QUANTITATIVE STEPS IN AXER-LANDAU EQUIVALENCE THEOREM

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Abstract. Completing previous enquiries of the same nature, it is shown that, for every non-negative integer h, there exists a positive constant c such that, for $x \geq 10$, the inequality $|\sum_{n\le x}\mu(n)(\log n)^h/n| \ll \max_{y\sim x}|\sum_{n\le y}\mu(n)|(\log y)^h/y + x^{-c/\log\log x}$ holds. The main theorem applies to general problems of this kind.

1. Introduction

The error term in the Prime Number Theorem is a central quantity in multiplicative number theory. Other error terms appear, and for instance, if we define

$$\psi(x) = \sum_{n\le x}\Lambda(n), \quad \tilde{\psi}(x) = \sum_{n\le x}\Lambda(n)/n,$$

we would like to have access to $\tilde{\psi}(x) - (\log x - \gamma)$. A theorem of Landau from [10] which combines results of Axer and Landau, namely [9], [1] and [8], asserts that this latter goes to zero as soon as $\psi(x) - x$ does. After a cursory look, one could believe this problem to be readily solved by a partial summation, but this is not the case. Going from $\tilde{\psi}(x) - (\log x - \gamma)$ to $\psi(x) - x$ is indeed mechanical, but the reverse is more difficult. As a matter of fact Diamond & Zhang exhibited in [4] a system of Beurling generalized integers where $\psi_{\mathcal{P}}(x) \sim x$ but where $\tilde{\psi}_{\mathcal{P}}(x) - \log x$ does not have a limit, with an obvious notation. The Landau Theorem refered to above answers this question from a qualitative viewpoint. I addressed the question of a quantitative version of it in [13], and more completely together with D. Platt in [12], with applications to explicit estimates in view: the idea is to concentrate the work on $\psi(x) - x$ and to derive automatically estimates for $\tilde{\psi}(x) - (\log x - \gamma)$ (and for similar quantities with primes in arithmetic progressions). The results obtained are not (conjecturally) optimal but are still rather strong, both from a theoretical and numerical point of view, see below. The link with different error terms concerning $\sum_{p\le x} 1/p$ and the Euler products $\prod_{p\le x}(1 - z/p)$, for any complex number z of modulus not more than 2, has further been investigated by Vanlalngaia in [11].

We know since Landau in [10] that the error term of the Prime Number Theorem is linked with the one of the summatory function $M(x)$ of the

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Moebius function. From now on we use the notation

$$M(x)=\sum_{n\le x}\mu(n),\quad m(x)=\sum_{n\le x}\mu(n)/n. \tag{1.1}$$

Inferring a *quantitative* error term (here, simply a bound) for $M(x)$ from the one of $\psi(x)-x$ has received attention. Let us mention the work [7] of Kienast, the work [17] of Schoenfeld and the one of El Marraki in [5], though these two latter authors do not present their investigation in this perspective, and lately the paper [14]. The answers are up to now rather unsatisfactory.

One would also like to derive error terms for $m(x)$ from the one of $M(x)$. Following the well-known paper [1] of Axer, the question has later been addressed by Kienast using a tauberian argument and Lambert series in [6] and more precisely in [7]; it is initially this question that we set to investigate here. It is striking that, in Kienast's Satz 10 taken from [7], no direct derivation is given for what could be thought as a simple question, but an additionnal requirement concerning $\psi(x)$ has to be made.

We end this survey of previous researches with two remarks. First and as noticed by Landau, inferring an error term for $M(x)$ once we have one for $m(x)$ is routine. Secondly, we mention that a path using identities has been investigated by Balazard in [3] (see [2] for a french translation) and in [14].

The methods I used with Platt to go from $\psi(x)$ to $\tilde{\psi}(x)$ relied heavily on two explicit formulae that were compared; such material is missing when dealing with the Moebius function. However after analysing this proof, we discovered that similar information could be obtained from the Mellin transform directly, and this is the subject of the present work. Here is a corollary of the main Theorem.

