime factors and that we have assumed that $\alpha \operatorname{Log} D^* \ll 1$. As a consequence, the bound above is

$$\ll \|w'\|_\infty^2 Z^2 \alpha (\operatorname{Log} P_0)/P_0.$$

We set

$$Z^2 S_0^{(1)}((a_{d^*})_{d^* \leqslant D^*}) = \sum_{\substack{d^*, \delta, \\ (d^*, \mathfrak{f}_0) = 1}} \gamma(d^*) h(\delta) a_{d^*} \left(\sum_{\substack{\ell' \leqslant Q/\delta, \\ \ell' | d^*}} \mu(\ell') \zeta_{\delta \ell'} \right)^2, \tag{5.2}$$

and we can replace $S_0((a_{d^*})_{d^* \leqslant D^*})$ by $S_0^{(1)}((a_{d^*})_{d^* \leqslant D^*})$ up to an error term of size at most (up to a multiplicative constant)

$$\|w'\|_\infty^2 Z\alpha((\text{Log } P_0)^2 + (\text{Log } P_0)/P_0) \ll \|w'\|_\infty^2 Z\alpha(\text{Log } P_0)^2. \tag{5.3}$$

6. Summing over δ

We open the square in (5.2) to get

$$
\begin{aligned}
Z^2 S_0^{(1)}((a_{d^*})_{d^* \leqslant D^*}) &= \sum_{\substack{d^*, \\ (d^*,\mathfrak{f}_0)=1}} \gamma(d^*) a_{d^*} \sum_{\ell_1,\ell_2 | d^*} \mu(\ell_1)\mu(\ell_2) \sum_\delta h(\delta)\zeta_{\ell_1\delta}\zeta_{\ell_2\delta} \\
&= \int_{0 \leqslant t_1, t_2 \leqslant 1} w'(t_1)w'(t_2) \sum_{\substack{d^*, \\ (d^*,\mathfrak{f}_0)=1}} \gamma(d^*) a_{d^*} \sum_{\ell_1,\ell_2 | d^*} \mu(\ell_1)\mu(\ell_2) \\
&\quad \times \sum_{\delta \leqslant \min(Q^{t_1}/\ell_1, Q^{t_2}/\ell_2)} h(\delta) \, dt_1 \, dt_2
\end{aligned}
$$

on invoking (4.12). We appeal to (2.8) at this level. This leads to an error term which is

$$\ll \alpha Z \|w\|_1^2 \sum_{\substack{d^*, \\ (d^*,\mathfrak{f}_0)=1}} \gamma(d^*)\tau(d^*)^2 |a_{d^*}| = o(\|w'\|_1^2 Z)$$

by (H_5). We have thus approximated $Z^2 S_0^{(1)}((a_{d^*})_{d^* \leqslant D^*})$ up to an error term of size $o(\|w'\|_1^2 Z)$ by

$$\boxed{Z^2 S_0^{(2)}((a_{d^*})_{d^* \leqslant D^*}) = A \sum_{\substack{\ell_1,\ell_2 \leqslant D^*, \\ (\ell_1\ell_2,\mathfrak{f}_0)=1}} \mu(\ell_1)\mu(\ell_2)\Theta(\ell_1,\ell_2) \sum_{\substack{d^*, \\ (d^*,\mathfrak{f}_0)=1, \\ [\ell_1,\ell_2]|d^*}} \gamma(d^*) a_{d^*}} \tag{6.1}$$

with

$$\Theta(\ell_1,\ell_2) = \int_{0 \leqslant t_1, t_2 \leqslant 1} w'(t_1)w'(t_2) \text{Log}^{+\kappa} \min(Q^{t_1}/\ell_1, Q^{t_2}/\ell_2) \, dt_1 \, dt_2, \tag{6.2}$$

and where $\text{Log}^{+\kappa} x = (\text{Log} \max(1,x))^\kappa$.

7. The influence of $\mathcal{K}^*$

The influence of the compact set $\mathcal{K}^*$ in (6.1) is contained in the function γ. In our main application, we have $\gamma(p) = 1/p$, but we can encompass a more general case without much more difficulty. We assume that there exists $\kappa' > 0$ such that

$$\sum_{d^*/(d^*,\mathfrak{f}_0)=1} \left| \gamma(d^*) - \frac{\kappa'^{\omega(d^*)}}{d^*} \right| \tau(d^*)^2 |a_{d^*}| = \mathcal{O}(1/R). \tag{H_6}$$

We could put $o(1)$ (that is, a function that goes to 0 when P_0 and P tend to infinity) instead of $\mathcal{O}(1/R)$, but we would be blocked at Section 13. Hypothesis (H_6) implies that $S_0^{(2)}((a_{d^*})_{d^* \leqslant D^*})$ equals $S_0^{(3)}((a_{d^*})_{d^* \leqslant D^*})$ up to an error term of size $\mathcal{O}(\|w'\|_\infty^2/(RZ))$, where

$$\boxed{Z^2 S_0^{(3)}((a_{d^*})_{d^* \leqslant D^*}) = A \sum_{\substack{\ell_1,\ell_2 \leqslant D^*, \\ (\ell_1\ell_2,\mathfrak{f}_0)=1}} \mu(\ell_1)\mu(\ell_2)\Theta(\ell_1,\ell_2) \sum_{\substack{d^*, \\ (d^*,\mathfrak{f}_0)=1, \\ [\ell_1,\ell_2]|d^*}} \kappa'^{\omega(d^*)} a_{d^*}/d^*.} \tag{7.1}$$

8. *Some lemmas on classical sums over primes*

To save on typography, in this section, a sum with a $*$ superscript is restricted to prime variables $p > P_0$. We define classically

$$\text{li}(x) = \int_2^x \frac{dt}{\text{Log}\, t}. \tag{8.1}$$

This function is sometimes called the Eulerian logarithmic integral and should not be confused with the logarithmic integral Li (see [**1**, 5.1.3]) from which it differs by a constant. We prove some more results than necessary, but the general case is not much more difficult than the special one we shall require, and these estimates prepare the ground for not only using $a_r[P]$ but the sequence of square-free integers at most P that have exactly r prime factors, each of them being more than P_0.

8.1. *A general result*

In dimension 1, we set

$$W(\phi, t) = \max_{P_0 \leqslant y \leqslant t} \left(|\phi(y)| + \frac{\text{Log}\, y}{y} \int_{P_0}^y \frac{x|\phi'(x)|\, dx}{\text{Log}\, x} \right), \tag{8.2}$$

while, in dimension ν, we set $W_\nu(\phi, t)$ to be

$$\max_{\substack{P_0 \leqslant y_1, \ldots, y_\nu \leqslant t, \\ y_1 y_2 \ldots y_\nu \leqslant t}} \left(|\phi(y_1, \ldots, y_\nu)| + \sum_{1 \leqslant j \leqslant \nu} \frac{\text{Log}\, y_j}{y_j} \int_{P_0}^{y_j} \frac{x|D_j(\phi, y_1, \ldots, y_\nu; x)|\, dx}{\text{Log}\, x} \right),$$

where $D_j(\phi, y_1, \ldots, y_\nu; x)$ denotes $\partial\phi/\partial y_j(y_1, \ldots, y_{j-1}, x, y_{j+1}, \ldots, y_\nu)$.

LEMMA 8.1. *Let ϕ be a continuous, piecewise C^1 function. For every $B \geqslant 1$, we have*

$$\sum_{p \leqslant t}^* \phi(p) = \int_{P_0}^t \frac{\phi(t)dt}{\text{Log}\, t} + \mathcal{O}_B \left(\frac{tW(\phi, t)}{R^B \text{Log}\, t} \right).$$

Proof. Let us denote by E the (inside of the) error term above.

$$\sum_{p \leqslant t}^* \phi(p) = \sum_{p \leqslant t}^* \left(-\int_p^t \phi'(x)\, dx + \phi(t) \right)$$

$$= (\text{li}(t) - \text{li}(P_0))\phi(t) - \int_{P_0}^t (\text{li}(x) - \text{li}(P_0))\phi'(x)\, dx + \mathcal{O}(E)$$

$$= \int_{P_0}^t \phi(t)\frac{dt}{\text{Log}\, t} + \mathcal{O}(E). \qquad \square$$

LEMMA 8.2. *Let $\phi(x_1, x_2, \ldots, x_\nu)$ be a continuous function over $\mathbb{R}^{+\nu}$ which is piecewise C^1. We have, for every $B \geqslant 1$,*

$$\sum_{p_1 \ldots p_\nu \leqslant t}^* \phi(p_1, \ldots, p_\nu) = \int_{\substack{P_0 \leqslant x_1, \ldots, x_\nu \leqslant t, \\ x_1 \ldots x_\nu \leqslant t}} \frac{\phi(x_1, \ldots, x_\nu)\, dx_1 \ldots dx_\nu}{\text{Log}\, x_1 \ldots \text{Log}\, x_\nu}$$

$$+ \mathcal{O}_B \left(\frac{t(\text{Log Log}\, t)^{\nu-1}}{R^B \text{Log}\, t} W_\nu(\phi, t) \right).$$

Proof. We proceed by induction.

Here is the induction step. We write $t = T'T''$ and it is enough to select $T' = T'' = \sqrt{t}$. Let $S_{\nu+1}(\phi, t)$ be the sum to evaluate. The Dirichlet hyperbola principle gives us

$$S_{\nu+1}(\phi, t) = \sum_{p_1 p_2 \ldots p_\nu \leqslant T'}^{*} S_1(\phi(p_1, \ldots, p_\nu, \cdot), t/(p_1 p_2 \ldots p_\nu))$$

$$+ \sum_{p_{\nu+1} \leqslant T''}^{*} (S_\nu(\phi(\cdot, \ldots, \cdot, p_{\nu+1}), t/p_{\nu+1}) - S_\nu(\phi(\cdot, \ldots, \cdot, p_{\nu+1}), T')). \qquad (8.3)$$

Concerning the first sum, say Σ_1, we use Lemma 8.1 to get

$$\Sigma_1 = \int_{P_0}^{t} S_\nu(\phi(\cdot, \ldots, \cdot, x_{\nu+1}, \min(T', t/x_{\nu+1})) \frac{dx_{\nu+1}}{\operatorname{Log} x_{\nu+1}}$$

$$+ \mathcal{O}_B \left(\sum_{p_1 \ldots p_\nu \leqslant T'}^{*} \frac{t}{R^B p_1 \ldots p_\nu \operatorname{Log} T''} W_{\nu+1}(\phi, t) \right).$$

The error term is admissible. We use the induction hypothesis on the main term, say $\Sigma_{1,1}$, and get

$$\Sigma_{1,1} = \int_{P_0}^{t} \int_{\substack{P_0 \leqslant x_1, \ldots, x_\nu \leqslant \min(T', t/x_{\nu+1}), \\ x_1 \ldots x_\nu x_{\nu+1} \leqslant t}} \phi(x_1, \ldots, x_\nu, x_{\nu+1}) \prod_{1 \leqslant i \leqslant \nu+1} \frac{dx_i}{\operatorname{Log} x_i}$$

$$+ \mathcal{O}_B \left(\int_{P_0}^{t} \frac{t}{x_{\nu+1} \operatorname{Log} T' \operatorname{Log} x_{\nu+1}} \frac{(\operatorname{Log} \operatorname{Log} t)^{\nu-2}}{R^B} W_{\nu+1}(\phi, t) \, dt \right).$$

The error term is again admissible. We handle the other part in pretty much the same manner and get the claimed lemma. $\qquad\square$

LEMMA 8.3.   Let $\phi(x_1, x_2, \ldots, x_\nu)$ be a continuous function over $\mathbb{R}^{+\nu}$ which is C^1 by piece. We have, for every $B \geqslant 1$,

$$\sum_{p_1 \ldots p_\nu \leqslant t}^{*} \frac{\phi(p_1, p_2, \ldots, p_\nu)}{p_1 \ldots p_\nu} = \int_{\substack{P_0 \leqslant x_1, x_2, \ldots, x_\nu \leqslant t, \\ x_1 x_2 \ldots x_\nu \leqslant t}} \frac{\phi(x_1, x_2, \ldots, x_\nu) \, dx_1 \ldots dx_\nu}{x_1 \ldots x_\nu \operatorname{Log} x_1 \ldots \operatorname{Log} x_\nu}$$

$$+ \mathcal{O}_B \left(\frac{(\operatorname{Log} \operatorname{Log} t)^{\nu-1}}{R^B} W_\nu(\phi, t) \right).$$

8.2.   Application

THEOREM 8.1.   We have, with $R = \operatorname{Log} P_0$ and for every $B \geqslant 1$,

$$\sum_{\substack{p_1 p_2 \ldots p_\nu \leqslant t, \\ P_0 < p_1, \ldots, p_\nu}} 1 = \frac{t}{\operatorname{Log} t} A_\nu \left(\frac{\operatorname{Log} t}{R} \right) + \mathcal{O}_B \left(\frac{t(\operatorname{Log} \operatorname{Log} t)^{\nu-1}}{R^B \operatorname{Log} t} \right),$$

where A_ν is a $C^\infty[\nu, \infty[$-function such that $A_\nu(u) = 0$ when $u \leqslant \nu$ and

$$0 \leqslant A_\nu(u) \ll_\nu (\operatorname{Log} u)^{\nu-1}, \quad 0 \leqslant A'_\nu(u) \ll_\nu (\operatorname{Log} u)^{\nu-1}/u.$$

It is defined recursively by

$$A_{\nu+1}(u) = \int_\nu^{u-1} A_\nu(v) \frac{u \, dv}{v(u-v)} = \int_\nu^{u-1} \left(\frac{A_\nu(v)}{v} + \frac{A_\nu(v)}{u-v} \right) dv. \qquad (8.4)$$

We have, when $u \geqslant 1$,

$$A_1(u) = 1,$$

and, when $u \geqslant 2$,

$$A_2(u) = 2\operatorname{Log}(u-1).$$

THEOREM 8.2. *We have, when $\nu \in \{1, 2, 3\}$ and for every $B \geqslant 1$,*

$$\sum_{\substack{p_1 \ldots p_\nu \leqslant t, \\ p_1, \ldots, p_\nu > P_0}} \frac{1}{p_1 \ldots p_\nu} = \nu! \mathscr{F}_\nu \left(\frac{\operatorname{Log} t}{\operatorname{Log} P_0} \right) + \mathcal{O}_B \left(\frac{(\operatorname{Log}\operatorname{Log} t)^\nu}{R^B} \right),$$

where $\mathscr{F}_\nu$ is given, when $h \geqslant \nu$, by

$$\begin{cases} \mathscr{F}_1(h) = \operatorname{Log} h, \\[2mm] \mathscr{F}_2(h) = \displaystyle\int_2^h \operatorname{Log}(v-1) \frac{dv}{v}, \\[2mm] \mathscr{F}_3(h) = \dfrac{1}{3} \displaystyle\int_2^{h-1} \operatorname{Log}(v-1) \operatorname{Log}(h-v) \frac{dv}{v}. \end{cases}$$

We further set $\mathscr{F}_0(h) = 1$.

