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Quotient and Product Sets of Thin Subsets of the Positive Integers

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Abstract—We study the cardinalities of A/A and AA for thin subsets A of the set of the first n positive integers. In particular, we consider the typical size of these quantities for random sets A of zero density and compare them with the sizes of A/A and AA for subsets of the shifted primes and the set of sums of two integral squares.

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1. INTRODUCTION

Let A be a set of positive integers. The quotient and the product set of A are defined by $A/A=\{a/a'\colon a,a'\in A\}$ and $AA=\{aa'\colon a,a'\in A\}$, respectively. In this article we will be concerned with estimating the cardinality of AA and A/A when $A\subset\{1,\dots,n\}$ and $|A|=o(n)$, as $n\to+\infty$.

We begin in Section 2 with general lower and upper bounds for $|A/A|$ and $|AA|$, depending on n and the density $\alpha=|A|/n$ of the set A. In Section 3 we consider a probabilistic model $B(n,\alpha)$, where each integer in the set $\{1,\dots,n\}$ is chosen independently to be in A with probability α. We then determine the typical behavior of the cardinality of A/A and AA for random sets A in this probabilistic model. Our first result concerns A/A.

Theorem 1.1. *Let A be a random set in $B(n,\alpha)$. If $\alpha=o(1)$, then $|A/A|\sim|A|^2$ with probability $1-o(1)$.*

Erdős showed that if $A\subset\{1,\dots,n\}$ is a multiplicative Sidon set, that is, if all products aa', $a,a'\in A$, $a\le a'$, are distinct, then $|A|\le\pi(n)(1+o(1))$. In view of this it is interesting that Theorem 1.1 tells us that for almost all sets with $|A|=o(n)$ we have $|A/A|\sim|A|^2$.

It can be shown that the condition $\alpha=o(1)$ cannot be removed in Theorem 1.1. More precisely, it is shown in [1] that when α is a fixed real number and $n\to\infty$, we have $|A/A|\sim c_\alpha|A|^2$ with probability $1-o(1)$, for an explicit $c_\alpha<1$. Moving to $|AA|$, we now state our second result.

Theorem 1.2. *Let A be a random set in $B(n,\alpha)$. If $\alpha=o((\log n)^{-1/2})$, then we have $|AA|\sim|A|^2/2$ with probability $1-o(1)$.*

It is natural to ask if the behavior of some notable sets A is typical in the sense of Theorems 1.1 and 1.2. In Section 4 we study this question when A is one of the sets

$$P_n=\{p\le n\colon p\text{ a prime}\},\qquad P_n-1=\{p-1\colon p\text{ a prime}\le n\},$$
$$Q_n=\{a^2+b^2\colon a^2+b^2\le n\},\qquad Q_n-1=\{a^2+b^2-1\colon a^2+b^2\le n\}.$$

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While our question is easily answered for the set P_n because it is a multiplicative Sidon set, it is more interesting in the other cases. Indeed, the set of shifted primes $P_n - 1$ is the most natural set of zero density for which this question is not trivial. Thus we may ask if the cardinalities of $|(P_n-1)/(P_n-1)|$ and $|(P_n-1)(P_n-1)|$ are typical according to our probabilistic model $B(n,\alpha)$, when $\alpha \sim (\log n)^{-1}$, the density of the set of prime numbers less than or equal to n. Our next result says that this is the case.

Theorem 1.3. *We have* $|(P_n-1)/(P_n-1)| \sim 2|(P_n-1)(P_n-1)| \sim |P_n|^2$.

An easy modification of the proof of Theorem 1.2 proves that $|AA| \gg |A|^2$ with probability $1-o(1)$ when $\alpha \ll (\log n)^{-1/2}$. Notable sets having this density are the sets Q_n and $Q_n - 1$. Both sets have density $\alpha \sim \lambda/\sqrt{\log n}$ where λ is the Landau–Ramanujan constant. The first of these sets has a multiplicative structure, and for this reason it does not behave like a random set.

Theorem 1.4. *With*

$$c = \frac{6}{\pi^2} \prod_{p \equiv 3 \pmod 4} \left(1 + \frac{1}{p^2}\right) < 1$$

we have $|Q_n/Q_n| \sim c|Q_n|^2$.

For the product set of Q_n, the difference with a random set is more dramatic. On account of the multiplicative structure of Q_n, what we have is a generalization of the classical multiplication table problem of Erdős, and, in fact, it easily follows from Theorem 1.4 of A. Mangerel [8] that

$$|Q_nQ_n| \ll |Q_n|^2 (\log n)^{-\delta_0/2} (\log\log n)^{-3/2},$$

where $\delta_0 = 1 - (1 + \log\log 2)/\log 2$, the exponent that appears in the classical multiplication table problem. The shifted set $Q_n - 1$, however, does not retain the multiplicative structure, and then the sizes of the quotient and product sets are closer to what we expect for random sets.

Theorem 1.5. *We have* $|(Q_n-1)(Q_n-1)| \asymp |Q_n|^2$ *and* $|(Q_n-1)/(Q_n-1)| \asymp |Q_n|^2$.

We believe that $|(Q_n-1)(Q_n-1)| \sim |Q_n|^2/2$ and that $|(Q_n-1)/(Q_n-1)| \sim |Q_n|^2$, as with random sets, but we are unable to prove these assertions. We end this section with four other questions[1] that we could not answer.

Problem 1.1. Is it true that if $|A/A| \sim |A|^2$ and $A \subset \{1,\dots,n\}$ then $|A| = o(n)$?

Problem 1.2. Is it true that if $|AA| \sim |A|^2/2$ and $A \subset \{1,\dots,n\}$ then $|A| = o(n/(\log n)^{1/2})$?

Problem 1.3. Is it true that if $|A/A| \sim 6\pi^{-2}n^2$ and $A \subset \{1,\dots,n\}$ then $|A| \sim n$?

Problem 1.4. Is it true that if $|AA| \sim |\{1,\dots,n\}\{1,\dots,n\}|$ and $A \subset \{1,\dots,n\}$ then $|A| \sim n$?