Corollary. *There exists a positive constant c_1 such that, for $x\ge 10$, and notation being defined in* (1.1), *we have*

$$|m(x)|\ll \max_{x\le y\le 3x}\frac{|M(y)|}{y}+\exp\Bigl(-\frac{c_1\log x}{\log\log x}\Bigr).$$

Expliciting numerically the proof we give of this corollary would this time be clearly of no use to infer numerical estimates, at least at the time of writing. Indeed the latest best bound for $1/\zeta(s)$ is due to Trudgian in [18] and would put a constant at least of size 10^7 in front of the $x^{-1/(8\log\log x)}$. When the emphasis rests on this aspect, the path of identities taken in [3] and [14] remains a better choice.

Since our method is very general and we want to encompass several cases in one statement, our first call is to define a general setting. The reader may want to keep the case of the Moebius function in mind. We start with a Dirichlet series

$$F(s)=\sum_{n\ge 1}a_n/n^s\quad(\Re s>1) \tag{1.2}$$

which we assume to be absolutely convergent in the half-plane $\Re s > 1$. We further assume that there exists five constants $r, \ell \in \mathbb{Z}^+$, $c \in (0, 1]$, $A \in [2, \infty)$ and $B \in [2, \infty)$, and a polynomial R of degree at most $r - 1$ such that the function $F(s) - R(s-1)/(s-1)^r$ extends holomorphically to the domain

$$\Re s \geq 1 - \frac{c}{\log(A + |\Im s|)} \tag{1.3}$$

(where $\Re s$ and $\Im s$ denotes respectively the real and imaginary part of the complex number s), and where it satisfies

$$\bigl|F(s) - R(s-1)/(s-1)^r\bigr| \leq B \log^{\ell}(A + |\Im s|). \tag{1.4}$$

The choice $r = 0$ is allowed and is indeed the case of the Moebius function, i.e. when $F(s) = 1/\zeta(s)$; We further check that $R = 0$ and $\ell = 1$ in this case. For the primes, we use $F(s) = -\zeta'(s)/\zeta(s)$ together with $r = \ell = 1$ and $R = 1$. The above hypotheses imply classically that, for any non-negative integer $h \geq 0$, there exists a polynomial $P_{r,h}$ of degree at most $r + h$ such that $\sum_{n \leq x} a_n/n - P_{r,h}(\log x)$ goes to zero, and also a polynomial Q of degree at most $r - 1$ such that $\sum_{n \leq x} a_n - xQ(\log x)$ is bounded above in absolute value by a multiplicative constant times $x \exp(-c\sqrt{\log(A + x)}/2)$. When $r = 0$, we set $Q = 0$. Here is our central result.

Theorem 1.1. *Let $x \geq 10$. Notation being as above, for any non-negative integer h, we have*

$$\left|\sum_{n \leq x} \frac{a_n \log^h n}{n} - P_{r,h}(\log x)\right| \ll \max_{x \leq y \leq 3x} \frac{|\sum_{n \leq y} a_n - yQ(\log y)|(\log y)^h}{y} + \exp\left(-\frac{c \log x}{7 \log(A + \log x)}\right).$$

When $r = 0$, the polynomial $P_{r,h}$ is constant.

The main point is that the last error term above is better than the one (recalled above) that we know how to get for $\sum_{n \leq x} \frac{a_n \log^h n}{n} - P_{r,h}(\log x)$. The reader will see in the proof that we only use the zero-free region up to height $\log x$.

Though case $h = 0$ has been more advertised, Axer already showed that proving that $\sum_{n \leq x} (\mu(n) \log n)/n$ be asymptotic to -1 is equivalent to showing that $M(x)$ is $o(x/\log x)$, an assertion that Kienast generalized to any power of $\log n$, *provided* a similar condition concerning ψ is satisfied. The above theorem settles the problem is full generality in a quantitative manner.

We should maybe be more precise when comparing results over such a span of times. From 1900 until 1950-1960, there was a belief that the so-called "elementary" methods had a distinct (and lesser) power than "analytic" methods. As time went, more and more results from the "analytic" world were converted to the "elementary" world, and it is now clear that there are no differences in the results obtainable. However one setting or another (or a mix of both as often in sieve context) offers more efficiency to tackle a given set of problems.