A direct approach (without using Theorem 8.1) seems to give one less $\operatorname{Log}\operatorname{Log} t$ in the error term. This is irrelevant for our applications.

9. *Specialization of a_{d^*} and summing over d^**

The expression (7.1) for our main term still contains three variables, ℓ_1, ℓ_2 and d^*, and we discuss here our treatment of the variable d^*. We have to restrict a_{d^*} to $a_r[P]$ to do so. To simplify notation, we tag our summation signs with a $\star$ to indicate that all the prime factors of the variables therein are within $(P_0, P]$. We add the simplistic hypothesis

$$\begin{cases} \gamma(p) \ll 1/p, \\[2mm] \displaystyle\sum_{p \geqslant P_0} \left| \gamma(p) - \frac{\kappa'}{p} \right| = \mathcal{O}(1/\operatorname{Log} P_0). \end{cases} \tag{H_7}$$

Hypotheses (H_5) and (H_6) are then trivially verified. Since $\operatorname{Log} P/\operatorname{Log} P_0 = \tau/\varepsilon$, we find that

$$\sum_{\substack{d^*, \\ (d^*, \mathfrak{f}_0)=1, \\ m|d^*}} \kappa'^{\omega(d^*)} a_{d^*}/d^* = \frac{\kappa'^r}{m} \frac{\operatorname{Log}(\tau/\varepsilon)^{r-\omega(m)}}{(r-\omega(m))!} + \mathcal{O}(\operatorname{Log}(\tau/\varepsilon)^{r-\omega(m)}/(mP_0))$$

(this is valid even if $\omega(m) = r$) where $H = (\operatorname{Log} P)/R$. As a consequence, and replacing ℓ_1 by $\ell_0 \ell_1$ and ℓ_2 by $\ell_0 \ell_2$, with $\gcd(\ell_0 \ell_1, \ell_0 \ell_2) = \ell_0$, we find that

$$Z^2 S_0^{(3)}(a_r[P]) = A\kappa'^r \sum_{\substack{\ell_0, \ell_1, \ell_2, \\ \omega(\ell_0 \ell_1 \ell_2) \leqslant r}}^{\star} \frac{\mu(\ell_1)\mu(\ell_2)\mu^2(\ell_0 \ell_1 \ell_2)}{\ell_0 \ell_1 \ell_2} \Theta(\ell_0 \ell_1, \ell_0 \ell_2) \frac{\operatorname{Log}(\tau/\varepsilon)^{r-\omega(\ell_0 \ell_1 \ell_2)}}{(r-\omega(\ell_0 \ell_1 \ell_2))!}$$
$$+ \mathcal{O}(Z \operatorname{Log}(\tau/\varepsilon)^{2r}/P_0).$$

10. *Summing over ℓ_0, ℓ_1 and ℓ_2*

We only have the last three variables ℓ_0, ℓ_1 and ℓ_2 to take care of. The fact that the weight w is kept general complicates things somewhat but in fact selecting $w(t) = \max(0, 1-t)$ would

not lead to a notoriously simpler expression. Let us define

$$\mathscr{G}_s^{(0)}(P) = \sum_{\substack{\ell_0,\ell_1,\ell_2, \\ \omega(\ell_0\ell_1\ell_2)=s}}^{\star} \frac{\mu(\ell_1)\mu(\ell_2)\mu^2(\ell_0\ell_1\ell_2)}{\ell_0\ell_1\ell_2} \Theta(\ell_0\ell_1, \ell_0\ell_2).$$

On recalling (6.2), we find that

$$\mathscr{G}_s^{(0)}(P) = (\operatorname{Log} Q)^\kappa \int_{0\leqslant t_1,t_2\leqslant 1} w'(t_1)w'(t_2)\mathscr{G}_s^{(1)}(t_1,t_2,P)\, dt_1\, dt_2$$

with

$$\mathscr{G}_s^{(1)}(t_1,t_2,P) = \sum_{\substack{\ell_0\ell_1\leqslant Q^{t_1} \\ \ell_0\ell_2\leqslant Q^{t_2}, \\ \omega(\ell_0\ell_1\ell_2)=s}}^{\star} \frac{\mu(\ell_1)\mu(\ell_2)\mu^2(\ell_0\ell_1\ell_2)}{\ell_0\ell_1\ell_2} \min(t_1 - \alpha\operatorname{Log}(\ell_0\ell_1), t_2 - \alpha\operatorname{Log}(\ell_0\ell_2))^\kappa.$$

$$(10.1)$$

Lemma 8.3 is tailored to handle this sum. We thus find that

$$\begin{aligned}
&\mathscr{G}_s^{(1)}(t_1,t_2,P) \\
&= \sum_{b+c+d=s} \frac{(-1)^{c+d}}{b!c!d!} \int_{P_0\leqslant x_1,\ldots,x_b\leqslant P} \int_{P_0\leqslant y_1,\ldots,y_c\leqslant P} \int_{P_0\leqslant z_1,\ldots,z_d\leqslant P} \\
&\quad \frac{\min(t_1 - \alpha\operatorname{Log}(\prod_i x_i \prod_i y_i), t_2 - \alpha\operatorname{Log}(\prod_i x_i \prod_i z_i))^{+\kappa} \prod_i dx_i \prod_i dy_i \prod_i dz_i}{\prod_i(x_i \log x_i) \prod_i(y_i \log y_i) \prod_i(z_i \log z_i)} \\
&\quad + \mathcal{O}_B\left(\frac{(\operatorname{Log}\operatorname{Log} P)^{s-1}}{R^B}\right).
\end{aligned}$$

This calls for several changes of variables, which when done, lead to

$$\mathscr{G}_s^{(1)}(t_1,t_2,P) = \sum_{b+c+d=s} \frac{(-1)^{c+d}}{b!c!d!}\mathscr{G}_{b,c,d}^{(2)}(t_1,t_2,\varepsilon,\tau) + \mathcal{O}_B\left(\frac{(\operatorname{Log}\operatorname{Log} P)^{s-1}}{R^B}\right),$$

where

$$\begin{aligned}
\mathscr{G}_{b,c,d}^{(2)}(t_1,t_2,\varepsilon,\tau) &= \int_{\varepsilon\leqslant u_1,\ldots,u_b\leqslant\tau} \int_{\varepsilon\leqslant v_1,\ldots,v_c\leqslant\tau} \int_{\varepsilon\leqslant w_1,\ldots,w_d\leqslant\tau} \\
&\quad \frac{(\min(t_1 - \Sigma_i v_i, t_2 - \Sigma_i w_i) - \Sigma_i u_i)^{+\kappa} \prod_i du_i \prod_i dv_i \prod_i dw_i}{\prod_i u_i \prod_i v_i \prod_i w_i}.
\end{aligned} \qquad (10.2)$$

We thus find that $S_0^{(3)}(a_r[P]))$ equals $S_0^{(4)}(a_r[P])$ up to an error term of size $\mathcal{O}_B((\operatorname{Log}\operatorname{Log} P)^r/(R^B Z))$, where

$$\boxed{ZS_0^{(4)}(a_r[P]) = A\kappa'^r \int_{0\leqslant t_1,t_2\leqslant 1} w'(t_1)w'(t_2)\mathfrak{G}_{\kappa,r}(t_1,t_2,\varepsilon,\tau)\, dt_1\, dt_2} \qquad (10.3)$$

and

$$\mathfrak{G}_{\kappa,r}(t_1,t_2,\varepsilon,\tau) = \sum_{a+b+c+d=r} \frac{(-1)^{c+d}}{a!b!c!d!}\mathscr{G}_{b,c,d}^{(2)}(t_1,t_2,\varepsilon,\tau)\operatorname{Log}(\tau/\varepsilon)^a. \qquad (10.4)$$

This quantity depends on P_0, while the quantity we computed is independent of it. This problem is cleared in the subsequent sections.

11. Average decay rate of the Fourier transforms of piecewise affine linear functions on $\mathbb{R}^n$

This part is due to D.S. Ramana (personal communication) whom we thank warmly for his input as well as for the authorization to reproduce his argument below.

A k-simplex in $\mathbb{R}^n$ is the convex hull of $k + 1$ affine linearly independent points in $\mathbb{R}^n$, where k and n are integers. If a simplex P is spanned by a set S, then the simplexes spanned by the proper subsets of S are called the faces of P.

The union of the faces of a simplex is called the boundary of the simplex. The complement in a simplex of its boundary is called its interior. When the simplex is an n-simplex in $\mathbb{R}^n$, its boundary and interior are the same as the boundary and interior of a subset of the topological space $\mathbb{R}^n$.

By a simplicial complex in $\mathbb{R}^n$ we mean a finite family $\mathcal{K}$ of simplexes such that every face of a simplex in $\mathcal{K}$ is also in $\mathcal{K}$ and the intersection of any two simplexes A and B in $\mathcal{K}$ is a face of both A and B. This implies, in particular, that interiors of distinct simplexes in $\mathcal{K}$ are disjoint. The union of the simplexes in a simplicial complex $\mathcal{K}$ is denoted by $|\mathcal{K}|$. A compactly supported complex-valued function on $\mathbb{R}^n$ is said to piecewise affine linear if its support is contained in $|\mathcal{K}|$, for some simplicial complex $\mathcal{K}$, and if its restriction to any simplex in $\mathcal{K}$ is affine linear.

THEOREM 11.1. *If $n \geqslant 1$ is an integer and f is a continuous compactly supported complex-valued function on $\mathbb{R}^n$ that is piecewise affine linear on a simplicial complex $\mathcal{K}$, we then have that the Fourier transform $\hat{f}$ of f satisfies*

$$\int_{\mathbb{R}^n} \frac{\|x\|^n |\hat{f}(x)|}{\mathrm{Log}(2 + \|x\|)^{n+1}} \, dx \ll Mm, \qquad (11.1)$$

where M is the essential supremum of $\|\nabla f\|$ on $|\mathcal{K}|$, m is the number of n-simplexes in $\mathcal{K}$ and the implicit constant depends only on n.

The Fejér kernel in dimension 1 shows that the power n is optimal.

We infer this theorem from [**3**, Theorem 2.1(ii)], which gives the average rate of decay of the Fourier transform of the characteristic function of a polyhedron, by an application of the divergence theorem.

Preliminaries. Let us verify that in any simplicial complex $\mathcal{K}$ every $(n-1)$-simplex of $\mathcal{K}$ is either the common face of exactly two n-simplexes of $\mathcal{K}$ or lies on the boundary of $|\mathcal{K}|$.

For any $(n-1)$-simplex F in $\mathbb{R}^n$ there is a unique affine linear form H such that F lies on the (affine) hyperplane defined by $H(x) = 0$. By abuse of notation, we call the half-spaces defined by the inequalities $H(x) \geqslant 0$ and $H(x) \leqslant 0$ the sides of F. Now suppose that P and P' are n-simplexes in $\mathbb{R}^n$ whose intersection is an $(n-1)$-simplex F that is a face of both P and P'. If both these simplexes lie on the same side of F, then their interiors have a non-empty intersection. Indeed, there are $n + 1$ affine linear forms H_i and H_i' such that P and P' are the sets of x in $\mathbb{R}^n$ that satisfy $H_i(x) \geqslant 0$ and $H_i'(x) \geqslant 0$, respectively, for $1 \leqslant i \leqslant n + 1$. Suppose that $H_1 = H_1'$ and that the face F lies on the hyperplane determined by these linear forms. Let p be a point in the interior of F. Then F is the unique face of either P or P' that contains p. Thus, $x = p$ satisfies the inequalities $H_i(x) > 0$ and $H_i'(x) > 0$ for all $i \geqslant 2$. These strict inequalities define an open set U in $\mathbb{R}^n$ that contains p. Since p lies on the boundary of the open half-space determined by $H_1(x) > 0$, U has a non-empty intersection V with this half-space. Plainly, V is contained in the interior of both P and P'. It follows that if P and P' are n-simplexes of a simplicial complex and if they have a common face F that is an $(n-1)$-simplex, then, since their interiors do not intersect, P and P' must necessarily lie on distinct sides of the face F. This implies that an $(n-1)$-simplex F can be the common face of no more than two n-simplexes of a simplicial complex.

Let us now suppose that F is an $(n-1)$-simplex that is the face of a unique n-simplex P of a simplicial complex $\mathcal{K}$. Let p be a point in the interior of F. Suppose that p lies in the interior of $|\mathcal{K}|$ and let U be any open ball in $\mathbb{R}^n$ centred at p and contained in $|\mathcal{K}|$. Since p

lies on the boundary of P, the intersection V of U with the complement of P in $\mathbb{R}^n$ is a non-empty open subset of $\mathbb{R}^n$. Since V is contained in the complement of P in $|\mathcal{K}|$, it follows that this complement has a non-empty interior and therefore contains an n-simplex whose interior intersects U. Since the number of n-simplexes in $\mathcal{K}$ is finite, we see, on taking for U any open ball in a decreasing sequence of such balls centred at p and contained in $|\mathcal{K}|$, that there is an n-simplex P' contained in $\mathcal{K}$, distinct from P, and containing p. Since p is in the interior of F, this means that F is a face of P', contradicting our hypothesis on F. Thus, every point in interior of F lies on the boundary of $|\mathcal{K}|$ and consequently F lies on the boundary of $|\mathcal{K}|$ as well, as required.

Finally, we recall the divergence theorem. If v is a C^1 function from an open set Ω in $\mathbb{R}^n$ to $\mathbb{R}$, its gradient is the function from Ω into $\mathbb{R}^n$ defined by $t \mapsto (\partial v/\partial t_1, \ldots, \partial v/\partial t_n)$ and is denoted by ∇v. If v is a C^1 function from an open set Ω into $\mathbb{R}^n$, its divergence is the function from Ω into $\mathbb{R}$ defined by $t \mapsto \partial v/\partial t_1 + \ldots + \partial v/\partial t_n$ and is denoted by $\nabla \cdot v$. Please note the presence of the '$\cdot$' symbol.

Let F be an $(n-1)$-simplex F that is the face of an n-simplex P in $\mathbb{R}^n$. Suppose that F lies on a hyperplane H in $\mathbb{R}^n$ determined by the relation $\langle t, c \rangle + a = 0$, for some c of norm 1 in $\mathbb{R}^n$ and some a in $\mathbb{R}^n$. If P lies on the side given by $\langle t, c \rangle + a \leqslant 0$, then the unit outward normal n_F to F is defined to be c. When P lies on the other side of F, we set $n_F = -c$.