2. LOWER AND UPPER BOUNDS FOR $|AA|$ AND $|A/A|$

The following trivial bounds always hold and cannot be improved in general:

$$\begin{aligned} 2|A| - 1 &\le |A/A| \le |A|^2 - |A| + 1, \\ 2|A| - 1 &\le |AA| \le \frac{|A|^2 + |A|}{2}. \end{aligned} \tag{2.1}$$

The upper bounds are reached when A is a multiplicative Sidon set; for example, when each element of A is a prime number. The lower bounds are reached when A is a geometric progression.

[1] Yurii Shteinikov has solved Problems 1.1 and 1.3. His solutions appear in the article [12] in this volume.

Let us define $r_{AA}(x)$ and $r_{A/A}(x)$ as the number of representations of x as $x = aa'$ and $x = a/a'$, respectively, with $a, a' \in A$. These representation functions have the following properties:

(i) $|A|^2 = \sum_x r_{AA}(x) = \sum_x r_{A/A}(x)$ and

(ii) $\sum_x r_{AA}^2(x) = \sum_x r_{A/A}^2(x)$.

If $\max_x r_{AA}(x) = o(|A|)$, the lower bound in (2.1) for $|AA|$ and $|A/A|$ can be improved.

Proposition 2.1. *For any set A of positive integers we have*

$$|AA|, |A/A| \geq \frac{|A|^2}{\max_x r_{AA}(x)}. \tag{2.2}$$

Proof. This lower bound is trivial for $|AA|$:

$$|A|^2 = \sum_{x \in AA} r_{AA}(x) \leq |AA| \max_x r_{AA}(x).$$

For the quotient set, (2.2) follows by using property (ii) above and remarking by means of the Cauchy–Schwarz inequality that

$$(|A|^2)^2 = \left(\sum_x r_{A/A}(x)\right)^2 \leq |A/A| \sum_x r_{A/A}^2(x) = |A/A| \sum_x r_{AA}^2(x)$$
$$\leq |A/A| \max_x r_{AA}(x) \sum_x r_{AA}(x) \leq |A/A| \cdot |A|^2 \max_x r_{AA}(x). \qquad \square$$

When $A = \{1, \dots, n\}$, the quantity $|A/A|$ is the number of visible lattice points from the origin in the square $[1, n] \times [1, n]$, and it is well known that

$$|\{1, \dots, n\}/\{1, \dots, n\}| \sim \frac{6}{\pi^2} n^2.$$

On the other hand, the quantity $|\{1, \dots, n\}\{1, \dots, n\}|$ is the number of distinct integers that appear in the multiplication table of order n. The problem of estimating this quantity is the classical multiplication table problem. Erdős [5] proved that $|\{1, \dots, n\}\{1, \dots, n\}| = o(n^2)$ and later that $|\{1, \dots, n\}\{1, \dots, n\}| \ll n^2 (\log n)^{-\delta_0}$ where $\delta_0 = 1 - (1 + \log\log 2)/\log 2$. Finally K. Ford [6] proved that

$$|\{1, \dots, n\}\{1, \dots, n\}| \asymp n^2 (\log n)^{-\delta_0} (\log\log n)^{-3/2}.$$

If $A \subset \{1, \dots, n\}$, then $\max_x r_{AA}(x) \leq \max_{m \leq n^2} \tau(m)$ where $\tau(m)$ is the divisor function. Inequality (2.2) and the well-known upper bound for $\tau(m)$ give the lower bounds

$$|AA|, |A/A| \geq |A|^2 \exp\left\{-\log 4 \frac{\log n}{\log\log n}(1 + o(1))\right\}. \tag{2.3}$$

The preceding lower bounds cannot be greatly improved in general, as the following example shows. Let k be the largest integer such that $Q = p_1 \dots p_k \leq n$ and consider the set $A = \{d\colon d \mid Q\}$. It is clear that $|A| = 2^k$, and the prime number theorem implies that $k \sim \log n/\log\log n$. On the other hand, it is easy to see that $AA = \{p_1^{\epsilon_1} \dots p_k^{\epsilon_k}\colon\ 0 \leq \epsilon_j \leq 2\}$ and that $A/A = \{p_1^{\epsilon_1} \dots p_k^{\epsilon_k}\colon\ -1 \leq \epsilon_j \leq 1\}$. In both cases we have

$$|AA| = |A/A| = 3^k = |A|^2 \left(\frac{3}{4}\right)^k = |A|^2 \exp\left\{-\log\frac{4}{3} \cdot \frac{\log n}{\log\log n}(1 + o(1))\right\}.$$

It is possible to take advantage of the density of A to obtain a better lower bound for $|AA|$, at least when the density α is not too small. We will show this by means of the following simple lemma.

Lemma 2.1. *Let $\tau_n(x)$ the number of the representations of x in the form $x = jk$ with $j,k \le n$. Then*

$$\sum_x \tau_n^2(x) \ll n^2 \log n.$$

Proof. We observe that if $k_1 j_1 = k_2 j_2$ then there exist a, b, c and d such that $k_1 = ab$, $j_1 = cd$, $k_2 = ac$ and $j_2 = bd$. To see this, take $a = (k_1, k_2)$, $b = k_1/(k_1, k_2)$, $c = (j_1, j_2)$ and $d = j_1/(j_1, j_2)$.

Consequently, the sum in the lemma is bounded above by

$$T = \bigl|\bigl\{(a,b,c,d)\colon\ ab, cd, ac, bd \le n,\ a \le b,\ c \le d\bigr\}\bigr|.$$

Given $b, c \le n$, we have $a, b \le \min\{n/b, n/c\}$. Thus we have

$$T \le \sum_{b,c\le n} \bigl(\min\{n/b, n/c\}\bigr)^2 \le 2 \sum_{b\le c\le n} \frac{n^2}{c^2} \ll n^2 \log n. \qquad \square$$

Proposition 2.2. *If $A \subset \{1,\dots,n\}$ and $|A| = \alpha n$, then $|AA| \gg |A|^2\alpha^2/\log n$.*

Proof.

$$|A|^4 = \Biggl(\sum_{x\in AA} r_{AA}(x)\Biggr)^2 \le |AA| \sum_x r_{AA}^2(x) \le |AA| \sum_x \tau_n^2(x) \ll |AA| n^2 \log n. \qquad \square$$

Elekes and Ruzsa [4] considered sets A of real numbers of cardinality m and studied lower bounds for $|AA|$ and $|A/A|$ when $|A+A| = Km$. Under these conditions they proved that

$$|AA|, |A/A| \gg \frac{|A|^2}{K^4 \log m}.$$

It is easy to check that their result implies the lower bounds $|AA|, |A/A| \gg |A|^2\alpha^4/\log n$ in our situation. Although this lower bound is worse than what we have given in Proposition 2.2, their result is much more general. Note that we have used the fact that A is a set of integers.