If needed, the proof of Theorem 1.1 provides us with a localization in $[x,(1+\kappa)x]$ for y, for any positive κ. There remains the problem of converting the last error term in a power saving in x, and we state formally:

Question. *Is it true that for any function F verifying the above hypotheses, there exists two positive constants a and b such that*

$$\left|\sum_{n\le x}\frac{a_n}{n}-P_{r,0}(\log x)\right|\ll \max_{x/a\le y\le ax}\frac{|\sum_{n\le y}a_n-yQ(\log y)|}{y}+x^{-b}\ ?$$

In case of the von Mangoldt function or of the Moebius function, this can be turned into a conjecture which is readily shown to be true under the Riemann Hypothesis (on taking b to be any positive real number below 1/2). In general, and for instance for Beurling numbers, we are undecided.

Though Theorem 1.1 of [12] is a consequence of the above, getting a statement numerically as good as Theorem 1.2 of the same paper is difficult, if at all possible, with the method we develop here.

We can of course get similar corollaries with $\mu(n)\chi(n)$ rather than $\mu(n)$, where χ is a Dirichlet character, and also replace the Moebius function $\mu(n)$ by the Liouville function $\lambda(n)$. We can further see these functions as related to the number field $\mathbb{Q}$ and consider similar quantities but related to a different number field, or consider L-series associated to Hecke grossen Charakteren or to some modular form, and, finally, to Beurling numbers.

2. Preliminary lemmas

We first recall a handy integral Gorny's inequality we proved in [15, Theorem 1.4].

Lemma 2.1. *Let $\mathcal{C}_k(a,b)$ be the class of functions f over an interval (a,b) (both a and b can be infinite) that are k-times differentiable, such that all $f^{(h)}$ when $h\in\{0,\cdots,k\}$ are in L^2 and such that, for all index $h\in\{0,\cdots,k-1\}$, we have $f^{(h)}(a)=f^{(h)}(b)=0$. Let f be in $\mathcal{C}_k(a,b)$. For any $h\in\{0,\cdots,k\}$, we have*

$$\int_a^b\left|f^{(h)}(v)\right|^2dv\le\left(\int_a^b\left|f^{(k)}(v)\right|^2dv\right)^{\frac{h}{k}}\left(\int_a^b\left|f(v)\right|^2dv\right)^{1-\frac{h}{k}}.$$

We rely on [16] for the smoothing process. We define, for any integer $m\ge 1$, the function over $[0,1]$

$$f_m(t)=(4t(1-t))^m. \tag{2.1}$$

This function f_m satisfies

$$f_m^{(k)}(0)=f_m^{(k)}(1)=0\quad(0\le k\le m-1). \tag{2.2}$$

We recall part of [16, Lemma 6].

Lemma 2.2.

$$\|f_m\|_1 = \frac{2^{2m} m!^2}{(2m+1)!} \quad , \quad \|f_m^{(m)}\|_2 = \frac{2^{2m} m!}{\sqrt{2m+1}}, \tag{2.3}$$

$$\|f_m^{(k)}\|_2 \ll \sqrt{m}(4m/e)^k, \quad (k \le m). \tag{2.4}$$

Proof. The first statements come from [16, Lemma 6] and we deduce the last one by appealing to Lemma 2.1:

$$\|f_m^{(k)}\|_2 \le \left(2^{2m}\frac{m!}{\sqrt{2m+1}}\right)^{k/m} \left(\frac{2^{4m}(2m)!^2}{(4m+1)!}\right)^{(m-k)/(2m)}$$

since $\|f_m\|_2 = \sqrt{\|f_{2m}\|_1}$. Hence, on using Lemma 2.1,

$$\begin{aligned}
\|f_m^{(k)}\|_2/\|f_m\|_1 &\le \left(\frac{m!^2}{(2m+1)}\frac{(4m+1)!}{(2m)!^2}\right)^{k/(2m)} \frac{(2m)!(2m+1)!}{\sqrt{(4m+1)!m!^2}} \\
&\ll \left(\frac{m^{2m}}{e^{2m}}\frac{(4m)^{4m}e^{4m}}{e^{4m}(2m)^{4m}}\right)^{k/(2m)} \frac{(2m/e)^{4m}m^2}{m^{5/2}(4m/e)^{2m}(m/e)^{2m}} \\
&\ll \sqrt{m}\left(\frac{4m}{e}\right)^k
\end{aligned}$$

as claimed. □

Lemma 2.3. *We have, when $u > 1$ and $k \ge 0$,*

$$\frac{d^k}{du^k}\left(\frac{1}{\log^2 u}\right) = (-1)^k \sum_{2\le j\le k+2} \frac{c_k(j)}{j!u^k(\log u)^j}$$

for some non-negative coefficients $c_k(j)$ that satisfy $\sum_j c_k(j) = k!$.