Suppose that P is an n-simplex in $\mathbb{R}^n$ and $v(t_1, \ldots, t_n)$ is a C^1 function from an open neighbourhood of P into $\mathbb{R}^n$. Then the divergence theorem is the relation

$$\int_P \nabla \cdot v \, dt_1 dt_2 \ldots dt_n = \sum_{F \in \mathcal{F}(P)} \int_F \langle v, n_F \rangle \, dS_F, \tag{11.2}$$

where $\mathcal{F}(P)$ denotes the set of $(n-1)$ simplexes that are the faces of P, n_F denotes the unit outward normal and dS_F denotes the element of $(n-1)$-dimensional surface measure for each such face F. To describe dS_F in terms of a parametrization (co-ordinate chart) of a given face $F \in \mathcal{F}(P)$ of a simplex P, let ϕ from $\mathbb{R}^{n-1}$ to the hyperplane containing F be a parametrization. Then, from [**22**, p. 86], we have

$$dS_F = \sqrt{\det((D\phi)^t(D\phi))} \, dt_1 \, dt_2 \ldots dt_{n-1}. \tag{11.3}$$

If g is an orthogonal matrix that transports n_F to the vector $(0, 0, \ldots, 0, 1)$ and let a in $\mathbb{R}^n$ be such that $t \mapsto gt + a$ is an affine linear isomorphism from the co-ordinate plane $t_n = 0$ to the hyperplane containing F, then composing this map with the embedding $t \mapsto (t, 0)$ from $\mathbb{R}^{n-1}$ onto the co-ordinate hyperplane $t_n = 0$, we get a parametrization ϕ. With this parametrization we have

$$dS_F = dt_1 \, dt_2 \ldots dt_{n-1}, \tag{11.4}$$

since $D\phi$ is given by the matrix of $n-1$ columns and n rows whose columns are the first $n-1$ columns of g, which is orthogonal. Let us note that since ϕ is affine linear, $\phi^{-1}(F)$ is a simplex in $\mathbb{R}^{n-1}$.

Proof of Theorem 11.1.   We begin by noting that if $\phi(t)$ and $f(t)$, with $t = (t_1, t_2, \ldots, t_n)$ are C^1 functions on an open set in $\mathbb{R}^n$, then, at any point t in this open set with $\nabla(\phi(t)) \neq 0$, we have

$$\nabla \cdot \left(\frac{\nabla \phi}{\|\nabla \phi\|^2} f(t) e(\phi(t)) \right) = 2\pi i f(t) e(\phi(t)) + \nabla \cdot \left(\frac{\nabla \phi}{\|\nabla \phi\|^2} f(t) \right) e(\phi(t)). \tag{11.5}$$

We apply the above relation with $\phi(t) = \langle x, t \rangle$, where $x \neq 0$ is a given point in $\mathbb{R}^n$. Then $\nabla \phi$ is x and we have from (11.5) that

$$2\pi i f(t) e(\langle x, t \rangle) = \nabla \cdot \left(\frac{x}{\|x\|^2} f(t) e(\langle x, t \rangle) \right) - \frac{\langle x, \nabla f \rangle}{\|x\|^2} e(\langle x, t \rangle). \tag{11.6}$$

Let P denote any n-simplex in $\mathcal{K}$. We then integrate the above relation over P and sum over all n-simplexes P in $\mathcal{K}$. On now recalling that the support of f is in $|\mathcal{K}|$, that the interiors of distinct simplexes in $\mathcal{K}$ are disjoint and that the complement in $\mathcal{K}$ of the union of the n-simplexes in $\mathcal{K}$ is of measure zero, we conclude that

$$2\pi i \hat{f}(x) = \sum_{P \in \mathcal{K}} \int_P \nabla \cdot \left(\frac{x f(t)}{\|x\|^2} e(\langle x, t \rangle) \right) dt - \sum_{P \in \mathcal{K}} \int_P \frac{\langle x, \nabla f \rangle}{\|x\|^2} e(\langle x, t \rangle) \, dt. \tag{11.7}$$

The divergence theorem in the form given, for example, by Schwartz [**22**, Théorème 6.10.10, p. 365], applied to each P in $\mathcal{K}$, gives the following expression for the left-hand side of (11.7):

$$\sum_{P \in \mathcal{K}} \int_P \nabla \cdot \left(\frac{x f(t)}{\|x\|^2} e(\langle x, t \rangle) \right) dt = \sum_{P \in \mathcal{K}} \sum_{F \in \mathcal{F}(P)} \int_F \frac{\langle x, n_F \rangle}{\|x\|^2} f(t) e(\langle x, t \rangle) \, dS_F, \tag{11.8}$$

where, for each n-simplex in $\mathcal{K}$, we have written $\mathcal{F}(P)$ to denote the set of $(n-1)$-simplexes in $\mathcal{K}$ that are faces of P, and used n_F for the unit outward normal and dS_F for the surface measure of a given face F in $\mathcal{F}(P)$.

Let F be an $(n-1)$-simplex in $\mathcal{K}$. If F is a common face of n-simplexes P and P' and n_F is the unit outward normal to F as a face of P, then the unit outward normal to F as a face of P' is $-n_F$. We have already seen that F can be a common face of exactly two n-simplexes in $\mathcal{K}$. Since f is a continuous function, it follows that the sum of the contributions to the right-hand side of (11.8) from F that are common faces of pairs of n-simplexes is zero. If F is not the common face of two n-simplexes, then it must necessarily lie on the boundary of $|\mathcal{K}|$, where f vanishes, since it is continuous and its support is in $|\mathcal{K}|$. Thus, the left-hand side of (11.7) is zero. This proof is in fact the adaptation in several variables of the usual one concerning the Fejér kernel $(1 - |t|)^+$: to show that its Fourier transform is small enough, we use integration by parts and show that the integrated term vanishes because of continuity.

Turning to the second term on the right-hand side of (11.7), we note that since f is piecewise linear on $\mathcal{K}$, ∇f is a constant on each P. Moreover, the Cauchy–Schwarz inequality gives $|\langle x, \nabla f \rangle| \leqslant \|x\| \|\nabla f\| \leqslant M\|x\|$. It now follows from (11.7) and the triangle inequality that

$$2\pi |\hat{f}(x)| \leqslant \frac{M}{\|x\|} \sum_{P \in \mathcal{K}} \left| \int_P e(\langle x, t \rangle) \, dt \right|. \tag{11.9}$$

This equation yields a good bound for $\hat{f}(x)$ when x is not orthogonal to any P, or close to be so, but fails in that case. We need an L^1 estimate, and in that case, the measure of the set of 'bad cases' is small enough. We proceed to give a rigorous content to this remark. Integrating (11.9) over $\mathbb{R}^n$ and passing to polar co-ordinates, that is, recalling that $\mathbb{R}^n = \mathbb{R}_+ \times S^{n-1}(\mathbb{R})$ and applying Fubini's theorem, we deduce that

$$\int_{\mathbb{R}^n} \frac{\|x\|^n |\hat{f}(x)|}{\mathrm{Log}(2 + \|x\|)^{n+1}} \, dx \ll M \sum_{P \in \mathcal{K}} \int_0^\infty \frac{\lambda^{n-1}}{\mathrm{Log}(2 + \lambda)^{n+1}} \int_{S^{n-1}(\mathbb{R})} \left| \int_P e(\langle \lambda\sigma, t \rangle) \, dt \right| d\sigma \, d\lambda, \tag{11.10}$$

where $d\sigma$ is the surface measure on $S^{n-1}(\mathbb{R})$. From (ii), Theorem 2.1 of [**3**], as also from [**25**, pp. 124–125], we have, for any simplex P in $\mathcal{K}$, that

$$\int_{S^{n-1}(\mathbb{R})} \left| \int_P e(\langle \lambda\sigma, t \rangle) \, dt \right| d\sigma \ll \frac{\log(1 + \lambda)^{n-1}}{(1 + \lambda)^n}. \tag{11.11}$$

On substituting this bound into (11.10) and integrating over λ, we conclude the proof of the theorem except for one detail it is necessary to check that the method of proof in [**3**] gives us (11.11) with an implied constant that depends only on n. This is easily seen by following the argument in [**3**, pp. 257–259], which is an induction on n through an application of the divergence theorem to the integral over P in (11.11), and remarking that, for any $n \geqslant 1$, the left-hand side of (11.11) is invariant when P is transformed by affine isometries of $\mathbb{R}^n$. $\qquad\square$

12. Using a Fourier transform

We define

$$f_{t_1,t_2,\kappa}(u,v,w) = \max(0, \min(t_1 - |u| - |v|, t_2 - |u| - |w|))^\kappa, \tag{12.1}$$

and $f_{t_1,t_2} = f_{t_1,t_2,1}$. We expand f_{t_1,t_2} as a Fourier transform and we need to bound this Fourier transform uniformly in t_1 and t_2. It is possible to get such a proof on relying on the special form of f. Indeed Maxima [16] (which we have used via its graphical interface Wxmaxima [26]) tells us that (with $X = 2\pi x$, $Y = 2\pi y$, $Z = 2\pi z$ and $t_3 = t_2 - t_1$)

$$XYZ\hat{f}_{t_1,t_2}(x,y,z) = \frac{\begin{array}{c} 4Y(Z + X - Y)(Z + X + Y)\cos(t_3 Z + t_1 X) \\ -4Y(Z - X - Y)(Z - X + Y)\cos(t_3 Z - t_1 X) \\ +4X(Z - X - Y)(Z + X - Y)\cos(t_2 Z + t_1 Y) \\ -4X(Z - X + Y)(Z + X + Y)\cos(t_2 Z - t_1 Y) \end{array}}{(Z - X - Y)(Z + X + Y)(Z - X + Y)(Z + X - Y)} \tag{12.2}$$

and

$$\hat{f}_{t_1,t_2}(x,y,z) = \frac{\begin{array}{c} 16\cos(t_3 Z)\cos(t_1 X) - 16\cos(t_2 Z)\cos(t_1 Y) \\ -8(Z^2 + Y^2 - X^2)\sin(t_3 Z)\sin(t_1 X)/(ZX) \\ -8(Z^2 - Y^2 + X^2)\sin(t_2 Z)\sin(t_1 Y)/(ZY) \end{array}}{(Z - X - Y)(Z + X + Y)(Z - X + Y)(Z + X - Y)}. \tag{12.3}$$

This last expression has the advantage of showing clearly what happens when X, Y or Z is close to 0. Such a proof is however clumsy at best, rather lengthy and would not support any slight change in the initial function. In the course of this study, I formulated a conjecture that has been since proved by D.S. Ramana (personal communication) and whose proof we reproduced in Section 11. Having this at hand, the general case is readily handled.

LEMMA 12.1.   *We have*

$$\int_{x,y,z} |\hat{f}_{t_1,t_2,\kappa}(x,y,z)|(1 + |x|)^\alpha (1 + |y|)^\alpha (1 + |z|)^\alpha \, dx \, dy \, dz \ll 1,$$

provided $\alpha \in [0, 1/3)$, and uniformly in t_1 and t_2 belonging to $[0,1]^2$. Note, furthermore, that $f_{t_1,0,\kappa} = f_{0,t_2,\kappa} = 0$.

Proof.   We show that the convolution of any two functions that verify this inequality still verifies this inequality by induction, on appealing to the inequality

$$(1 + |h|)^\alpha \ll (1 + |u|)^\alpha + (1 + |h - u|)^\alpha.$$

The initial step is provided by Theorem 11.1 and the remark that we can bound $|\hat{f}_{t_1,t_2}(x,y,z)|$ by 1 when $x^2 + y^2 + z^2 \leqslant 1$.

$$(1 + |h|)^\alpha \ll (1 + |u|)^\alpha + (1 + |h - u|)^\alpha. \qquad \square$$

On expressing $f_{t_1,t_2,\kappa}$ in terms of its Fourier transform, we find that the function $\mathscr{G}^{(2)}_{b,c,d}(t_1, t_2, \varepsilon, \tau)$ is equal to

$$\int_{x,y,z} \hat{f}_{t_1,t_2,\kappa}(x,y,z) \int_{\varepsilon \leqslant u_1,\ldots,u_b \leqslant \tau} \int_{\varepsilon \leqslant v_1,\ldots,v_c \leqslant \tau} \int_{\varepsilon \leqslant w_1,\ldots,w_d \leqslant \tau}$$

$$= \prod_i \frac{e(xu_i)\, du_i}{u_i} \prod_i \frac{e(yv_i)\, dv_i}{v_i} \prod_i \frac{e(zw_i)\, dw_i}{w_i} \, dx \, dy \, dz$$

$$= \int_{x,y,z} \hat{f}_{t_1,t_2,\kappa}(x,y,z) E(\varepsilon,\tau,x)^b E(\varepsilon,\tau,y)^c E(\varepsilon,\tau,z)^d \, dx \, dy \, dz$$

with

$$E(\varepsilon, \tau, x) = \int_{\varepsilon \leqslant u \leqslant \tau} \frac{e(xu)\, du}{u} = \operatorname{Log} \frac{\tau}{\varepsilon} + E_0(\varepsilon, \tau, x), \tag{12.4}$$

where

$$E_0(\varepsilon, \tau, x) = \int_{\varepsilon \leqslant u \leqslant \tau} \frac{e(xu) - 1}{u}\, du. \tag{12.5}$$

The change of variable $u \mapsto v = xu$ shows that $E_0(\varepsilon, \tau, x)$ is bounded in absolute value by $\mathcal{O}(\operatorname{Log}(2 + |x|))$. Replacing E by E_0 yields the following formula:

$$\mathscr{G}_{b,c,d}^{(2)}(t_1, t_2, \varepsilon, \tau) = \int_{x,y,z} \hat{f}_{t_1, t_2, \kappa}(x, y, z) \sum_{\ell, m, n \geqslant 0} \binom{b}{\ell}\binom{c}{m}\binom{d}{n}$$
$$\left(\operatorname{Log} \frac{\tau}{\varepsilon}\right)^{b+c+d-\ell-m-n} \mathscr{E}_{\ell, m, n}(\varepsilon, \tau, x, y, z)\, dx\, dy\, dz$$

with the simplification

$$\mathscr{E}_{\ell, m, n}(\varepsilon, \tau, x, y, z) = E_0(\varepsilon, \tau, x)^\ell E_0(\varepsilon, \tau, y)^m E_0(\varepsilon, \tau, z)^n. \tag{12.6}$$

This leads to (since $H = \tau/\varepsilon$)

$$\mathfrak{G}_{\kappa, r}(t_1, t_2, \varepsilon, \tau) = \int_{x,y,z} \hat{f}_{t_1, t_2, \kappa}(x, y, z) \sum_{a+b+c+d=r} \frac{(-1)^b}{a!b!c!d!} \sum_{\ell, m, n \geqslant 0}$$
$$\binom{b}{\ell}\binom{c}{m}\binom{d}{n} \left(\operatorname{Log} \frac{\tau}{\varepsilon}\right)^{r-\ell-m-n} \mathscr{E}_{\ell, m, n}(\varepsilon, \tau, x, y, z)\, dx\, dy\, dz.$$

It is almost immediate to replace $E_0(\varepsilon, \tau, \cdot)$ by $E_0(0, \tau, \cdot)$, by using Lemma 12.1 and the bound

$$E_0(0, \tau, t) - E_0(\varepsilon, \tau, t) \ll \min(\varepsilon t, \operatorname{Log}(2 + |t|)).$$