Pomerance and Sárközy [11] proved that if $A \subset \{1,\dots,n\}$ and $|A| = \alpha n$ then

$$|AA| \ge |A|^2\alpha^2 \left(\log\frac{1}{\alpha}\right)^{-2} (\log n)^{-(2\log 2 - 1) + o(1)}. \tag{2.4}$$

They also proved that this result is sharp, at least when $\alpha \gg 1$, in the sense that there is a set $A \subset \{1,\dots,n\}$ with $|A| \sim n$ and $|AA| = |A|^2(\log n)^{-(2\log 2-1)+o(1)}$. Inequality (2.4) evidently gives a better lower bound than Proposition 2.2 when we have $\alpha \gg \exp\bigl\{-(\log n)^{1-\log 2 + o(1)}\bigr\}$.

It turns out that we can get better lower bounds for $|A/A|$ than those given above for $|AA|$. More precisely, it was proved in [2] that $|A/A| \gg |A|^2\alpha^2$ for any set $A \subset \{1,\dots,n\}$ with $|A| > \alpha n$. Later this lower bound was improved by the present authors in [3]. We showed that for any $\epsilon > 0$ we have

$$|A/A| \gg_\epsilon |A|^2\alpha^\epsilon. \tag{2.5}$$

This estimate is optimal in the sense that the factor α^ϵ cannot be removed. This is also shown in [3] by means of an example that is very similar to the one we gave in the paragraph immediately below (2.3).

Turning to upper bounds, since $AA \subset \{1,\dots,n\}\{1,\dots,n\}$, we get the upper bound

$$|AA| \ll \frac{n^2}{(\log n)^{\delta_0}(\log\log n)^{3/2}} \ll \frac{|A|^2}{\alpha^2(\log n)^{\delta_0}(\log\log n)^{3/2}}. \tag{2.6}$$

In the following theorem we summarize all of the estimates discussed above.

Theorem 2.1. *If* $A\subset\{1,\dots,n\}$ *and* $|A|=\alpha n$, *then*

$$\max\left\{\frac{1}{\alpha n}, \frac{1}{D(n)}, \frac{\alpha^2}{\log n}, \frac{(\alpha/\log(1/\alpha))^2}{(\log n)^{2\log 2-1+o(1)}}\right\} \ll \frac{|AA|}{|A|^2} \ll \min\left\{1, \frac{1}{\alpha^2(\log n)^{\delta_0+o(1)}}\right\},$$

$$\max\left\{\frac{1}{\alpha n}, \frac{1}{D(n)}, c_\epsilon\alpha^\epsilon\right\} \ll \frac{|A/A|}{|A|^2} \ll 1,$$

where

$$\delta_0 = 1-\frac{1+\log\log 2}{\log 2} \qquad \textit{and} \qquad D(n)=\max_{m\le n^2}\tau(m)=\exp\left\{\frac{\log 4\cdot\log n}{\log\log n}(1+o(1))\right\}.$$

Proof. The upper bounds $|AA|/|A|^2, |A/A|/|A|^2 \ll 1$ are trivial. The upper bound

$$\frac{|AA|}{|A|^2} \ll \alpha^{-2}(\log n)^{-\delta_0}(\log\log n)^{-3/2}$$

is explained in (2.6). The lower bounds $|AA|/|A|^2, |A/A|/|A|^2 \gg (\alpha n)^{-1}$ are trivial. The lower bounds $|A/A| \gg |A|^2/D(n)$ and $|AA| \gg |A|^2/D(n)$ are explained in (2.3). The lower bound $|AA|/|A|^2 \gg \alpha^2(\log n)^{-1}$ is proved in Proposition 2.2. The lower bound

$$\frac{|AA|}{|A|^2} \gg \left(\frac{\alpha}{\log(1/\alpha)}\right)^2(\log n)^{1-2\log 2+o(1)}$$

is explained in (2.4). The lower bound $|A/A|/|A|^2 \gg c_\epsilon\alpha^\epsilon$ is explained in (2.5). $\square$

3. RANDOM SETS

Let n be a large enough integer and α a real number in $[0,1]$ with $\alpha n\ge 1$. Here we consider the probabilistic model $B(n,\alpha)$. In this model A denotes a random subset of the set $\{1,\dots,n\}$, with each element of this set independently chosen to belong to A with a fixed probability $\alpha\in[0,1]$. One may then verify that for the random variable $|A|=\sum_{1\le i\le n}\mathbf{1}_{\{i\in A\}}$ we have

(i) $\mathbb{E}(|A|)=\alpha n$ and $\mathrm{Var}(|A|)=n\alpha(1-\alpha)$;

(ii) $\mathbb{E}(|A|^2)=(\alpha n)^2+n\alpha(1-\alpha)$ and $\mathrm{Var}(|A|^2)=4n^3\alpha^3(1-\alpha)+O(n^2\alpha^2)$.