Proof. We prove the existence of the coefficients $c_k(j)$ by recursion:

$$\left(\frac{1}{\log^2 u}\right)^{(k+1)} = (-1)^{k+1} \sum_{2\le j\le k+2} \left(\frac{kc_k(j)}{j!u^{k+1}(\log u)^j} + \frac{jc_k(j)}{j!u^{k+1}(\log u)^{j+1}}\right)$$

and thus $c_{k+1}(j) = kc_k(j) + c_k(j-1)$, with $c_k(1) = c_k(k+3) = 0$. Define $P_k(X) = \sum_{2\le j\le k+2} c_k(j)X^j$. The recursion gives us that

$$P_{k+1}(X) = (X+k)P_k(X), \quad P_0(X) = X^2,$$

whence $P_k(X) = (X+k-1)(X+k-2)\cdots X^2$. The Taylor expansion of $P_k(X)/X$ is given by the (unsigned) Stirling numbers of the first kind $|s(k,j)|$, but we do not need this. It is enough for us to see that $P_k(1) = k! = \sum_j c_k(j)$. This does not lose much since $c_k(k+2) = (k-1)!$. □

From f_m, we define (almost as in section 4 of [12])

$$g_m(t) = \begin{cases} 1 & \text{when } 0 < t \le 2, \\ 1 - \|f_m\|_1^{-1}\int_0^{t-2} f_m(u)du & \text{when } 2 \le t \le 3, \\ 0 & \text{when } t \ge 3. \end{cases} \tag{2.5}$$

Note that the function g_m satisfies $0 \le g_m(t) \le 1$.

We consider, for $\Re s \leq 1$, the Mellin transform

$$H_m(s) = \int_2^\infty \frac{1 - g_m(u)}{u \log^2 u} u^{s-1} du. \tag{2.6}$$

We need the denominator $\log^2 u$ at infinity, but since this function vanishes at $u = 1$, we have shifted the smoothing from 2 onwards.

Lemma 2.4. *When $\Re s \leq 1$, we have $|H_m(s)| \ll_m 1$. The function H_m extends in a holomorphic function in the half plane $\Re s \leq 1$. When $|\Im s| \geq 1$, we have, uniformly in m:*

$$|H_m(s)| \ll 1/|s|, \quad |H_m(s)| \ll (2m)^m/|s|^{m+1}.$$

Proof. We define $1 - g_m(u) = a_m(u)$ so that, by Lemma 2.2 when $1 \leq k \leq m+1$, and directly otherwise, we have $a_m^{(k)}(u) \ll (2m/e)^k$ for $k \in \{0, \cdots, , m+1\}$. We further set $a_m(u)/(\log u)^2 = b_m(u)$. Notice that, for any $k \in \{0, \cdots, m\}$, we have $b_m^{(k)}(2) = 0$. Hence we can use $k \in \{0, \cdots, m+1\}$ integrations by parts to reach

$$H_m(s) = \frac{(-1)^k}{(s-1)s \cdots (s-1+k)} \int_2^\infty b_m^{(k)}(u) u^{s-2+k} du.$$

We use this expression for $k = m+1$. Leibniz's formula together with Lemma 2.3 give us that

$$\begin{aligned}
b_m^{(m+1)}(u) &= \sum_{0 \leq k \leq m+1} \binom{m+1}{k} a_m^{(k)}(u) \sum_{2 \leq j \leq 2+m+1-k} \frac{c_{m+1-k}(j)}{(m+1-k)! u^{m+1-k} (\log u)^j}. \\
&= \frac{a_m(u)}{u^{m+1}(m+1)!} \sum_{2 \leq j \leq m+3} \frac{c_{m+1}(j)}{(\log u)^j} \\
&\quad + \sum_{1 \leq k \leq m+1} \binom{m+1}{k} a_m^{(k)}(u) \sum_{2 \leq j \leq m+3-k} \frac{c_{m+1-k}(j)}{(m+1-k)! u^{m+1-k} (\log u)^j}.
\end{aligned}$$