We also need to note that, when $H \geqslant 1$,

$$\int_{\substack{x,y,z, \\ |x| \geqslant H}} |\hat{f}_{t_1, t_2}(x, y, z)| (1 + |x|)^{1/4} \operatorname{Log}(2 + |x|)\, dx\, dy\, dz \ll 1,$$

which gives us the bound

$$\int_{\substack{x,y,z, \\ |x| \geqslant H}} |\hat{f}_{t_1, t_2}(x, y, z)| \operatorname{Log}(2 + |x|)\, dx\, dy\, dz \ll 1/H^{1/4}.$$

The same bound holds true with the condition $|x| \geqslant H$ replaced by $|y| \geqslant H$ or $|z| \geqslant H$. We thus get

$$\int_{x,y,z} \hat{f}_{t_1, t_2, \kappa}(x, y, z)(E_0(\varepsilon, \tau, x) - E_0(0, \tau, x))\mathscr{E}_{\ell-1, m, n}(\varepsilon, \tau, x, y, z)\, dx\, dy\, dz$$
$$\ll \int_{\substack{x,y,z, \\ |x| \leqslant H}} + \int_{\substack{x,y,z, \\ |x| \geqslant H}} \cdots$$
$$\ll \int_{\substack{x,y,z, \\ |x| \leqslant H}} \varepsilon H^{3/4} |x|^{1/4} |\hat{f}_{t_1, t_2, \kappa}(x, y, z) \mathscr{E}_{\ell-1, m, n}(\varepsilon, \tau, x, y, z)|\, dx\, dy\, dz$$
$$+ \int_{\substack{x,y,z, \\ |x| \geqslant H}} \operatorname{Log}(2 + |x|)|\hat{f}_{t_1, t_2, \kappa}(x, y, z) \mathscr{E}_{\ell-1, m, n}(\varepsilon, \tau, x, y, z)|\, dx\, dy\, dz$$
$$\ll \varepsilon H^{3/4} + H^{-1/4}.$$

We select $H = 1/\varepsilon$ and repeat the argument for each factor. This leads to

$$
\mathfrak{G}_{\kappa,r}(t_1, t_2, \varepsilon, \tau) = \int_{x,y,z} dx\, dy\, dz \hat{f}_{t_1,t_2,\kappa}(x, y, z) \sum_{a+b+c+d=r} \frac{(-1)^b}{a!b!c!d!} \sum_{\ell,m,n \geqslant 0}
$$
$$
\binom{b}{\ell}\binom{c}{m}\binom{d}{n}\left(\operatorname{Log}\frac{\tau}{\varepsilon}\right)^{r-\ell-m-n} \mathscr{E}_{\ell,m,n}(0, \tau, x, y, z) + \mathcal{O}((\operatorname{Log}\varepsilon)^r \varepsilon^{1/4}).
$$
$$(12.7)$$

There cannot be any dependence of the main term in ε. It means that each coefficient of $\operatorname{Log}(\tau/\varepsilon)^d$ for positive d has to vanish. This is not so obvious because of two obstructions: P_0 has to be somewhat large but restricted in size with respect to P; furthermore, the limit has to be taken in some L^p-sense. We treat both problems one after another in the next section.

13. A divertimento

We have found that

$$
\int_{0 \leqslant t_1, t_2 \leqslant 1} w'(t_1) w'(t_2) \mathfrak{G}_{\kappa,r}(t_1, t_2, \varepsilon, \tau)\, dt_1\, dt_2 = \frac{S^{(4)} Z}{(-1)^r A}
$$
$$
+ \mathcal{O}_B\left(\frac{(\operatorname{Log}\operatorname{Log} P)^r}{R^B} + \frac{\operatorname{Log}(\tau/\varepsilon)^{2r}}{P_0} + \alpha^2 (\operatorname{Log}\operatorname{Log} P)^r (\operatorname{Log} P_0)^2 + \frac{1}{R}\right).
$$

Let us constrain τ to be bounded: $1 \leqslant \tau \leqslant T$. We bound above $1/P_0$ by $1/R^B$, and H by $\operatorname{Log} P$, then replace $\operatorname{Log} P_0$ by $\varepsilon \operatorname{Log} Q$. We set $q' = \operatorname{Log} Q$, select $B = 1$ and get that the error term above is

$$
\ll_T \frac{(\operatorname{Log} q')^r}{\varepsilon q'} + \varepsilon^2 (\operatorname{Log} q')^r + \frac{1}{\varepsilon q'} \ll_T \frac{1}{\varepsilon q} + \varepsilon^2 (\operatorname{Log} q)^r \ll_T \frac{1}{\varepsilon q} + \varepsilon^2 q
$$

with $q = q'/(\operatorname{Log} q')^r$. We have thus at our disposal two functions f and g such that

$$
|f(\varepsilon) - g(q)| \ll \frac{1}{\varepsilon q} + \varepsilon^2 q,
$$
$$(13.1)$$

when q goes to infinity and ε goes to 0. We infer below from this hypothesis that f has a limit when ε goes to 0.

LEMMA 13.1.   *Let $\delta > 0$. When $1/\delta \leqslant \varepsilon q \leqslant \delta/\varepsilon$, we have*

$$
\frac{1}{\varepsilon q} + \varepsilon^2 q \leqslant 2\delta.
$$

Proof.   Set $x = \varepsilon q$. The last inequality is verified if and only if

$$
1 - 2\delta x + \varepsilon x^2 \leqslant 0.
$$

The discriminant of this trinomial is $4\delta^2 - 4\varepsilon \geqslant 0$, for otherwise the interval we consider is empty. The variable x has to lie between the two roots

$$
\frac{\delta - \sqrt{\delta^2 - \varepsilon}}{\varepsilon} = \frac{1}{\delta + \sqrt{\delta^2 - \varepsilon}} \leqslant \frac{1}{\delta} \quad \text{and} \quad \frac{\delta}{\varepsilon} \leqslant \frac{\delta + \sqrt{\delta^2 - \varepsilon}}{\varepsilon}.
$$

The lemma follows readily. $\square$

LEMMA 13.2.   *When $\varepsilon^2/\delta^2 \leqslant \varepsilon' \leqslant \varepsilon \leqslant 1$, we have $|f(\varepsilon) - f(\varepsilon')| \leqslant 4\delta$.*

Proof. Let $q = \delta/\varepsilon^2$. We have $|f(\varepsilon) - g(q)| \leqslant 2\delta$. On the other hand, $1/\delta \leqslant \varepsilon' q \leqslant \delta/\varepsilon'$, and thus $|f(\varepsilon') - g(q)| \leqslant 2\delta$. The triangle inequality concludes. $\square$

We select $\delta = 2\sqrt{\varepsilon}$. Let $\varepsilon' \in]0, \varepsilon]$. There exists a non-negative integer k such that $4^{-k-1}\varepsilon < \varepsilon' \leqslant 4^{-k}\varepsilon$. We write the difference $f(\varepsilon') - f(\varepsilon)$ as

$$f(\varepsilon') - f(4^{-k}\varepsilon) + f(4^{-(k-1)}\varepsilon) - f(4^{-(k-2)}\varepsilon) + \ldots + f(\varepsilon/4) - f(\varepsilon).$$

We can thus bound the difference by $8(2^{-k} + \ldots + 1)\sqrt{\varepsilon} \leqslant 16\sqrt{\varepsilon}$. The function f verifies Cauchy's criterion in the neighbourhood of 0 and thus admits a limit there, as required.

As a conclusion, we can restrict the summation in (12.7) to $r + c - \ell - m - n = 0$. This means $a = 0$, $\ell = b$, $m = c$ and $n = d$, and thus we have

$$\int_{0 \leqslant t_1, t_2 \leqslant 1} w'(t_1)w'(t_2)\mathfrak{G}_{\kappa,r}(t_1, t_2, \varepsilon, \tau) \, dt_1 \, dt_2 = \int_{0 \leqslant t_1, t_2 \leqslant 1} w'(t_1)w'(t_2) \int_{x,y,z} dx \, dy \, dz$$
$$\hat{f}_{t_1,t_2,\kappa}(x, y, z) \sum_{b+c+d=r} \frac{(-1)^b}{b!c!d!}\mathscr{E}_{b,c,d}(0, \tau, x, y, z)$$
$$+ \mathcal{O}((\mathrm{Log}\,\varepsilon)^r \varepsilon^{1/4}).$$

This equality is valid for every w' bounded, and say continuous (measurable would do). We have assumed w' to be non-positive only to have an easy bound on λ_d, but this hypothesis is otherwise irrelevant. By polarization, we extend it to $w_1'(t_1)w_2'(t_2)$ for any two distinct functions w_1' and w_2'. The relation holds also for linear combinations of such (elementary tensor) products, which happen to be dense in $L^2([0,1]^2)$. Indeed, each monomial is an elementary tensor product and the Weierstrass approximation theorem applies; this enables us to extend our relation from $w_1' \otimes w_2'$ to any continuous function over $[0,1]^2$. Such functions are dense in any L^p for $1 \leqslant p \leqslant \infty$ and this is enough to show that

$$\mathfrak{G}_{\kappa,r}(t_1, t_2, \epsilon, \tau) + \mathcal{O}((\mathrm{Log}\,\varepsilon)^r \varepsilon^{1/4})$$
$$= \int_{x,y,z} \hat{f}_{t_1,t_2,\kappa}(x, y, z) \sum_{b+c+d=r} \frac{(-1)^{c+d}}{b!c!d!}\mathscr{E}_{b,c,d}(0, \tau, x, y, z) \, dx \, dy \, dz. \qquad (13.2)$$

We have finally reached the existence of the following function:

$$\mathfrak{G}_{\kappa,r}(t_1, t_2, \tau) = \lim_{\epsilon \to 0} \mathfrak{G}_{\kappa,r}(t_1, t_2, \epsilon, \tau)$$
$$= \int_{x,y,z} \hat{f}_{t_1,t_2,\kappa}(x, y, z) \sum_{b+c+d=r} \frac{(-1)^{c+d}}{b!c!d!}\mathscr{E}_{b,c,d}(0, \tau, x, y, z) \, dx \, dy \, dz. \qquad (13.3)$$

This expression has also the following consequences.

LEMMA 13.3. *The function* $\mathfrak{G}_{\kappa,r}(t_1, t_2, \tau)$ *is symmetrical in* t_1 *and* t_2. *Moreover, we have* $\mathfrak{G}_{\kappa,r}(0, t_2, \tau) = 0$.

14. A general formula for weights carried on integers with r prime factors

We can take the limit as ε goes to 0 in (13.3) and eliminate the Fourier transform, obtaining

$$\mathfrak{G}_{\kappa,r}(t_1, t_2, \tau) = \lim_{\epsilon \to 0} \sum_{b+c+d=r} \sum_{\substack{\mathcal{B} \subset \{1,\ldots,b\}, \\ \mathcal{C} \subset \{1,\ldots,c\}, \\ \mathcal{D} \subset \{1,\ldots,d\}}} \frac{(-1)^{b+|\mathcal{B}|+|\mathcal{C}|+|\mathcal{D}|}}{b!c!d!} \int_{\substack{\epsilon \leqslant u_1,\ldots,u_b \leqslant \tau, \\ \epsilon \leqslant v_1,\ldots,v_c \leqslant \tau, \\ \epsilon \leqslant w_1,\ldots,w_d \leqslant \tau}}$$

$$\frac{(\min(t_1 - \sum_{i\in\mathcal{C}} v_i, t_2 - \sum_{i\in\mathcal{D}} w_i) - \sum_{i\in\mathcal{B}} u_i)^{+\kappa} \prod_i du_i \prod_i dv_i \prod_i dw_i}{\prod_i u_i \prod_i v_i \prod_i w_i}. \tag{14.1}$$

This formula *is* explicit. We can also take the limit as ϵ goes to 0 in (13.3) (replace ε by ϵ), or, after the same manipulation, in (10.4). It also reveals the following property.

LEMMA 14.1. We have $\mathfrak{G}_{\kappa,r}(t_1, t_2, \tau) = \tau^\kappa \mathfrak{G}_{\kappa,r}(t_1/\tau, t_2/\tau, 1)$.

This leads to the main theorem, namely, the following.

THEOREM 14.1. We have, when $P = Q^\tau$, and with S_0 being defined in (3.2),

$$(\mathrm{Log}\, Q)^\kappa S_0(a_r[P]; \mathcal{K}, \mathcal{K}^*)$$
$$= A\kappa'^r \int_{0 \leqslant t_1, t_2 \leqslant 1} w'(t_1) w'(t_2) \mathfrak{G}_{\kappa,r}(t_1, t_2, \tau)\, dt_1\, dt_2 + \mathcal{O}(\|w'\|_\infty^2 (\mathrm{Log}\, Q)^{-1/9}),$$

where $\mathfrak{G}_{\kappa,r}$ is the bounded continuous function given above.

The error term $\mathcal{O}((\mathrm{Log}\, Q)^{-1/9})$ is achieved by selecting $\mathrm{Log}\, P_0 = \sqrt{\mathrm{Log}\, Q}$. It is uniform in τ between 1 and some fixed bound, say T (the error term depending on T). It depends on $\mathcal{K}$ and on $\mathcal{K}^*$. We have not tried to optimize the exponent $1/9$.

CONJECTURE 14.1. The function $\mathfrak{G}_{\kappa,r}$ is twice continuously differentiable in the domain $0 \leqslant t_1 \leqslant t_2 \leqslant \tau$.

We can split the domain of integration in (14.1), once $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ are fixed, into subdomains in which the expression $(\min(t_1 - \sum_{j\in\mathcal{C}} v_j, t_2 - \sum_{k\in\mathcal{D}} w_k) - \sum_{i\in\mathcal{B}} u_i)^{+\kappa}$ reduces to a polynomial. This implies conditions like

$$\sum_{i\in\mathcal{B}} u_i \leqslant t_2 - \sum_{k\in\mathcal{D}} w_k \leqslant t_1 - \sum_{j\in\mathcal{C}} v_j.$$

In front of such a condition, we would fix $(u_i)_{i\in\mathcal{B}}$, then $(w_k)_{k\in\mathcal{D}}$ and finally $(v_j)_{j\in\mathcal{C}}$. The remaining variables are in fact independent. We would then (try to) resolve each singularity by using Lemma 20.1. This is the process we will follow when $r = 0, 1$ and 2. If this programme carries through, then the conjecture would be true. But we are not sure additional conditions would not pop in. Note that a discussion as to whether $t_1 \leqslant t_2$ comes in immediately when $\mathcal{D} = \mathcal{C} = \emptyset$. It is also necessary to handle the second condition above:

$$t_2 - t_1 + \sum_{j\in\mathcal{C}} v_j \leqslant -\sum_{k\in\mathcal{D}} w_k.$$

When $t_2 - t_1 < 0$, this condition would compete with the non-negativity conditions on $(w_k)_{k\in\mathcal{D}}$. This discussion justifies the conjecture.