Indeed, the random variables $\mathbf{1}_{\{i\in A\}}$, with i varying over the set $\{1,\dots,n\}$, are mutually independent. Therefore, for any $\lambda\in\mathbb{R}$ we have

$$\mathbb{E}(\exp\{\lambda|A|\}) = \prod_{1\le i\le n}\mathbb{E}\big(\exp\{\lambda\,\mathbf{1}_{\{i\in A\}}\}\big). \tag{3.1}$$

Also, for any i we have $\mathbb{E}\big(\exp\{\lambda\,\mathbf{1}_{\{i\in A\}}\}\big)=1+\alpha\sum_{k\ge1}\lambda^k/k!$ for all $\lambda\in\mathbb{R}$. Then it follows that

$$\sum_{k\ge0}\frac{\lambda^k\,\mathbb{E}(|A|^k)}{k!} = \mathbb{E}(\exp\{\lambda|A|\}) = (1+\alpha(\exp\{\lambda\}-1))^n \tag{3.2}$$

for all real λ. Thus we may obtain $\mathbb{E}(|A|^k)$ for any given integer $k\ge0$ by differentiating k times $(1+\alpha(\exp\{\lambda\}-1))^n$ with respect to λ and putting $\lambda=0$ in the result. Doing this for $k=1,2$

and 4 by means of the classical formula of Faà di Bruno (see [10, p. 807, Theorem 2]) applied to the composition of functions $\lambda \to f(g(\lambda))$ with $f(\lambda) = \lambda^n$ and $g(\lambda) = 1 + \alpha(\exp(\lambda) - 1)$ gives the formulae for $\mathbb{E}(|A|)$ and $\mathbb{E}(|A|^2)$ listed above as well as

$$\begin{aligned}\mathbb{E}(|A|^4) &= \alpha^4 n(n-1)(n-2)(n-3) + 6\alpha^3 n(n-1)(n-2) + 7\alpha^2 n(n-1) + \alpha n \\ &= n^4\alpha^4 + 6n^3\alpha^3(1-\alpha) + O(n^2\alpha^2),\end{aligned}$$

which readily implies the formulae for the variances in (i) and (ii). These formulae then yield the following lemma.

Lemma 3.1. *If $\alpha n \to \infty$, then $|A| \sim \alpha n$ and $|A|^2 \sim (\alpha n)^2$ with probability $1 - o(1)$.*

To prove Theorems 1.1 and 1.2, we will need two further observations given by the propositions below.

Proposition 3.1. *If $\alpha = o(1)$ and $\alpha n \to \infty$, then $\mathbb{E}(|A/A|) \sim (\alpha n)^2$ in $B(n, \alpha)$.*

Proof. Given a/b, $1 \le a < b \le n$, $(a, b) = 1$, we observe that $a/b \in A/A$ if and only if $at, bt \in A$ for some $t \le n/b$. Denote by E_t the event $(at, bt \in A)$ and by E_t^{c} the complementary event. We have

$$\mathbb{P}(a/b \in A/A) = 1 - \mathbb{P}\biggl(\bigcap_{t \le n/b} E_t^{\mathrm{c}}\biggr).$$

We now note that the events E_t, $t \le n/b$, are independent if $b > \sqrt{n}$. Indeed, if not, there are $t_1 \ne t_2 \le n/b$ such that $at_1 = bt_2$. Then $t_1 = bz$ and $t_2 = az$ for some positive integer z. The condition $t_1 \le n/b$ now implies that $bz \le n/b$ and hence that $b \le \sqrt{n}$, justifying our assertion. It follows that the events E_t^{c}, $t \le n/b$, are also independent events when $b > \sqrt{n}$. Further, we have $\mathbb{P}(E_t) = \alpha^2$ and $\mathbb{P}(E_t^{\mathrm{c}}) = 1 - \alpha^2$. With these remarks we obtain the following assertions:

(i) If $b > \sqrt{n}$ and $(a, b) = 1$, then

$$\mathbb{P}(a/b \in A/A) = 1 - \prod_{t \le n/b} \mathbb{P}(E_t^{\mathrm{c}}) = 1 - \prod_{t \le n/b} (1 - \alpha^2) = 1 - (1 - \alpha^2)^{\lfloor n/b \rfloor}.$$

(ii) If $n\alpha^2 < b \le \sqrt{n}$, we have

$$\mathbb{P}(a/b \in A/A) \le \sum_{t \le n/b} \mathbb{P}(E_t) \le \frac{n\alpha^2}{b}.$$

(iii) If $b \le n\alpha^2$, we use the trivial bound $\mathbb{P}(a/b \in A/A) \le 1$.

By means of the classical inequality $1 - (1 - \theta)^\ell \le \ell\theta$, valid for θ in $[0, 1]$ and integer $\ell \ge 0$, we see that $1 - (1 - \alpha^2)^{\lfloor n/b \rfloor} \le \min\{1, n\alpha^2/b\}$. This gives

$$\mathbb{P}(a/b \in A/A) = 1 - (1 - \alpha^2)^{\lfloor n/b \rfloor} + O\biggl(\min\biggl\{1, \frac{n\alpha^2}{b}\biggr\}\biggr)$$

when $b \le \sqrt{n}$. Thus

$$\begin{aligned}\sum_{\substack{1 \le a < b \le \sqrt{n} \\ (a,b)=1}} \mathbb{P}(a/b \in A/A) &= \sum_{\substack{1 \le a < b \le \sqrt{n} \\ (a,b)=1}} 1 - (1 - \alpha^2)^{\lfloor n/b \rfloor} + O\biggl(\sum_{b \le \sqrt{n}} \min\{b, n\alpha^2\}\biggr) \\ &= \sum_{\substack{1 \le a < b \le \sqrt{n} \\ (a,b)=1}} 1 - (1 - \alpha^2)^{\lfloor n/b \rfloor} + O(n^{3/2}\alpha^2),\end{aligned}$$

on using $\min\{b, n\alpha^2\} \le n\alpha^2$. The preceding calculations allows us to write down an expression for $\mathbb{E}(|A/A|)$:

$$\mathbb{E}(|A/A|) = 1 + 2 \sum_{\substack{1\le a<b\le\sqrt{n}\\(a,b)=1}} \mathbb{P}(a/b \in A/A) + 2 \sum_{\substack{1\le a<b\le n\\(a,b)=1,\ b>\sqrt{n}}} \mathbb{P}(a/b \in A/A)$$

$$= 2S + O(1 + n^{3/2}\alpha^2), \tag{3.3}$$

where

$$S = \sum_{\substack{1\le a\le b\le n\\(a,b)=1}} \bigl(1 - (1-\alpha^2)^{\lfloor n/b\rfloor}\bigr).$$