The second part of this expression vanishes when $u \geq 3$ and is otherwise bounded above by

$$\begin{aligned}
&\ll \sum_{1 \leq k \leq m+1} \binom{m+1}{k} m^k \frac{(m+1-k)!}{(\log 2)^{m+3-k}} \\
&\ll \frac{(m+1)!\kappa}{(\log 2)^m} \sum_{0 \leq k \leq m+1} \frac{m^k (\log 2)^k}{k!} \\
&\ll \frac{(m+1)!}{(\log 2)^m} e^{m \log 2} \ll m^{3/2} \Big(\frac{2m}{e \log 2} \Big)^m \ll (2m)^m.
\end{aligned}$$

Therefore, when $|\Im s| \geq 1$, we get

$$H_m(s) \ll \frac{1}{|s|^{m+1}} \bigg(\int_2^\infty \frac{u^{\Re s - 2 + m + 1} du}{u^{m+1} (\log u)^2} + (2m)^m \bigg) \ll (2m)^m/|s|^{m+1}$$

as required. □

3. The decomposition

To ease the typographical work, we define

$$S_h(x) = \sum_{n\le x} a_n, \quad \tilde{S}_h(x) = \sum_{n\le x} a_n(\log n)^h/n. \tag{3.1}$$

It will also be helpful to use the shortcut

$$\rho_h(t) = (\log t)^h/t. \tag{3.2}$$

We first use an integration by parts to remove the $(\log n)^h/n$:

$$\begin{aligned}\tilde{S}_h(x) &= \frac{S(x)(\log x)^h}{x} - \int_1^x S(t)\rho_h'(t)dt \\ &= \frac{S((x) - xQ(\log x))(\log x)^h}{x} + Q(\log x)(\log x)^h \\ &\quad - \int_1^x Q(\log t)t\rho_h'(t)dt - \int_1^\infty (S(t) - tQ(\log t))\rho_h'(t)dt \\ &\quad + \int_x^\infty (S(t) - tQ(\log t))\rho_h'(t)dt\end{aligned}$$

since our hypotheses ensure that $S(t) - tQ(\log t) = \mathcal{O}(t/\log(2t)^{h+2})$. Though much more is true, such an estimate is enough to prove that the integral $\int_1^\infty (S(t) - tQ(\log t))\rho_h'(t)dt$ is absolutely convergent. By unicity, we have

$$\begin{aligned}P_{r,h}(\log x) = Q(\log x)(\log t)^h + \int_1^x Q(\log t)t\rho_h'(t)dt \\ + \int_1^\infty (S(t) - tQ(\log t))\rho_h'(t)dt.\end{aligned}$$

At this level, we take new notation:

$$\Delta(x) = \sum_{n\le x} a_n - xQ(\log x), \tilde{\Delta}_h(x) = \sum_{n\le x} \frac{a_n(\log n)^h}{n} - P_{r,h}(\log x). \tag{3.3}$$

The decomposition we will use is the following one:

$$\begin{aligned}\tilde{\Delta}_h(x) = \frac{\Delta(x)(\log x)^h}{x} - \int_x^{3x} \Delta(t)g_m(t/x)\rho_h'(t)dt \\ - \int_x^\infty \Delta(t)(1 - g_m(t/x))\rho_h'(t)dt.\end{aligned} \tag{3.4}$$

Our next task is to express the last summand in this formula in terms of Mellin transforms. On recalling (2.6), Mellin inversion formula gives us, when $u > 0$,

$$\frac{1 - g_m(u)}{u\log^2 u} = \frac{1}{2i\pi}\int_{1-i\infty}^{1+i\infty} H_m(s)u^{-s}ds \tag{3.5}$$

and thus

$$\int_x^\infty \Delta(t)(1-g_m(t/x))\rho_h'(t)dt = \int_1^\infty \Delta(t)(1-g_m(t/x))\rho_h'(t)dt$$
$$= \frac{1}{2i\pi}\int_{1-i\infty}^{1+i\infty} H_m(s)x^s \int_1^\infty \Delta(t)t(\log t)^2\rho_h'(t)\frac{dt}{xt^s}ds.$$