Note that the expression (14.1) implies that this function is twice continuously differentiable per pieces. Assuming this conjecture and recalling that $\mathfrak{G}_{\kappa,r}(0,t_2,\tau)$ vanishes, we find that

$$\int_{0\leqslant t_1\leqslant t_2\leqslant 1} w'(t_1)w'(t_2)\mathfrak{G}_{\kappa,r}(t_1,t_2,\tau)\,dt_1\,dt_2$$
$$= \int_0^1 w'(t_2)\left[0 + \int_0^{t_2}(w(t_2)-w(t_1))\frac{d\mathfrak{G}_{\kappa,r}(t_1,t_2,\tau)}{dt_1}\,dt_1\right]dt_2$$
$$= \int_0^1 w'(t_2)\int_0^{t_2}(w(t_2)-w(t_1))\frac{d\mathfrak{G}_{\kappa,r}(t_1,t_2,\tau)}{dt_1}\,dt_1\,dt_2.$$

We use a second integration by parts, with respect to t_2, and integrate the factor $w'(t_2)(w(t_2) - w(t_1))$ as $\frac{1}{2}(w(t_2) - w(t_1))^2$. On using the fact that $w(1) = 0$, we reach by this way a main formula:

$$\boxed{\begin{aligned}\int_{0\leqslant t_1,t_2\leqslant 1} & w'(t_1)w'(t_2)\mathfrak{G}_{\kappa,r}(t_1,t_2,\tau)\,dt_1\,dt_2 \\ &= \int_0^1 w(t_1)^2\frac{d\mathfrak{G}_{\kappa,r}(t_1,1,\tau)}{dt_1}\,dt_1 \\ &\quad -\frac{1}{2}\int_{0\leqslant t_1,t_2\leqslant 1}(w(t_2)-w(t_1))^2\frac{d^2\mathfrak{G}_{\kappa,r}(t_1,t_2,\tau)}{dt_1\,dt_2}\,dt_1\,dt_2.\end{aligned}} \qquad (14.2)$$

14.1. Proof of Theorem 1.2

We restrict our attention to $\mathcal{K} = \mathcal{K}(h_1,\ldots,h_\kappa)$. As already noted at the end of Section 2, the constants A and $\mathscr{C}$ are then equal and strictly positive. We select, for each i in $\{1,\ldots,\kappa\}$, the additional compact set $\mathcal{K}^* = \mathcal{U} - h_i$, so that $\kappa' = 1$. Theorem 1.2 is thus simply a rewriting in this special case of Theorem 14.1 since

$$S_i(a_r[Q^\tau]) = S(a_r[Q^\tau];\mathcal{K}(h_1,\ldots,h_\kappa),\mathcal{U} - h_i).$$

15.   On a generalization of the Euler beta function

We consider the family of functions defined, for every $k \geqslant 0$ and every real number $a,b > 0$, by

$$B_k(a,b) = (-1)^k\int_0^1 t^{a-1}(1-t)^{b-1}\operatorname{Log}^k(1-t)\,dt \geqslant 0. \qquad (15.1)$$

We need to evaluate this function for $k = 1$ and $k = 2$ but it is easier to take a general path. When $k = 0$, we have

$$B(a,b) = B_0(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}. \qquad (15.2)$$

We note that for non-negative $z < b$, by the Lebesgue dominated convergence theorem, we have

$$\sum_{k\geqslant 0}\frac{B_k(a,b)z^k}{k!} = \int_0^1 t^{a-1}(1-t)^{b-1}\sum_{k\geqslant 0}\frac{(-z\operatorname{Log}(1-t))^k}{k!}\,dt = B(a,b-z). \qquad (15.3)$$

As a consequence, we find the following lemma.

LEMMA 15.1.

$$B_k(a,b) = \frac{d^k}{dz^k}B(a,b-z)\bigg|_{z=0}.$$

We recall the definition of the digamma function:

$$\psi(x) = \Gamma'(x)/\Gamma(x). \tag{15.4}$$

We recall the following lemma that we take from [**1**, 6.3.2, 6.4.2, 6.4.3].

LEMMA 15.2.   *When $m \geqslant 1$ and $n \geqslant 1$ are integers, we have*

$$\psi^{(m)}(n) = (-1)^{m+1} m! \left(-\zeta(m+1) + \sum_{1 \leqslant c \leqslant n-1} \frac{1}{c^{m+1}} \right).$$

In case $m = 0$, the same formula holds but with $-\gamma$ instead of $-\zeta(1)$.

Lemma 15.1 leads to the following recursive formula.

LEMMA 15.3.   *When $k \geqslant 0$, we have*

$$B_{k+1}(a, b) = \sum_{0 \leqslant m \leqslant k} \binom{k}{m} B_{k-m}(a, b)(-1)^{m+1}(\psi^{(m)}(a+b) - \psi^{(m)}(b)).$$

Proof.   We appeal to Lemma 15.1 to infer that

$$\frac{B_1(a, b-z)}{B(a, b-z)} = \psi(a+b-z) - \psi(b+z).$$

We then use the Leibniz law concerning the kth derivative of a product and evaluate the resulting expression at $z = 0$ to get our lemma.   □

As a consequence, we find the following.

LEMMA 15.4.   *When a and b are positive integers, we have*

$$B_1(a, b) = (\mathcal{H}_{a+b-1} - \mathcal{H}_{b-1})B(a, b).$$

LEMMA 15.5.   *When a and b are positive integers, we have*

$$B_2(a, b) = \left((\mathcal{H}_{a+b-1} - \mathcal{H}_{b-1})^2 + \sum_{b \leqslant c \leqslant a+b-1} \frac{1}{c^2} \right) B(a, b).$$

Proof.   We have, by Lemma 15.3,

$$B_2(a, b) = B_1(a, b)(\mathcal{H}_{a+b-1} - \mathcal{H}_{b-1}) + B(a, b) \sum_{b \leqslant c \leqslant a+b-1} \frac{1}{c^2},$$

and the lemma follows readily.   □

16. Derivatives with respect to τ

On using (10.4) and (10.2), we readily find that

$$\tau \frac{d\mathfrak{G}_{\kappa,r}(t_1, t_2, \epsilon, \tau)}{d\tau} = \mathfrak{G}_{\kappa,r-1}(t_1, t_2, \epsilon, \tau) - \mathfrak{G}_{\kappa,r-1}(t_1 - \tau, t_2, \epsilon, \tau)$$

$$- \mathfrak{G}_{\kappa,r-1}(t_1, t_2 - \tau, \epsilon, \tau) + \mathfrak{G}_{\kappa,r-1}(t_1 - \tau, t_2 - \tau, \epsilon, \tau), \qquad (16.1)$$

and the same holds with $\epsilon = 0$ which is (1.3). This gives us the following formula:

$$\int_{t_1,t_2} w'(t_1)w'(t_2) \frac{d\mathfrak{G}_{\kappa,r}(t_1, t_2, \tau)}{d\tau} \, dt_1 \, dt_2$$

$$= \int_{t_1,t_2} (w'(t_1) - w'(t_1 + \tau))(w'(t_2) - w'(t_2 + \tau)) \frac{\mathfrak{G}_{\kappa,r-1}(t_1, t_2, \tau)}{\tau} \, dt_1 \, dt_2. \qquad (16.2)$$

We can then apply (14.2) or (18.2) when $r - 1 = 1$. There are two ways to introduce the derivative with respect to κ. The first one, which initiated this line of enquiries, consists in adding a differentiable weight $W(\mathrm{Log}\max(p_1, \ldots, p_r))$ to $a_r[P]$. We then see that we can let this weight approximate the step function $1_{x \leqslant \mathrm{Log}\, P}$. There is, however, a much simpler way by directly noting that

$$\mathfrak{G}_{\kappa,r}(t_1, t_2, \tau_0) = \int_0^{\tau_0} \frac{d\mathfrak{G}_{\kappa,r}(t_1, t_2, \tau)}{d\tau}$$

since $\mathfrak{G}_{\kappa,r}(t_1, t_2, 0) = 0$.

17. Special case $r = 0$

We use (10.3) together with (10.4) and (10.2) and get

$$\mathfrak{G}_{\kappa,0}(t_1, t_2, \tau) = \min(t_1, t_2)^\kappa. \qquad (17.1)$$

We find that

$$\int_{0 \leqslant t_1, t_2 \leqslant 1} w'(t_1)w'(t_2)\mathfrak{G}_{\kappa,0}(t_1, t_2, \tau) \, dt_1 \, dt_2 = \kappa \int_0^1 w(t)^2 t^{\kappa-1} \, dt. \qquad (17.2)$$

Proof. The following lines are straightforward:

$$\int_{0 \leqslant t_1 \leqslant t_2 \leqslant 1} w'(t_1)w'(t_2)t_1^\kappa \, dt_1 \, dt_2 = \int_0^{t_1} w'(t_1) \int_{t_1}^1 w'(t_2) \, dt_2 t_1^\kappa \, dt_1$$

$$= -\int_0^{t_1} w'(t_1)w(t_1)t_1^\kappa \, dt_1 = \frac{\kappa}{2} \int_0^{t_1} w(t_1)^2 t_1^{\kappa-1} \, dt_1,$$

as required. $\qquad\qquad\qquad\square$

18. Special case $r = 1$

A general expression. We again use (10.3) together with (10.4) and (10.2) and get, when $t_1 \leqslant t_2$,

$$\mathfrak{G}_{\kappa,1}(t_1, t_2, \varepsilon, \tau) = t_1^\kappa \mathrm{Log}(\tau/\varepsilon) - \int_\varepsilon^\tau \frac{\max(0, \min(t_1, t_2 - w))^\kappa \, dw}{w}.$$

On discussing the size of τ and letting ε go to zero, we get the following expression:

$$\mathfrak{G}_{\kappa,1}(t_1, t_2, \tau) = t_1^\kappa \mathrm{Log} \frac{\tau}{t_2 - t_1} - \int_{t_2-t_1}^{\max(\tau, t_2)} \frac{(t_2 - w)^\kappa \, dw}{w}, \qquad (18.1)$$

where the integral is *not* taken in the algebraical sense, that is, it vanishes when $\tau < t_2 - t_1$.

An integral form for the associated distribution. We can eliminate $\mathfrak{G}_{\kappa,1}$ altogether from our expressions by appealing to (14.2):

$$
\boxed{
\begin{aligned}
\int_{0\leqslant t_1,t_2\leqslant 1} & w'(t_1)w'(t_2)\mathfrak{G}_{\kappa,1}(t_1,t_2,\tau)\,dt_1\,dt_2/\kappa \\
&= \int_{\substack{0\leqslant t<1,\\1-t\leqslant\tau}} w(t)^2 t^{\kappa-1}\operatorname{Log}\frac{\tau}{1-t}\,dt \\
&\quad + \int_{\substack{0\leqslant t_1<t_2\leqslant 1,\\t_2-t_1\leqslant\tau}} \frac{(w(t_2)-w(t_1))^2}{t_2-t_1}t_1^{\kappa-1}\,dt_1\,dt_2.
\end{aligned}
}
\tag{18.2}
$$

Note that the right-hand side is a sum of two non-negative terms, so that no cancellations between them may occur. Note further that we have added the condition $t_1 < t_2$ on the right-hand side while no such condition appears on the left-hand side.

Proof. We use (18.1) and readily get

$$
\frac{d\mathfrak{G}_{\kappa,1}(t_1,1,\tau)}{dt_1} = \kappa t_1^{\kappa-1}\operatorname{Log}\frac{\tau}{1-t_1},
$$

as well as

$$
\frac{d^2\mathfrak{G}_{\kappa,1}(t_1,1,\tau)}{dt_1\,dt_2} = \frac{\kappa t_1^{\kappa-1}}{t_2-t_1},
$$

when $t_2 - t_1 \leqslant \tau$ and vanishes otherwise. $\qquad\square$

19. *Special case $r = 2$*

An integral form for the associated distribution. We first note that, when $\tau_0 > 0$,

$$
\begin{aligned}
\int_{0\leqslant t_1,t_2\leqslant 1} & w'(t_1)w'(t_2)\mathfrak{G}_{\kappa,2}(t_1,t_2,\tau_0)\,dt_1\,dt_2 \\
&= \int_0^{\tau_0}\int_{0\leqslant t_1,t_2\leqslant 1} w'(t_1)w'(t_2)\frac{d\mathfrak{G}_{\kappa,2}(t_1,t_2,\tau)}{d\tau}\,dt_1\,dt_2\,d\tau.
\end{aligned}
$$

At this stage, we employ (16.2):

$$
\begin{aligned}
\int_{0\leqslant t_1,t_2\leqslant 1} & w'(t_1)w'(t_2)\mathfrak{G}_{\kappa,2}(t_1,t_2,\tau_0)\,dt_1\,dt_2 \\
&= \int_0^{\tau_0}\int_{0\leqslant t_1,t_2\leqslant 1} (w'(t_1)-w'(t_1+\tau))(w'(t_2)-w'(t_2+\tau))\mathfrak{G}_{\kappa,1}(t_1,t_2,\tau)\,dt_1\,dt_2\frac{d\tau}{\tau}.
\end{aligned}
$$

We then use (18.2) and reach

$$
\boxed{
\begin{aligned}
\int_{0\leqslant t_1,t_2\leqslant 1} & w'(t_1)w'(t_2)\mathfrak{G}_{\kappa,2}(t_1,t_2,\tau_0)\,dt_1\,dt_2/\kappa \\
&= \int_0^{\tau_0}\int_{\substack{0\leqslant t<1,\\1-t\leqslant\tau}} (w(t)-w(t+\tau))^2 t^{\kappa-1}\operatorname{Log}\frac{\tau}{1-t}\,dt\frac{d\tau}{\tau} \\
&\quad + \int_0^{\tau_0}\int_{\substack{0\leqslant t_1<t_2\leqslant 1,\\t_2-t_1\leqslant\tau}} \frac{(w(t_2)-w(t_2+\tau)-w(t_1)+w(t_1+\tau))^2}{t_2-t_1}t_1^{\kappa-1}\,dt_1\,dt_2\frac{d\tau}{\tau}.
\end{aligned}
}
\tag{19.1}
$$

20. A useful lemma

The following is a trivial lemma that we will use over and over again.

LEMMA 20.1. *We have*

$$\frac{(x-y)^{\kappa} - x^{\kappa}}{y} = - \sum_{1 \leqslant \ell \leqslant \kappa} (x-y)^{\kappa-\ell} x^{\ell-1}.$$

This lemma is one of the reasons why we have to assume κ to be an integer. When r is 0, 1 or 2, this is the sole reason. When r is arbitrary, we represent in Section 12 a function depending on κ as a convolution product of κ factors, and this again requires κ to be an integer.

Proof. Indeed, we first note that

$$\sum_{1 \leqslant \ell \leqslant \kappa} u^{\kappa-\ell+1} v^{\ell-1} = u^{\kappa} \frac{1 - (v/u)^{\kappa}}{1 - (v/u)} = u \frac{u^{\kappa} - v^{\kappa}}{u - v},$$

where we select $u = x - y$ and $v = y$. $\qquad\square$

21. Special case $w(t) = (1 - t)^{+\nu}$ and $r \leqslant 2$.

We select here $w(t) = (1 - t)^{+\nu}$ and compute the main terms when $r = 0, 1$ and 2. Let us start with a simple consequence of (17.2).