We now transform S in the following manner:

$$S = \sum_{\substack{1\le a\le b\le n\\(a,b)=1}} \bigl(1 - (1-\alpha^2)^{\lfloor n/b\rfloor}\bigr) = \sum_{1\le b\le n} \varphi(b)\bigl(1 - (1-\alpha^2)^{\lfloor n/b\rfloor}\bigr)$$

$$= \sum_{1\le k} \bigl(1 - (1-\alpha^2)^k\bigr) \sum_{n/(k+1)<b\le n/k} \varphi(b).$$

We use the notation $\Phi(x) = \sum_{m\le x} \varphi(m)$ to write

$$S = \sum_{1\le k} \bigl(1 - (1-\alpha^2)^k\bigr)\left(\Phi\Bigl(\frac{n}{k}\Bigr) - \Phi\Bigl(\frac{n}{k+1}\Bigr)\right)$$

$$= \sum_{1\le k} \Phi\Bigl(\frac{n}{k}\Bigr)\bigl(1 - (1-\alpha^2)^k - \bigl(1 - (1-\alpha^2)^{k-1}\bigr)\bigr)$$

$$= \alpha^2 \sum_{1\le k} \Phi\Bigl(\frac{n}{k}\Bigr)(1-\alpha^2)^{k-1}$$

$$= \alpha^2 \sum_{1\le k\le\alpha^{-2}} \Phi\Bigl(\frac{n}{k}\Bigr)(1-\alpha^2)^{k-1} + O\left(\alpha^2 \sum_{k>\alpha^{-2}} \Bigl(\frac{n}{k}\Bigr)^2\right). \tag{3.4}$$

Now we use the well-known estimate $\Phi(x) = 3\pi^{-2}x^2 + O(x\log x)$ to get

$$\sum_{1\le k\le\alpha^{-2}} \Phi\Bigl(\frac{n}{k}\Bigr)(1-\alpha^2)^{k-1} = \frac{3n^2}{\pi^2} \sum_{1\le k\le\alpha^{-2}} \frac{(1-\alpha^2)^{k-1}}{k^2} + O\left(\sum_{1\le k\le\alpha^{-2}} \frac{n}{k}\log\frac{n}{k}\right). \tag{3.5}$$

Again using the inequality $1 - (1-\theta)^\ell \le \ell\theta$, we have

$$\sum_{1\le k\le\alpha^{-2}} \frac{(1-\alpha^2)^{k-1}}{k^2} = \sum_{1\le k\le\alpha^{-2}} \frac{1}{k^2} + \sum_{1\le k\le\alpha^{-2}} \frac{(1-\alpha^2)^{k-1} - 1}{k^2}$$

$$= \frac{\pi^2}{6} + O(\alpha^2) + O\left(\sum_{k\le\alpha^{-2}} \frac{\alpha^2}{k}\right). \tag{3.6}$$

Combining (3.6) with (3.5) and (3.4), we get

$$S = \frac{(\alpha n)^2}{2} + O\left(\alpha^4 n^2 \log\frac{1}{\alpha}\right) + O\bigl(\alpha^2 n \log^2 n\bigr). \tag{3.7}$$

Finally, combining (3.7) and (3.3), we have

$$\mathbb{E}(|A/A|) = (\alpha n)^2 + O\left(\alpha^4 n^2 \log \frac{1}{\alpha}\right) + O\big(\alpha^2 n \log^2 n\big) + O\big(1 + n^{3/2}\alpha^2\big).$$

Recalling that $\alpha = o(1)$, we then conclude that $\mathbb{E}(|A/A|) \sim (\alpha n)^2$ as $\alpha n \to \infty$. $\square$

Proposition 3.2. *If $\alpha n \to \infty$ and $\alpha = o(1/\sqrt{\log n}\,)$, then $\mathbb{E}(|AA|) \sim (\alpha n)^2/2$ in $B(n,\alpha)$.*

Proof. Given $x \le n^2$, let $\tau_n(x) = |\{(m_1, m_2)\colon m_1 m_2 = x,\ 1 \le m_1, m_2 \le n\}|$, the function introduced in Lemma 2.1.

If x is not a square, then

$$\mathbb{P}(x \in AA) = \mathbb{P}\Bigg(\bigcup_{m_1 m_2 = x,\ m_1 < m_2 \le n} (m_1 \in A,\ m_2 \in A)\Bigg),$$

$$\begin{aligned}\mathbb{P}(x \in AA) &= 1 - \mathbb{P}\Bigg(\bigcap_{m_1 m_2 = x,\ m_1 < m_2 \le n} (m_1 \notin A \text{ or } m_2 \notin A)\Bigg)\\ &= 1 - \prod_{m_1 m_2 = x,\ m_1 < m_2} (1-\alpha^2) = 1 - (1-\alpha^2)^{\tau_n(x)/2}.\end{aligned}$$

If x is a square, then

$$\mathbb{P}(x \in AA) = 1 - (1-\alpha^2)^{\tau_n(x)/2} + O(\alpha).$$

Thus we have

$$\mathbb{E}(|AA|) = \sum_x \big(1 - (1-\alpha^2)^{\tau_n(x)/2}\big) + O(\alpha n).$$

We use the inequality $|1 - zk - (1-z)^k| \ll (zk)^2$, the equality $\sum_x \tau_n(x) = n^2$ and the estimate $\sum_x \tau_n^2(x) \ll n^2 \log n$ proved in Lemma 2.1 to get

$$\mathbb{E}(|AA|) = \alpha^2 \sum_x \frac{\tau_n(x)}{2} + O\left(\alpha^4 \sum_x \tau_n^2(x)\right) + O(\alpha n) = \frac{\alpha^2 n^2}{2} + O\big(\alpha^4 n^2 \log n\big) + O(\alpha n) \sim \frac{(\alpha n)^2}{2},$$

since $\alpha = o((\log n)^{-1/2})$. $\square$

Proof of Theorems 1.1 and 1.2. Assume that $\alpha n \to \infty$ and consider the random variable

$$X_A = |A|^2 - |A| + 1 - |A/A|.$$

Since X_A is nonnegative, Markov's inequality and Proposition 3.1 give

$$\mathbb{P}\big(X_A \ge \epsilon(\alpha n)^2\big) \le \frac{\mathbb{E}(X_A)}{\epsilon(\alpha n)^2} = o(1)$$

for any $\epsilon > 0$ as $n \to \infty$. Thus, $X_A = o((\alpha n)^2)$ with probability $1 - o(1)$. Combining this with Lemma 3.1 and Proposition 3.1, we conclude that $|A/A| = |A|^2 + o(|A|^2)$ with probability $1 - o(1)$. It proves Theorem 1.1 when $\alpha n \to \infty$. The proof of Theorem 1.2 when $\alpha n \to \infty$ is similar. We only have to replace X_A with the random variable $Y_A = (|A|^2 + |A|)/2 - |AA|$. $\square$

4. QUOTIENT AND PRODUCT SETS OF NOTABLE SETS

We prove Theorems 1.3–1.5 in this section. We begin with a number of preliminary results.