We employ at this level the exact expression of ρ_h', i.e. $t^2\rho_t'(t) = -(\log t)^h + h(\log t)^{h-1}$. We note that, when $\Re s > 1$, we have

$$F(s)/s = \int_1^\infty S(t)dt/t^{s+1} \tag{3.6}$$

where the integral converges absolutely. As a consequence we find that, for any non-negative integer k, the following holds:

$$\frac{d^k}{ds^k}(F(s)/s) = (-1)^k \int_1^\infty S(t)(\log t)^k dt/t^{s+1}. \tag{3.7}$$

We infer from this formula that

$$(-1)^{h+1}\int_0^\infty S(t)t(\log t)^2\rho_h'(t)\frac{dt}{t^s} = \frac{d^{h+2}}{ds^{h+2}}(F(s)/s) + h\frac{d^{h+1}}{ds^{h+1}}(F(s)/s). \tag{3.8}$$

We define

$$G_h(s) = (-1)^h\frac{d^{h+2}}{ds^{h+2}}(F(s)/s) + (-1)^h h\frac{d^{h+1}}{ds^{h+1}}(F(s)/s) - \frac{R_h(s-1)}{(s-1)^{h+r+2}} \tag{3.9}$$

where R_h is a polynomial of degree at most $h+r+1$ and $R_h(s-1)/(s-1)^{h+r+2}$ is the polar part of $G_h(s)$ at $s=1$. The final thing we have to notice is that

$$\int_1^\infty \Delta(t)t(\log t)^2\rho_h'(t)\frac{dt}{t^s}ds = G_h(s) \tag{3.10}$$

for $\Re s \geq 1$. This is however obvious when $\Re s > 1$ by following the above reasoning, simply replacing (3.6) by

$$\left(F(s) - \frac{R_h(s-1)}{(s-1)^r}\right)/s = \int_1^\infty (S(t) - tQ(\log t))\frac{dt}{t^{s+1}}. \tag{3.11}$$

The extension to $\Re s \geq 1$ follows by unicity of analytic continuation, let us say. Here is thus the final result of this section

$$\tilde{\Delta}_h(x) = \frac{\Delta(x)(\log x)^h}{x} + \int_x^{3x} \Delta(t)g_m(t/x)\rho_h'(t)dt - \frac{1}{2i\pi}\int_{1-i\infty}^{1+i\infty} H_m(s)x^{s-1}G_h(s)\frac{ds}{s}. \tag{3.12}$$

4. Proof of Theorem 1.1

In order to use formula (3.12), we need some bounds for G_h. We take s in the region

$$\Re s \geq 1 - \frac{c}{2\log(A + |\Im s|)} \tag{4.1}$$

Then, to bound $G_h(s)$, we use Cauchy's Theorem on a circle centered in s and of radius $(c/2)/\log(A+|\Im s|)$, immediately getting that the modulus of $G_h(s)$ is bounded by a constant multiple of $\log^{\ell+h+2}(A+|\Im s|)$ in the region defined by (4.1).

Once this is established, we select a parameter $T \geq 2$, shift the line of integration to

$$\Re s = 1 - \frac{c}{2\log(A+T)} \tag{4.2}$$

when $|\Im s| \leq T$. When $|\Im s| \geq T$, we shift the line of integration to $\Re s = 1$ and we complete this path with the two horizontal segments $|\Im s| = T$ and $1 - \frac{c}{\log(A+T)} \leq \Re s \leq 1$. We get, on this line $\mathcal{L}$:

$$\int_{\mathcal{L}} \left| H_m(s) x^{s-1} G_h(s) \frac{ds}{s} \right| \ll \left(x^{-(c/2)/\log(A+T)} \log T + \frac{1}{T^2}\left(\frac{2m}{T}\right)^m \right) (\log(A+T))^{\ell+h+2}.$$

We select

$$T = 4m, \qquad m = [\log x].$$

When x is so large that

$$\frac{c}{3}\log x \geq (\ell+h+2)(\log\log(A+4\log x))^2,$$

say $x \geq x_0(A, \ell+h, c)$, the next inequality holds:

$$\int_{\mathcal{L}} \left| H_m(s,\kappa) x^{s-1} G_h(s) \frac{ds}{s} \right| \ll_\kappa \exp\left(-\frac{c\log x}{7\log(A+\log x)} \right).$$

When x is smaller than $x_0(A, \ell+h, c)$, we adjust the constant in the above inequality; indeed the left hand side is bounded is this range. This completes the proof of Theorem 1.1.

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