LEMMA 21.1. *When* $w(t) = (1 - t)^{+\nu}$, *with* $\nu \geqslant 1$, *and* $\tau > 0$, *we have*

$$\nu^2 \int_{0 \leqslant t_1, t_2 \leqslant 1} (1 - t_1)^{\nu-1} (1 - t_2)^{\nu-1} \frac{\mathfrak{G}_{\kappa,0}(t_1, t_2, \tau)}{\kappa} \, dt_1 \, dt_2 = B(2\nu + 1, \kappa).$$

In case $r = 1$, we have formula (18.2).

LEMMA 21.2. *When* $w(t) = (1 - t)^{+\nu}$ *and* $\tau \geqslant 1$, *we have*

$$\nu^2 \int_{0 \leqslant t_1, t_2 \leqslant 1} (1 - t_1)^{\nu-1} (1 - t_2)^{\nu-1} \frac{\mathfrak{G}_{\kappa,1}(t_1, t_2, \tau)}{\kappa} \, dt_1 \, dt_2 = (\mathcal{H}_{\kappa+2\nu} - 2\mathcal{H}_{2\nu} + 2\mathcal{H}_{\nu}) B(2\nu + 1, \kappa).$$

Proof. Let us call momentarily I the integral to compute. We start with (18.2) and get

$$I = B_1(\kappa, 2\nu - 1) + \int_{0 \leqslant t_1 \leqslant t_2 \leqslant 1} \frac{((1 - t_1)^{\nu} - (1 - t_2)^{\nu})^2}{t_2 - t_1} t_1^{\kappa-1} \, dt_2 \, dt_1.$$

We then expand one $(1 - t_1)^{\nu} - (1 - t_2)^{\nu}$ via Lemma 20.1:

$$I = B_1(\kappa, 2\nu + 1) + \sum_{1 \leqslant k \leqslant \nu} \int_{0 \leqslant t_1 \leqslant t_2 \leqslant 1} (1 - t_1)^{2\nu-k} (1 - t_2)^{k-1} t_1^{\kappa-1} \, dt_2 \, dt_1$$

$$- \sum_{1 \leqslant k \leqslant \nu} \int_{0 \leqslant t_1 \leqslant t_2 \leqslant 1} (1 - t_1)^{\nu-k} (1 - t_2)^{\nu+k-1} t_1^{\kappa-1} \, dt_2 \, dt_1$$

$$= (\mathcal{H}_{2\nu+\kappa} - 2\mathcal{H}_{2\nu} + 2\mathcal{H}_{\nu}) B(2\nu + 1, \kappa),$$

on using Lemma 15.4. The lemma follows readily. $\qquad\square$

We reach the main difficulty of this part, whose proof will take much of our energy. The reader will see that the proof is lengthy but essentially straightforward.

LEMMA 21.3.   *When $w(t) = (1-t)^{+\nu}$, we have*

$$\nu^2 \int_{0 \leqslant t_1, t_2 \leqslant 1} (1-t_2)^{\nu-1}(1-t_1)^{\nu-1} \frac{\mathfrak{G}_{\kappa,2}(t_1, t_2, 1)}{\kappa \cdot B(2\nu+1, \kappa)} \, dt_1 \, dt_2$$
$$= \frac{1}{2} \mathcal{H}_{\kappa+2\nu}(\mathcal{H}_{\kappa+2\nu} - 4\mathcal{H}_{2\nu} + 4\mathcal{H}_\nu) + \mathcal{O}(\mathrm{Log}(2\nu)).$$

The proof starts by (19.1). We then appeal to (22.1) and (22.7), which are proved in next section. We have already noted that $K(\nu) = \frac{1}{2} \mathrm{Log}^2 \nu + \mathcal{O}(\mathrm{Log}\,\nu)$. We shuffle the terms around and get the lemma.

22.   *Proof of Lemma 21.3 : usage of* (19.1) *when $\tau_0 = 1$ and $r = 2$*

We use (19.1) with $\tau_0 = 1$ and $w(t) = (1-t)^{+\nu}$. In the first term, the condition $1 \leqslant t + \tau$ implies that $-w(t+\tau) = 0$. On using Lemmas 21.1 and 21.2, we find that this first term equals $\frac{1}{2} B_2(\kappa, 2\nu+1)$; that is, by Lemma 15.5,

$$\int_0^1 \int_{\substack{0 \leqslant t < 1, \\ 1-t \leqslant \tau}} ((1-t)^\nu - (1-t-\tau)^{+\nu})^2 t^{\kappa-1} \, \mathrm{Log}\, \frac{\tau}{1-t} \, dt \frac{d\tau}{\tau}$$
$$= \frac{1}{2} \left((\mathcal{H}_{2\nu+\kappa} - \mathcal{H}_{2\nu})^2 + \sum_{2\nu+1 \leqslant c \leqslant 2\nu+\kappa} \frac{1}{c^2} \right) B(2\nu+1, \kappa). \tag{22.1}$$

Concerning the second term, there are several ranges to consider, according to the location of $1-\tau$ with respect to t_1 and t_2.

22.1.   *When $t_2 \leqslant 1-\tau$*

Remember that we should also have $t_2 \leqslant t_1 + \tau$. We use Lemma 20.1 and the decomposition

$$\frac{(1-t_2)^\nu - (1-t_2-\tau)^\nu}{\tau} - \frac{(1-t_1)^\nu - (1-t_1-\tau)^\nu}{\tau}$$
$$= \sum_{1 \leqslant k \leqslant \nu} \left[(1-t_2)^{\nu-k}(1-t_2-\tau)^{k-1} - (1-t_1)^{\nu-k}(1-t_1-\tau)^{k-1} \right]$$

as well as

$$\frac{(1-t_2)^\nu - (1-t_1)^\nu}{t_2 - t_1} - \frac{(1-t_2-\tau)^\nu - (1-t_1-\tau)^\nu}{t_2 - t_1}$$
$$= -\sum_{1 \leqslant \ell \leqslant \nu} [(1-t_2)^{\nu-\ell}(1-t_1)^{\ell-1} - (1-t_2-\tau)^{\nu-\ell}(1-t_1-\tau)^{\ell-1}].$$

We multiply both and get this product to equal $-(A_1 - A_2 - A_3 + A_4)$ where

$$A_1 = \sum_{1 \leqslant k, \ell \leqslant \nu} (1-t_2)^{2\nu-k-\ell}(1-t_2-\tau)^{k-1}(1-t_1)^{\ell-1}, \tag{22.2}$$

$$A_2 = \sum_{1 \leqslant k, \ell \leqslant \nu} (1-t_1)^{\nu-k+\ell-1}(1-t_1-\tau)^{k-1}(1-t_2)^{\nu-\ell}, \tag{22.3}$$

$$A_3 = \sum_{1 \leqslant k, \ell \leqslant \nu} (1-t_2)^{\nu-k}(1-t_2-\tau)^{\nu-\ell+k-1}(1-t_1-\tau)^{\ell-1} \tag{22.4}$$

and

$$A_4 = \sum_{1 \leqslant k, \ell \leqslant \nu} (1 - t_1)^{\nu - k}(1 - t_1 - \tau)^{k-1+\ell-1}(1 - t_2 - \tau)^{\nu - \ell}. \qquad (22.5)$$

We consider each summand separately.

22.1.1. Contribution of A_1. Note first that

$$\int_{t_2 - t_1 \leqslant \tau \leqslant 1 - t_2} A_1 \, d\tau = \sum_{1 \leqslant k, \ell \leqslant \nu} \frac{1}{k}(1 - t_2)^{2\nu - k - \ell}(1 - t_1)^{\ell - 1}(1 + t_1 - 2t_2)^k.$$

This also means that $t_2 \leqslant (1 + t_1)/2$, otherwise the range of integration is empty. This upper bound is always larger than t_1. We thus get

$$\int_{t_1}^{(1+t_1)/2} \int_{t_2 - t_1}^{1 - t_2} A_1 \, d\tau \, dt_2 = \sum_{1 \leqslant k, \ell \leqslant \nu} \frac{1}{k}(1 - t_1)^{\ell - 1} \int_{t_1}^{(1+t_1)/2} (1 - t_2)^{2\nu - k - \ell}(1 + t_1 - 2t_2)^k \, dt_2.$$

Concerning the inner integral, note that

$$\int_{t_1}^{(1+t_1)/2} (1 - t_2)^{2\nu - k - \ell}(1 + t_1 - 2t_2)^k \, dt_2 = 2^k \int_0^{(1-t_1)/2} \left(\frac{1 - t_1}{2} + t_3 \right)^{2\nu - k - \ell} t_3^k \, dt_3$$

$$= 2^k \left(\frac{1 - t_1}{2} \right)^{2\nu - \ell + 1} \int_0^1 (1 + t)^{2\nu - k - \ell} t^k \, dt,$$

and this means that we can write

$$\int_{t_1}^{(1+t_1)/2} \int_{t_2 - t_1}^{1 - t_2} A_1 \, d\tau \, dt_2 = K_1(\nu)(1 - t_1)^{2\nu},$$

for some coefficient $K_1(\nu)$. An expression of this coefficient is obtained by setting $t_1 = 0$:

$$K_1(\nu) = \int_0^{1/2} \int_{t_2}^{1 - t_2} \sum_{1 \leqslant k, \ell \leqslant \nu} (1 - t_2)^{2\nu - k - \ell}(1 - t_2 - \tau)^{k-1} \, d\tau \, dt_2$$

$$= \sum_{1 \leqslant k, \ell \leqslant \nu} \frac{1}{k} \int_0^{1/2} (1 - t_2)^{2\nu - k - \ell}(1 - 2t_2)^k \, dt_2$$

$$\leqslant \sum_{1 \leqslant k, \ell \leqslant \nu} \frac{1}{k} \int_0^{1/2} (1 - t_2)^{2\nu - \ell} \, dt_2 \leqslant \mathcal{H}_\nu (\mathcal{H}_{2\nu} - \mathcal{H}_\nu).$$

22.1.2. Contribution of A_2. We proceed as above and first integrate in τ

$$\int_{t_2 - t_1 \leqslant \tau \leqslant 1 - t_2} A_2 \, d\tau = \sum_{1 \leqslant k, \ell \leqslant \nu} \frac{1}{k}(1 - t_1)^{\nu - k + \ell - 1}\left[(1 - t_2)^k - (t_2 - t_1)^k\right](1 - t_2)^{\nu - \ell}.$$

We next note that

$$\int_{t_1}^{(1+t_1)/2} \sum_{1 \leqslant k, \ell \leqslant \nu} \frac{1}{k}(1 - t_1)^{\nu - k + \ell - 1}(1 - t_2)^{\nu + k - \ell} \, dt_2$$

$$= \sum_{1 \leqslant k, \ell \leqslant \nu} \frac{1}{\nu + k - \ell + 1}(1 - t_1)^{\nu - k + \ell - 1}(1 - 2^{-(\nu + k - \ell + 1)})(1 - t_1)^{\nu + k - \ell + 1}.$$

Concerning the other part, we first note that

$$\int_{t_1}^{1} (t_2 - t_1)^k (1 - t_2)^{\nu - \ell} \, dt_2 = (1 - t_1)^{\nu - \ell + k + 1} B(k + 1, \nu - \ell + 1).$$

As a conclusion, there exists a coefficient $K_2(\nu)$ such that

$$\int_0^1 \int_{t_1}^{(1+t_1)/2} \int_{t_2 - t_1}^{1 - t_2} A_2 \, d\tau \, dt_2 t_1^{\kappa - 1} \, dt_1 = K_2(\nu) B(2\nu + 1, \kappa).$$

We get an expression of this coefficient by specializing to $t_1 = 0$:

$$K_2(\nu) = \int_0^{1/2} \int_{t_2}^{1 - t_2} \sum_{1 \leqslant k, \ell \leqslant \nu} (1 - \tau)^{k-1} (1 - t_2)^{\nu - \ell} \, d\tau \, dt_2$$

$$= \int_0^{1/2} \sum_{1 \leqslant k, \ell \leqslant \nu} \frac{1}{k} ((1 - t_2)^k - t_2^k)(1 - t_2)^{\nu - \ell} dt_2$$

$$= \int_0^{1/2} \sum_{1 \leqslant k, \ell \leqslant \nu} \frac{1}{k} (1 - t_2)^{\nu + k - \ell} \, dt_2 + \mathcal{O}^* \left(\sum_{1 \leqslant k, \ell \leqslant \nu} \frac{1}{2^k k (\nu - \ell + 1)} \right)$$

$$= \sum_{1 \leqslant k, \ell \leqslant \nu} \frac{1}{k(k + \ell)} + \mathcal{O}(\text{Log}\, \nu).$$

Furthermore,

$$\sum_{1 \leqslant k, \ell \leqslant \nu} \frac{1}{k(k + \ell)} = \sum_{1 \leqslant k, \ell \leqslant \nu} \frac{1}{\ell} \left(\frac{1}{k} - \frac{1}{k + \ell} \right) = \mathcal{H}_\nu^2 - \sum_{1 \leqslant k, \ell \leqslant \nu} \frac{1}{\ell(k + \ell)}$$

so that

$$K_2(\nu) = \tfrac{1}{2} \mathcal{H}_\nu^2 + \mathcal{O}(\text{Log}\, \nu).$$

22.1.3. *Contribution of A_3.* The quantity A_3 is defined in (22.4). We start with an auxiliary formula.