When A is a subset of the positive integers, we write $\Delta(A)$ for the number of quadruples $(a_1,a_2,a_3,a_4)\in A^4$ satisfying $a_1/a_2=a_3/a_4$ and $a_1\neq a_2,a_3$.

Lemma 4.1. *For any subset A of the positive integers we have the following*:

(i) *If* $\Delta(A)=o(|A|^2)$, *then* $|A/A|\sim|A|^2\sim 2|AA|$.

(ii) *If* $\Delta(A)\ll|A|^2$, *then* $|A/A|\asymp|A|^2\asymp|AA|$.

Proof. Let A be any given subset of the positive integers. Then

$$\Delta(A)=\sum_{x\neq 1} r_{A/A}(x)(r_{A/A}(x)-1). \tag{4.1}$$

Also, we have

$$|A|^2-|A|=\sum_{x\in A/A,\ x\neq 1} r_{A/A}(x). \tag{4.2}$$

An application of the Cauchy–Schwarz inequality to the right-hand side of (4.2) gives

$$(|A|^2-|A|)^2\le |A/A|\sum_{x\neq 1} r^2_{A/A}(x).$$

On writing $r^2_{A/A}(x)$ as $r_{A/A}(x)+r_{A/A}(x)(r_{A/A}(x)-1)$ in the above relation and using (4.1) together with (4.2), we then deduce that

$$(|A|^2-|A|)^2\le |A/A|\big(|A|^2-|A|+\Delta(A)\big).$$

Thus,

$$|A/A|\ge \frac{(|A|^2-|A|)^2}{|A|^2-|A|+\Delta(A)}\begin{cases} =|A|^2(1+o(1)) & \text{if } \ \Delta(A)=o(|A|^2),\\ \gg |A|^2 & \text{if } \ \Delta(A)\ll |A|^2.\end{cases}$$

The first line of the above relation verifies the comparison between $|A/A|$ and $|A|^2$ in (i), while the second line together with the trivial inequality $|A/A|\le|A|^2$ gives the first assertion in (ii). For the remaining claims of the lemma we proceed analogously. Indeed, we have

$$\begin{aligned}|A|^4 &\le |AA|\sum_x r^2_{AA}(x)=|AA|\sum_x r^2_{A/A}(x)=|AA|\left(|A|^2+\sum_{x\neq 1} r^2_{A/A}(x)\right)\\ &=|AA|\big(2|A|^2-|A|+\Delta(A)\big),\end{aligned}$$

and consequently obtain

$$|AA|\ge \frac{|A|^4}{2|A|^2-|A|+\Delta(A)}\begin{cases} =\dfrac{1}{2}|A|^2(1+o(1)) & \text{if } \ \Delta(A)=o(|A|^2),\\ \gg |A|^2 & \text{if } \ \Delta(A)\ll |A|^2,\end{cases}$$

as required. $\square$

Proposition 4.1. *Given integers $a<b$, let E denote $ab(b-a)$ and let*

$$T_1(Y;a,b)=\big\{m\le Y\colon\ 1+am \ \textit{and} \ 1+bm \ \textit{are sums of two squares}\big\},$$

$$T_2(Y;a,b)=\big\{m\le Y\colon\ 1+am \ \textit{and} \ 1+bm \ \textit{are primes}\big\}.$$

Then for $j = 1, 2$ we have

$$T_j(Y; a, b) \ll \frac{Y}{\log^j Y} \prod_{p|E}\left(1 - \frac{1}{p}\right)^{-2}.$$

Proof. The proposition is a consequence of the Selberg sieve. For $j = 2$ it follows from the standard theorem of H. Halberstam and H.-E. Richert [7, p. 70, Theorem 2.3]. For $j = 1$, however, the proposition depends on an application of the version of the Selberg sieve that allows sieving out residue classes modulo prime powers. The required application is given by the theorem of W.G. Nowak [9, p. 72]. $\square$

Proof of Theorem 1.3. The theorem will follow from Lemma 4.1 applied with A taken to be $P_n - 1$ once we prove that $\Delta(P_n - 1) = o(|P_n|^2)$ as $n \to +\infty$, where

$$\Delta(P_n - 1) = \left|\left\{(p_1, p_2, p_3, p_4)\colon\ p_i \le n,\ \frac{p_1 - 1}{p_2 - 1} = \frac{p_3 - 1}{p_4 - 1},\ p_1 \ne p_2, p_3\right\}\right|. \tag{4.3}$$

Each quadruple (p_1, p_2, p_3, p_4) counted by the right-hand side of (4.3) satisfies

$$(p_1 - 1)(p_4 - 1) = (p_3 - 1)(p_2 - 1).$$

To each such quadruple there corresponds a quadruple (q_1, q_2, q_3, q_4) of integers such that

$$p_1 - 1 = q_1q_3, \qquad p_2 - 1 = q_1q_4, \qquad p_3 - 1 = q_2q_3, \qquad p_4 - 1 = q_2q_4,$$

where the integers q_i are defined by $q_1 = (p_1 - 1, p_2 - 1)$, $q_2 = (p_3 - 1, p_4 - 1)$ and $q_3 = (p_1 - 1)/q_1$, $q_4 = (p_4 - 1)/q_2$. Consequently, we have

$$\Delta(P_n - 1) = \left|\left\{(q_1, q_2, q_3, q_4)\colon\ q_1q_3, q_2q_3,\ q_1q_4, q_2q_4 \in P_n - 1,\ q_3 \ne q_4,\ q_1 \ne q_2\right\}\right|.$$