LEMMA 22.1.

$$\int_x^y (1 - t)^h (y - t)^b \, dt = \sum_{0 \leqslant g \leqslant h} \frac{(-1)^g h! b!}{(b + g + 1)!(h - g)!} (y - x)^{b+1+g} (1 - x)^{h - g}.$$

Proof. Let us call $I(h, b)$ the integral to evaluate. The formula is correct when $h = 0$. We readily get

$$I(h, b) = \left[-(1 - t)^h \frac{(y - t)^{b+1}}{b + 1} \right]_x^y - \frac{h}{b + 1} \int_x^y (1 - t)^{h-1} (y - t)^{b+1} \, dt$$

$$= (1 - x)^h \frac{(y - t_1)^{b+1}}{b + 1} - \frac{h}{b + 1} I(h - 1, b + 1). \qquad \square$$

We use the above lemma to get:

$$\int_{t_2-t_1}^{1-t_2} A_3\,d\tau = \sum_{1\leqslant k,\ell\leqslant\nu} (1-t_2)^{\nu-k}\int_{t_2-t_1}^{1-t_2}(1-t_1-\tau)^{\ell-1}(1-t_2-\tau)^{\nu-\ell+k-1}\,d\tau$$

$$= \sum_{1\leqslant k,\ell\leqslant\nu} (1-t_2)^{\nu-k}\int_{t_2}^{1-t_2+t_1}(1-\tau)^{\ell-1}(1-t_2+t_1-\tau)^{\nu-\ell+k-1}\,d\tau$$

$$= \sum_{\substack{1\leqslant k,\ell\leqslant\nu\\0\leqslant g\leqslant\ell-1}} \frac{(-1)^g(\ell-1)!(\nu-\ell+k-1)!}{(\nu-\ell+k+g)!(\ell-1-g)!}(1+t_1-2t_2)^{\nu-\ell+k+g}(1-t_2)^{\nu-k+\ell-1-g}$$

$$= \sum_{\substack{1\leqslant k,\ell\leqslant\nu\\0\leqslant g\leqslant\ell-1}} \frac{(-1)^{g+\ell+1}(\ell-1)!(\nu-\ell+k-1)!}{(\nu-1+g+k)!g!}(1+t_1-2t_2)^{\nu-1-g+k}(1-t_2)^{\nu-k+g}.$$

Our next step is to carry out the integration over t_2:

$$\int_{t_1}^{(1+t_1)/2}\int_{t_2-t_1}^{1-t_2} A_3\,d\tau\,dt_2 = \sum_{\substack{1\leqslant k,\ell\leqslant\nu\\0\leqslant g\leqslant\ell-1}} \frac{(-1)^{g+\ell+1}(\ell-1)!(\nu-\ell+k-1)!}{(\nu-1+g+k+b)!g!}$$

$$\int_{t_1}^{(1+t_1)/2}(1+t_1-2t_2)^{\nu-1-g+k}(1-t_2)^{\nu-k+g}\,dt_2 = K_3(\nu)(1-t_1)^{2\nu},$$

for some coefficient $K_3(\nu)$. Finally,

$$\int_0^1\int_0^{1-t_1}\int_{t_1}^{1-\tau} A_3\,dt_2\,d\tau\,t_1^{\kappa-1}\,dt_1 = K_3(\nu)B(2\nu+1,\kappa).$$

We again recover an expression of $K_3(\nu)$ by specializing to $t_1=0$:

$$K_3(\nu) = \int_0^{1/2}\int_{t_2}^{1-t_2}\sum_{1\leqslant k,\ell\leqslant\nu}(1-t_2)^{\nu-k}(1-t_2-\tau)^{\nu-\ell+k-1}(1-\tau)^{\ell-1}\,d\tau\,dt_2$$

$$= \int_{1/2}^1\int_{1-t_2}^{t_2}\sum_{1\leqslant k,\ell\leqslant\nu}t_2^{\nu-k}(t_2+\tau-1)^{\nu-\ell+k-1}\tau^{\ell-1}\,d\tau\,dt_2$$

$$\leqslant \int_{1/2}^1\int_{1-t_2}^{t_2}\sum_{1\leqslant k,\ell\leqslant\nu}t_2^{2\nu-\ell-1}\tau^{\ell-1}\,d\tau\,dt_2 \leqslant \int_{1/2}^1\int_{1-t_2}^{t_2}\sum_{1\leqslant k,\ell\leqslant\nu}t_2^{\nu-k}\tau^{\nu+k-2}\,d\tau\,dt_2$$

$$\leqslant \mathcal{H}_\nu(\mathcal{H}_{2\nu-1}-\mathcal{H}_{\nu-1}).$$

22.1.4. *Contribution of A_4*. The argument is routine now. We first note that

$$\int_{t_2-t_1}^{1-t_2} A_4\,d\tau = \sum_{1\leqslant k,\ell\leqslant\nu}(1-t_1)^{\nu-k}\int_{t_2-t_1}^{1-t_2}(1-t_1-\tau)^{k-1+\ell-1}(1-t_2-\tau)^{\nu-\ell}\,d\tau.$$

The next integration leads to

$$\int_{t_1}^{(1+t_1)/2}\int_{t_2-t_1}^{1-t_2} A_4\,d\tau\,dt_2 = K_4(\nu)(1-t_1)^{2\nu}$$

with

$$
\begin{aligned}
K_4(\nu) &= \int_0^{1/2} \int_{t_2}^{1-t_2} \sum_{1 \leqslant k,\ell \leqslant \nu} (1-\tau)^{k+\ell-2}(1-t_2-\tau)^{\nu-\ell} \, d\tau \, dt_2 \\
&= \int_0^1 \sum_{1 \leqslant k,\ell \leqslant \nu} \frac{(1-\tau)^{k-1+\ell-1}}{\nu-\ell+1}\left((1-\tau)^{\nu-\ell+1} - (1-\tau-\min(\tau,1-\tau))^{\nu-\ell+1}\right) d\tau \\
&\leqslant \int_0^1 \sum_{1 \leqslant k,\ell \leqslant \nu} \frac{(1-\tau)^{\nu+k-1}}{\nu-\ell+1} \, d\tau = \mathcal{H}_\nu(\mathcal{H}_{2\nu} - \mathcal{H}_\nu).
\end{aligned}
$$

22.2. When $0 \leqslant t_1 \leqslant 1-\tau \leqslant t_2 \leqslant t_1 + \tau$

This condition implies that $t_1 \geqslant \min(0, 1-2\tau)$ and thus $\tau \geqslant (1-t_1)/2$. We use Lemma 20.1 and the decomposition

$$
\frac{(1-t_2)^\nu - (1-t_1)^\nu + (1-t_1-\tau)^\nu}{\tau} = \frac{(1-t_2)^\nu}{\tau} - \sum_{1 \leqslant k \leqslant \nu} (1-t_1)^{\nu-k}(1-t_1-\tau)^{k-1}
$$

as well as

$$
\frac{(1-t_2)^\nu - (1-t_1)^\nu + (1-t_1-\tau)^\nu}{t_2 - t_1} = \frac{(1-t_1-\tau)^\nu}{t_2 - t_1} - \sum_{1 \leqslant \ell \leqslant \nu} (1-t_2)^{\nu-\ell}(1-t_1)^{\ell-1},
$$

and get that their product is equal to $C_1 - C_2 - C_3 + C_4$, with

$$
\begin{aligned}
C_1 &= \frac{(1-t_2)^\nu}{\tau} \frac{(1-t_1-\tau)^\nu}{t_2 - t_1} \\
&= \frac{(1-t_2)^\nu}{t_2 - t_1} \frac{(1-t_1)^\nu}{\tau} - \sum_{1 \leqslant k \leqslant \nu} (1-t_1)^{\nu-k}(1-t_1-\tau)^{k-1} \frac{(1-t_2)^\nu}{t_2 - t_1} \\
&= \frac{(1-t_1)^{2\nu}}{\tau(t_2 - t_1)} - \sum_{1 \leqslant k \leqslant \nu} (1-t_2)^{\nu-k} \frac{(1-t_1)^{\nu+k-1}}{\tau} \\
&\quad - \sum_{1 \leqslant k \leqslant \nu} (1-t_1)^{\nu-k}(1-t_1-\tau)^{k-1} \frac{(1-t_1)^\nu}{t_2 - t_1} \\
&\quad + \sum_{1 \leqslant k,\ell \leqslant \nu} (1-t_1)^{\nu-k+\ell-1}(1-t_1-\tau)^{k-1}(1-t_2)^{\nu-\ell} \\
&= C_{1,1} - C_{1,2} - C_{1,3} + C_4,
\end{aligned}
$$

say; we further have

$$
C_2 = \frac{(1-t_2)^\nu}{\tau} \sum_{1 \leqslant \ell \leqslant \nu} (1-t_2)^{\nu-\ell}(1-t_1)^{\ell-1},
$$

then

$$
C_3 = \frac{(1-t_1-\tau)^\nu}{t_2 - t_1} \sum_{1 \leqslant k \leqslant \nu} (1-t_1)^{\nu-k}(1-t_1-\tau)^{k-1},
$$

and finally

$$
C_4 = \sum_{1 \leqslant k,\ell \leqslant \nu} (1-t_1)^{\nu-k+\ell-1}(1-t_1-\tau)^{k-1}(1-t_2)^{\nu-\ell}.
$$

22.2.1. *Contribution of C_4.* We first note that

$$\int_{1-\tau}^{t_1+\tau} C_4\, dt_2 = \sum_{1\leqslant k,\ell\leqslant\nu} \frac{1}{\nu-\ell+1}(1-t_1)^{\nu-k+\ell-1}(1-t_1-\tau)^{k-1}\tau^{\nu-\ell+1}$$

$$- \sum_{1\leqslant k,\ell\leqslant\nu} \frac{1}{\nu-\ell+1}(1-t_1)^{\nu-k+\ell-1}(1-t_1-\tau)^{\nu-\ell+k}.$$

Our next step is to carry out the integration over τ, getting

$$\int_{(1-t_1)/2}^{1-t_1}\int_{1-\tau}^{t_1+\tau} C_4\, dt_2\, d\tau = \int_{(1-t_1)/2}^{1-t_1}\sum_{1\leqslant k,\ell\leqslant\nu}\frac{1}{\nu-\ell+1}(1-t_1)^{\nu-k+\ell-1}(1-t_1-\tau)^{k-1}\tau^{\nu-\ell+1}\, d\tau$$

$$- \sum_{1\leqslant k,\ell\leqslant\nu}\frac{1}{(\nu-\ell+1)(\nu-\ell+k+1)}(1-t_1)^{2\nu}$$

$$+ \sum_{1\leqslant k,\ell\leqslant\nu}\frac{1}{(\nu-\ell+1)2^{\nu-\ell+k+1}(\nu-\ell+k+1)}(1-t_1)^{2\nu}.$$

Concerning the inner integrals, we note that

$$\int_{(1-t_1)/2}^{1-t_1}(1-t_1-\tau)^{k-1}\tau^{\nu-\ell+1}\, d\tau = (1-t_1)^{\nu-\ell+1+k}B^\flat(k,\nu-\ell+2)$$

with

$$B^\flat(a,b) = \int_{1/2}^1 (1-t)^{a-1}t^{b-1}\, dt. \qquad (22.6)$$

As a conclusion, there exists a coefficient $K_{10}(\nu)$ such that

$$\int_0^1\int_{(1-t_1)/2}^{1-t_1}\int_{1-\tau}^{t_1+\tau} C_4\, dt_2\, d\tau t_1^{\kappa-1}\, dt_1 = K_{10}(\nu)B(2\nu+1,\kappa).$$

As before, we get an integral representation of $K_{10}(\nu)$ by selecting $t_1 = 0$:

$$K_{10}(\nu) = \int_{1/2}^1\int_{1-\tau}^\tau\sum_{1\leqslant k,\ell\leqslant\nu}(1-\tau)^{k-1}(1-t_2)^{\nu-\ell}\, d\tau\, dt_2$$

$$= \int_{1/2}^1\sum_{1\leqslant k,\ell\leqslant\nu}(1-\tau)^{k-1}\frac{\tau^\ell-(1-\tau)^\ell}{\ell}\, d\tau$$

$$= \int_{1/2}^1\sum_{1\leqslant\ell\leqslant\nu}\frac{\tau^{\ell-1}}{\ell}(1-(1-\tau)^\nu)\, d\tau - \sum_{1\leqslant k,\ell\leqslant\nu}\frac{1}{2^{k+\ell}\ell(k+\ell)} \ll 1.$$

22.2.2. *Contribution of C_1.* We have written $C_1 = C_{1,1} - C_{1,2} - C_{1,3} + C_4$ and we study each term individually. We first handle $C_{1,1}$. We note that

$$\int_{1-\tau}^{t_1+\tau}\frac{(1-t_1)^{2\nu}}{\tau(t_2-t_1)}\, dt_2 = \frac{(1-t_1)^{2\nu}}{\tau}\,\mathrm{Log}\,\frac{\tau}{1-\tau-t_1}.$$

Consequently, we find that

$$\int_{(1-t_1)/2}^{1-t_1}\int_{1-\tau}^{t_1+\tau} C_{1,1}\, dt_2\, d\tau = (1-t_1)^{2\nu}\int_{(1-t_1)/2}^{1-t_1}\,\mathrm{Log}\,\frac{\tau/(1-t_1)}{1-\tau/(1-t_1)}\,\frac{d\tau}{\tau}$$

$$= (1-t_1)^{2\nu}\int_{1/2}^1\,\mathrm{Log}\,\frac{\tau}{1-\tau}\,\frac{d\tau}{\tau},$$

and thus there exists a constant K_5 such that
$$\int_0^1 \int_{(1-t_1)/2}^{1-t_1} \int_{1-\tau}^{t_1+\tau} \frac{(1-t_1)^{2\nu}}{\tau(t_2-t_1)} \, dt_2 t_1^{\kappa-1} \, dt_1 = K_5 B(2\nu+1, \kappa).$$

We now handle the contribution of $C_{1,2}$. We start by noting that

$$\int_{1-\tau}^{t_1+\tau} C_{1,2} \, dt_2 = \sum_{0 \leqslant k \leqslant \nu-1} \frac{(1-t_1-\tau)^{\nu-k}}{\nu-k} \frac{(1-t_1)^{\nu+k}}{\tau} - \sum_{1 \leqslant k \leqslant \nu} \frac{\tau^{\nu-k}}{\nu-k+1}(1-t_1)^{\nu+k-1}$$

$$= \mathcal{H}_\nu \frac{(1-t_1)^{2\nu}}{\tau} - \sum_{\substack{0 \leqslant k \leqslant \nu-1, \\ 1 \leqslant \ell \leqslant \nu-k}} \frac{(1-t_1-\tau)^{\nu-k-\ell}}{\nu-k}(1-t_1)^{\nu+\ell+k-1}$$

$$- \sum_{1 \leqslant k \leqslant \nu} \frac{\tau^{\nu-k}(1-t_1)^{\nu+k-1}}{\nu-k+1}.$$

Hence, we can write

$$\int_{(1-t_1)/2}^{1-t_1} \int_{1-\tau}^{t_1+\tau} C_{1,2} \, dt_2 \, d\tau = \mathcal{H}_\nu(1-t_1)^{2\nu} \operatorname{Log} 2 - \sum_{\substack{1 \leqslant k \leqslant \nu, \\ 1 \leqslant \ell \leqslant \nu-k+1}} \frac{(1-2^{-(\nu-k+2-\ell)})(1-t_1)^{2\nu}}{(\nu-k+1)(\nu-k+2-\ell)}$$

$$- \sum_{1 \leqslant k \leqslant \nu} \frac{1-2^{-(\nu-k+1)}}{(\nu-k+1)^2}(1-t_1)^{2\nu}$$

$$= K_6(\nu)(1-t_1)^{2\nu},$$

for some coefficient $K_6(\nu)$. We get an integral expression for $K_6(\nu)$ by selecting $t_1 = 0$:

$$K_6(\nu) = \int_{1/2}^1 \int_{1-\tau}^{\tau} \sum_{1 \leqslant k \leqslant \nu} (1-t_2)^{\nu-k} \, dt_2 \frac{d\tau}{\tau} = \int_{1/2}^1 \sum_{1 \leqslant k \leqslant \nu} \frac{\tau^k - (1-\tau)^k}{k} \frac{d\tau}{\tau} \leqslant \pi^2/6.$$

We turn our attention to $C_{1,3}$, first getting

$$\int_{1-\tau}^{t_1+\tau} C_{1,3} \, dt_2 = \sum_{1 \leqslant k \leqslant \nu} (1-t_1)^{2\nu-k}(1-t_1-\tau)^{k-1} \operatorname{Log} \frac{\tau}{1-\tau-t_1}.$$

We note that

$$\int_{(1-t_1)/2}^{1-t_1} \sum_{1 \leqslant k \leqslant \nu} (1-t_1)^{2\nu-k}(1-t_1-\tau)^{k-1} \operatorname{Log} \frac{\tau/(1-t_1)}{1-\tau/(1-t_1)} \, d\tau$$

$$= \int_{1/2}^1 \sum_{1 \leqslant k \leqslant \nu} (1-\tau)^{k-1} \operatorname{Log} \frac{\tau}{1-\tau} \, d\tau (1-t_1)^{2\nu}.$$

Furthermore, we have

$$K_7(\nu) = \int_{1/2}^1 \sum_{1 \leqslant k \leqslant \nu} (1-\tau)^{k-1} \operatorname{Log} \frac{\tau}{1-\tau} \, d\tau$$

$$= \int_0^{1/2} \sum_{1 \leqslant k \leqslant \nu} \tau^{k-1} \operatorname{Log} \frac{\tau}{1-\tau} \, d\tau = \sum_{1 \leqslant k \leqslant \nu} \frac{1}{k} \int_0^{1/2} \tau^k \left(\frac{1}{\tau} + \frac{1}{1-\tau}\right) d\tau \ll 1.$$

As a conclusion, we find that

$$\int_0^1 \int_{(1-t_1)/2}^{1-t_1} \int_{1-\tau}^{t_1+\tau} C_{1,3} \, dt_2 \, d\tau t_1^{\kappa-1} \, dt_1 = K_7(\nu) B(2\nu+1, \kappa).$$

22.2.3. *Contribution of C_2.* We handle the integration with respect to t_2 at the beginning, obtaining

$$\int_{1-\tau}^{t_1+\tau} C_2 \, dt_2 = \sum_{1 \leqslant \ell \leqslant \nu} \frac{1}{2\nu - \ell + 1} \tau^{2\nu - \ell} (1 - t_1)^{\ell - 1}$$

$$- \sum_{1 \leqslant \ell \leqslant \nu} \frac{1}{2\nu - \ell + 1} \frac{(1 - t_1 - \tau)^{2\nu - \ell + 1}}{\tau} (1 - t_1)^{\ell - 1}$$

$$= \sum_{1 \leqslant \ell \leqslant \nu} \frac{1}{2\nu - \ell + 1} \tau^{2\nu - \ell} (1 - t_1)^{\ell - 1} - \sum_{1 \leqslant \ell \leqslant \nu} \frac{1}{2\nu - \ell + 1} \frac{(1 - t_1)^{2\nu}}{\tau}$$

$$+ \sum_{\substack{1 \leqslant \ell \leqslant \nu, \\ 1 \leqslant k \leqslant 2\nu - \ell + 1}} \frac{1}{2\nu - \ell + 1} (1 - t_1 - \tau)^{2\nu - \ell + 1 - k} (1 - t_1)^{k + \ell - 2}.$$

As before, we conclude from these lines that there exists a coefficient $K_8(\nu)$ such that

$$\int_{(1-t_1)/2}^{1-t_1} \int_{1-\tau}^{t_1+\tau} C_2 \, dt_2 \, d\tau = K_8(\nu)(1 - t_1)^{2\nu}.$$

On setting $t_1 = 0$ in the above equation, we get

$$K_8(\nu) = \int_{1/2}^{1} \int_{1-\tau}^{\tau} \sum_{1 \leqslant \ell \leqslant \nu} (1 - t_2)^{2\nu - \ell} \, dt_2 \frac{d\tau}{\tau} = \int_{1/2}^{1} \sum_{1 \leqslant \ell \leqslant \nu} \frac{(1 - \tau)^{2\nu - \ell + 1} - \tau^{2\nu - \ell + 1}}{2\nu - \ell + 1} \frac{d\tau}{\tau} \ll 1.$$

As a conclusion, we find that

$$\int_0^1 \int_{(1-t_1)/2}^{1-t_1} \int_{1-\tau}^{t_1+\tau} C_2 \, dt_2 \, d\tau t_1^{\kappa - 1} \, dt_1 = K_8(\nu) B(2\nu + 1, \kappa).$$

22.2.4. *Contribution of C_3.* Note first that

$$\int_{1-\tau}^{t_1+\tau} C_3 \, dt_2 = \sum_{1 \leqslant k \leqslant \nu} (1 - t_1)^{\nu - k} (1 - t_1 - \tau)^{\nu + k - 1} \operatorname{Log} \frac{\tau}{1 - t_1 - \tau},$$

and then integrate over τ, getting

$$\int_{(1-t_1)/2}^{1-t_1} \int_{1-\tau}^{t_1+\tau} C_3 \, dt_2 \, d\tau = \sum_{1 \leqslant k \leqslant \nu} (1 - t_1)^{\nu - k} \int_{(1-t_1)/2}^{1-t_1} (1 - t_1 - \tau)^{\nu + k - 1} \operatorname{Log} \frac{\tau/(1 - t_1)}{1 - \tau/(1 - t_1)} \, d\tau$$

$$= \sum_{1 \leqslant k \leqslant \nu} \int_{1/2}^{1} (1 - \tau)^{\nu + k - 1} \operatorname{Log} \frac{\tau}{1 - \tau} \, d\tau (1 - t_1)^{2\nu}.$$

Our conclusion is that there exists a coefficient $K_9(\nu)$ such that

$$\int_0^1 \int_{(1-t_1)/2}^{1-t_1} \int_{1-\tau}^{t_1+\tau} C_3 \, dt_2 \, d\tau t_1^{\kappa - 1} \, dt_1 = K_9(\nu) B(2\nu + 1, \kappa).$$

We again recover $K_9(\nu)$ by using the expression above with $t_1 = 0$ and get

$$K_9(\nu) = \sum_{1 \leqslant k \leqslant \nu} \int_{1/2}^{1} (1 - \tau)^{\nu + k - 1} \operatorname{Log} \frac{\tau}{1 - \tau} \, d\tau = \sum_{1 \leqslant k \leqslant \nu} \frac{1}{2^{\nu + k - 1}} \int_{1/2}^{1} \operatorname{Log} \frac{\tau}{1 - \tau} \, d\tau \ll 1.$$

22.3. **When $1 - \tau \leqslant t_1$**

We use

$$\frac{(1 - t_2)^{\nu} - (1 - t_1)^{\nu}}{t_2 - t_1} = - \sum_{1 \leqslant \ell \leqslant \nu} (1 - t_2)^{\nu - \ell} (1 - t_1)^{\ell - 1}.$$

We get next

$$\int_{t_1}^1 \sum_{1 \leqslant \ell \leqslant \nu} (1 - t_2)^{\nu - \ell}(1 - t_1)^{\ell - 1}((1 - t_1)^\nu - (1 - t_2)^\nu) \, dt_2$$

$$= \sum_{1 \leqslant \ell \leqslant \nu} (1 - t_1)^{2\nu} \frac{1}{\nu - \ell + 1} - \sum_{1 \leqslant \ell \leqslant \nu} (1 - t_1)^{2\nu} \frac{1}{2\nu - \ell + 1}$$

and

$$\int_{1 - t_1}^1 \int_{t_1}^1 \sum_{1 \leqslant \ell \leqslant \nu} (1 - t_2)^{\nu - \ell}(1 - t_1)^{\ell - 1}((1 - t_1)^\nu - (1 - t_2)^\nu) \, dt_2 \frac{d\tau}{\tau}$$

$$= -(2\mathcal{H}_\nu - \mathcal{H}_{2\nu})(1 - t_1)^{2\nu} \mathrm{Log}(1 - t_1)$$

so that

$$\int_0^1 \int_{1 - t_1}^1 \int_{t_1}^1 \sum_{1 \leqslant \ell \leqslant \nu} (1 - t_2)^{\nu - \ell}(1 - t_1)^{\ell - 1}((1 - t_1)^\nu - (1 - t_2)^\nu) \, dt_2 \frac{d\tau}{\tau} t_1^{\kappa - 1}$$

$$= (2\mathcal{H}_\nu - \mathcal{H}_{2\nu})B_1(\kappa, 2\nu + 1) = (2\mathcal{H}_\nu - \mathcal{H}_{2\nu})(\mathcal{H}_{\kappa + 2\nu} - \mathcal{H}_{2\nu})B(\kappa, 2\nu + 1).$$

22.4. *Collecting the various contributions*

We have proved that (with $w(t) = (1 - t)^{+\nu}$)

$$\int_0^1 \int_{\substack{0 \leqslant t_1 < t_2 \leqslant 1, \\ t_2 - t_1 \leqslant \tau}} \frac{(w(t_2) - w(t_2 + \tau) - w(t_1) + w(t_1 + \tau))^2}{t_2 - t_1} t_1^{\kappa - 1} \, dt_1 \, dt_2 \frac{d\tau}{\tau}$$

$$= (K(\nu) + (2\mathcal{H}_\nu - \mathcal{H}_{2\nu})(\mathcal{H}_{\kappa + 2\nu} - \mathcal{H}_{2\nu}))B(\kappa, 2\nu + 1). \qquad (22.7)$$

with

$$K(\nu) = -K_1(\nu) + K_2(\nu) + K_3(\nu) - K_4(\nu)$$
$$+ K_5 - K_6(\nu) - K_7(\nu) - K_8(\nu) - K_9(\nu) + 2K_{10}(\nu).$$

Note that, when ν is fixed, like $\nu = 1$, the dependence in κ appears already clearly. The material above leads to an evaluation of $K(\nu)$: we have shown that $K_1(\nu)$, $K_3(\nu)$, $K_4(\nu)$, $K_{10}(\nu)$, $K_6(\nu)$, $K_7(\nu)$, $K_8(\nu)$ and $K_9(\nu)$ are all bounded by $\mathcal{O}(\mathrm{Log}\,\nu)$. Furthermore, the constant $K_2(\nu)$ is $\frac{1}{2}\mathrm{Log}^2\nu + \mathcal{O}(\mathrm{Log}\,\nu)$ so that the same evaluation holds true for $K(\nu)$.

23. *On prime κ-tuples. Proof of Theorem 1.1*

We select parameters $P = N^\beta$ and $Q = P$ with $2(\beta + 1) < 1$. Working with $Q = P^{1/\tau_0}$ for some parameter τ_0 to be chosen would in fact lead to the same bound. We set $B = \lceil \beta^{-1} \rceil$. We consider

$$S = \sum_{1 \leqslant i \leqslant \kappa} (\Delta_0 S_i(a_0[P]) + 2U S_i(a_1[P]) - U^2(2S_i(a_2[P]) + S_i(a_1[P]))), \qquad (23.1)$$

where $U = \vartheta / \mathcal{H}_{\kappa + 2}$ for some general real parameter ϑ. Recalling (1.2) and after some shuffling, we find that

$$S = \sum_{n \leqslant N} \beta(n) \sum_{1 \leqslant i \leqslant \kappa} \left(\Delta_0 + 1 - \left(1 - U \sum_{\substack{p | n + h_i, \\ p \leqslant P}} 1 \right) \right)^2.$$

Let us now evaluate the sum S when $w(t) = (1-t)^+$, that is, we take $\nu = 1$. By Theorem 1.2, and because of our choice of P and Q, the sum S goes to infinity as soon as

$$\sum_{1 \leqslant i \leqslant \kappa} \int_0^1 \int_{t_1}^1 \left(\Delta_0 \mathfrak{G}_{\kappa,0}(t_1, t_2, 1) + (2U - U^2)\mathfrak{G}_{\kappa,1}(t_1, t_2, 1) - 2U^2\mathfrak{G}_{\kappa,2}(t_1, t_2, 1) \right) dt_1\, dt_2$$

is at least η, for some strictly positive η independent of N, P and Q, but which may depend on κ. We appeal to Lemmas 21.1–21.3 and get that the required inequality holds when $\Delta_0 + 2\vartheta(1 - \vartheta(2\mathcal{H}_{\kappa+2})^{-1})(1 - \mathcal{H}_{\kappa+2}^{-1}) - \vartheta^2(1 - 2\mathcal{H}_{\kappa+2}^{-1} + 2\mathfrak{B}\mathcal{H}_{\kappa+2}^{-2}) \geqslant \eta$, on writing the right-hand side of Lemma 21.3 in the form $\frac{1}{2}\mathcal{H}_{\kappa+2}(\mathcal{H}_{\kappa+2} - 2) + \mathfrak{B}$. Hence, the following inequality is to be fulfilled:

$$\Delta_0 \geqslant \eta - 2\vartheta + \vartheta^2 + \frac{2\vartheta}{\mathcal{H}_{\kappa+2}} + \frac{\vartheta^2(\mathfrak{B} - 1)}{\mathcal{H}_{\kappa+2}^2}.$$

We select $\eta = 1/\mathcal{H}_{\kappa+2}^2$ and

$$\Delta_0 = -2\vartheta + \vartheta^2 + \frac{2\vartheta}{\mathcal{H}_{\kappa+2}} + \frac{1 + \vartheta^2(\mathfrak{B} - 1)}{\mathcal{H}_{\kappa+2}^2}. \tag{23.2}$$

We denote by $\infty(N)$ a function that goes to infinity with N. With the choice (23.2), there are $\infty(N)$ many integers n such that

$$\kappa(\Delta_0 + 1)\mathcal{H}_{\kappa+2}^2 - \sum_{1 \leqslant i \leqslant \kappa} \left(\mathcal{H}_{\kappa+2} - \vartheta\omega(n + h_i; P) \right)^2 > 0,$$

where $\omega(m; P)$ denotes the number of prime factors of m that are not more than P counted without multiplicity. The quantity $1 + \Delta_0$ goes to zero only with the choice $\vartheta = 1$. We continue with this choice. We replace $\mathcal{H}_{\kappa+2}$ by $\operatorname{Log} \kappa$ as follows. Put $\Theta_\kappa = \mathcal{H}_{\kappa+2} - \operatorname{Log} \kappa = \mathcal{O}(1)$. We expand the quantity above as

$$\sum_{1 \leqslant i \leqslant \kappa} [(\operatorname{Log} \kappa - \omega(n + h_i; P))^2 + 2(\mathcal{H}_{\kappa+2} - \omega(n + h_i; P))\Theta_\kappa - \Theta_\kappa^2].$$

We treat the second sum by using Cauchy's inequality. Our last task is to replace $\omega(m; P)$ by $\omega(m)$. We first notice that an integer $m \leqslant N$ can have at most B prime factors greater than P, which implies that $\omega(m) \geqslant \omega(m; P) \geqslant \omega(m) - B$. We set $E_\kappa(m; P) = \omega(m) - \omega(m; P)$ and use the decomposition

$$\sum_{1 \leqslant i \leqslant \kappa} [(\operatorname{Log} \kappa - \omega(n + h_i))^2 + 2(\mathcal{H}_{\kappa+2} - \omega(n + h_i))E_\kappa(n + h_i; P) + E_\kappa(n + h_i; P)^2].$$

We again treat the second sum by using Cauchy's inequality. We finally select $P = Q = N^{2/9}$ and the Theorem 1.1.

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