Since each $p_i - 1 \le n$, we have that either $\max\{q_1, q_2\} \le \sqrt{n}$ or $\max\{q_3, q_4\} \le \sqrt{n}$. Therefore,

$$\begin{aligned}\Delta(P_n - 1) &\le 4 \sum_{1 \le q_1 < q_2 \le \sqrt{n}} \left|\left\{(q_3, q_4)\colon\ q_1q_3 + 1, q_2q_3 + 1, q_1q_4 + 1, q_2q_4 + 1 \in P_n\right\}\right| \\ &\le 4 \sum_{1 \le q_1 < q_2 \le \sqrt{n}} \left|\left\{m\colon\ q_1m + 1, q_2m + 1 \in P_n\right\}\right|^2.\end{aligned}$$

We use the notation and the conclusion of Proposition 4.1 for $j = 2$ to get

$$\begin{aligned}\Delta(P_n - 1) &\ll \sum_{1 \le q_1 < q_2 \le \sqrt{n}} T_2^2\left(\frac{n}{q_2}; q_1, q_2\right) \\ &\ll \sum_{1 \le q_1 < q_2 \le \sqrt{n}} \frac{n^2}{q_2^2 \log^4(n/q_2)} \prod_{p|q_1q_2(q_1 - q_2)} \left(1 - \frac{1}{p}\right)^{-4} \\ &\ll \frac{n^2}{\log^4 n} \sum_{1 \le q_1 < q_2 \le \sqrt{n}} \frac{f(q_1q_2(q_2 - q_1))}{q_2^2},\end{aligned}$$

where we have set $f(n) = \prod_{p|n}(1 - 1/p)^{-4}$ for any integer $n \ge 1$. We now have the following lemma.

Lemma 4.2. *For any integers $m, n \ge 1$ and $q_2 \ge 2$ we have*

$$f(nm) \le f(n)f(m), \tag{4.4}$$

$$f(n) \le \sum_{d|n,\ \mu(d)\ne 0} \frac{30^{\omega(d)}}{d}, \tag{4.5}$$

$$\sum_{1\le q_1<q_2} f(q_1(q_2-q_1)) \ll q_2. \tag{4.6}$$

Proof. Inequality (4.4) is clear. Also, equality holds in (4.4) when $(m, n) = 1$. To prove (4.5), we observe that $(1 - 1/p)^{-4} \le 1 + 30/p$ for any prime p. Thus if n' is the squarefree part of an integer n, we have

$$f(n) = f(n') = \prod_{p|n'}\left(1-\frac{1}{p}\right)^{-4} \le \prod_{p|n'}\left(1+\frac{30}{p}\right) = \sum_{d|n'} \frac{30^{\omega(d)}}{d} = \sum_{d|n,\ \mu(d)\ne 0} \frac{30^{\omega(d)}}{d}.$$

It then follows that for a given $q_2 \ge 2$ we have

$$\sum_{1\le q_1<q_2} f(q_1(q_2-q_1)) \le \sum_{d\le q_2^2,\ \mu(d)\ne 0} \frac{30^{\omega(d)}}{d}\big|\{q_1\colon\ q_1<q_2,\ q_1(q_2-q_1)\equiv 0 \pmod d\}\big|.$$

Since for squarefree d the number of solutions of the congruence $x(q_2 - x) \equiv 0 \pmod d$, with x ranging over a complete system of residue classes modulo d, does not exceed $2^{\omega(d)}$, we obtain

$$\sum_{1\le q_1<q_2} f(q_1(q_4-q_1)) \le \sum_{\substack{d\le q_2^2\\ \mu(d)\ne 0}} \frac{30^{\omega(d)}}{d}\Big(\frac{q_2}{d}+1\Big)2^{\omega(d)} \le 2q_2\prod_p\left(1+\frac{60}{p^2}\right) + 2\prod_{p\le q_2^2}\left(1+\frac{60}{p}\right) \ll q_2,$$

where for the penultimate inequality we split $d \le q_2^2$ into $d \le q_2$ and $q_2 < d \le q_2^2$. □

Returning to the proof of Theorem 1.3, we then have

$$\begin{aligned}
\Delta(P_n-1) &\ll \frac{n^2}{\log^4 n}\sum_{q_2\le\sqrt{n}} \frac{f(q_2)}{q_2^2}\sum_{q_1<q_2} f(q_1(q_2-q_1))\\
&\ll \frac{n^2}{\log^4 n}\sum_{q_2\le\sqrt{n}} \frac{f(q_2)}{q_2} \ll \frac{n^2}{\log^4 n}\sum_{d\le\sqrt{n},\ \mu(d)\ne 0} \frac{30^{\omega(d)}}{d}\sum_{q_2\le\sqrt{n},\ d|q_2}\frac{1}{q_2}\\
&\ll \frac{n^2}{\log^4 n}\sum_{d\le\sqrt{n},\ \mu(d)\ne 0} \frac{30^{\omega(d)}}{d^2}\sum_{q\le\sqrt{n}/d}\frac{1}{q} \ll \frac{n^2}{\log^3 n}\prod_p\left(1+\frac{30}{p^2}\right)\\
&\ll \frac{n^2}{\log^3 n} = o(|P_n|^2).
\end{aligned}$$

With this last bound Lemma 4.1 gives

$$|(P_n-1)/(P_n-1)| \sim 2|(P_n-1)(P_n-1)| \sim |P_n|^2,$$

which completes the proof of Theorem 1.3. □

Proof of Theorem 1.4. The first observation is that if $(a, b) = 1$, then $a/b \in Q_n/Q_n$ if and only if $a, b \in Q_n$. Thus

$$|Q_n/Q_n| = \sum_{a,b\in Q_n,\ (a,b)=1} 1 = \sum_{a,b\in Q_n}\ \sum_{d|(a,b)} \mu(d) = \sum_{d\le n}\mu(d)\Bigg(\sum_{a\in Q_n,\ d|a} 1\Bigg)^2.$$

Let $E_1 = \{2\} \cup \{\text{primes } p \equiv 1 \pmod 4\}$ and $E_2 = \{\text{primes } p \equiv 3 \pmod 4\}$. For $i = 1, 2$ define $N_i = \{n\colon p \mid n \Rightarrow p \in E_i\}$.

We can write any integer d in only one way in the form $d = d_1 d_2$ with $d_1 \in N_1$ and $d_2 \in N_2$. Note that if $d_2 \mid a$, $\mu(d_2) \neq 0$ and $a \in Q_n$, then $d_2^2 \mid a$.

Then if $a \in Q_n$, $d \mid a$ and $\mu(d) \neq 0$, we have $d_1 d_2^2 \mid a$ and $a/(d_1 d_2^2) \in Q_{n/(d_1 d_2^2)}$. Thus, taking $T = \log^2 n$, we have

$$\begin{aligned} |Q_n/Q_n| &= \sum_{\substack{d_1 d_2^2 \le n \\ d_1 \in N_1,\, d_2 \in N_2}} \mu(d_1 d_2) \bigl|Q_{n/(d_1 d_2^2)}\bigr|^2 \\ &= \sum_{\substack{d_1 d_2^2 \le T \\ d_1 \in N_1,\, d_2 \in N_2}} \mu(d_1 d_2) \bigl|Q_{n/(d_1 d_2^2)}\bigr|^2 + \sum_{\substack{d_1 d_2^2 > T \\ d_1 \in N_1,\, d_2 \in N_2}} \mu(d_1 d_2) \bigl|Q_{n/(d_1 d_2^2)}\bigr|^2. \end{aligned} \tag{4.7}$$

In the last sum we use the trivial estimate $\bigl|Q_{n/(d_1 d_2^2)}\bigr| \le n/(d_1 d_2^2)$ obtaining

$$\left| \sum_{\substack{d_1 d_2^2 > T \\ d_1 \in N_1,\, d_2 \in N_2}} \mu(d_1 d_2) \bigl|Q_{n/(d_1 d_2^2)}\bigr|^2 \right| \le n^2 \sum_{\substack{d_1 d_2^2 > T \\ d_1 \in N_1,\, d_2 \in N_2}} \frac{1}{d_1^2 d_2^4} \ll \frac{n^2}{T} = o(|Q_n|^2). \tag{4.8}$$

In the first sum we use the equality $Q_x = \lambda x/\sqrt{\log x} + E(x)$ with $E(x) = O(x/\log^{3/2} x)$ where λ is the Landau–Ramanujan constant. We observe that if $d_1 d_2^2 \le T$ then

$$\begin{aligned} Q_{n/(d_1 d_2^2)} &= \frac{\lambda n}{d_1 d_2^2 \sqrt{\log n - \log(d_1 d_2^2)}} + O\biggl(\frac{n}{d_1 d_2^2} \frac{1}{\log^{3/2}(n/(d_1 d_2^2))}\biggr) \\ &= \frac{\lambda n}{d_1 d_2^2 \sqrt{\log n}} \biggl(1 + O\biggl(\frac{\log T}{\log n}\biggr)\biggr). \end{aligned}$$

Thus we have

$$\begin{aligned} \sum_{\substack{d_1 d_2^2 \le T \\ d_1 \in N_1,\, d_2 \in N_2}} \mu(d_1 d_2) \bigl|Q_{n/(d_1 d_2^2)}\bigr|^2 &= \frac{\lambda^2 n^2}{\log n} \sum_{\substack{d_1 d_2^2 \le T \\ d_1 \in N_1,\, d_2 \in N_2}} \frac{\mu(d_1 d_2)}{d_1^2 d_2^4} \biggl(1 + O\biggl(\frac{\log T}{\log n}\biggr)\biggr) \\ &= \frac{\lambda^2 n^2}{\log n} \sum_{\substack{d_1 d_2^2 \le T \\ d_1 \in N_1,\, d_2 \in N_2}} \frac{\mu(d_1 d_2)}{d_1^2 d_2^4} + O\biggl(\frac{n^2 \log T}{\log^2 n}\biggr) \\ &= \frac{\lambda^2 n^2}{\log n} \sum_{\substack{d_1 d_2^2 \\ d_1 \in N_1,\, d_2 \in N_2}} \frac{\mu(d_1 d_2)}{d_1^2 d_2^4} + O\biggl(\frac{n^2}{T \log n}\biggr) + O\biggl(\frac{n^2 \log T}{\log^2 n}\biggr). \end{aligned} \tag{4.9}$$

Putting together (4.7), (4.8) and (4.9), we get

$$|Q_n/Q_n| \sim |Q_n|^2 \sum_{\substack{d_1 d_2^2 \\ d_1 \in N_1,\, d_2 \in N_2}} \frac{\mu(d_1 d_2)}{d_1^2 d_2^4}.$$

Finally we observe that

$$\sum_{\substack{d_1 d_2^2 \\ d_1\in N_1,\, d_2\in N_2}} \frac{\mu(d_1 d_2)}{d_1^2 d_2^4} = \sum_{d_1\in N_1} \frac{\mu(d_1)}{d_1^2} \sum_{d_2\in N_2} \frac{\mu(d_2)}{d_2^4} = \prod_{p\not\equiv 3 \ (\mathrm{mod}\ 4)} \left(1-\frac{1}{p^2}\right) \prod_{p\equiv 3 \ (\mathrm{mod}\ 4)} \left(1-\frac{1}{p^4}\right)$$

$$= \prod_{p}\left(1-\frac{1}{p^2}\right) \prod_{p\equiv 3 \ (\mathrm{mod}\ 4)} \left(1+\frac{1}{p^2}\right) = \frac{6}{\pi^2} \prod_{p\equiv 3 \ (\mathrm{mod}\ 4)} \left(1+\frac{1}{p^2}\right). \quad \square$$

Proof of Theorem 1.5. The proof is similar to that of Theorem 1.3. We take A to be $Q_n - 1$ instead of $P_n - 1$ and consider $T_1(n/q_2; q_1, q_2)$ in place of $T_2(n/q_2; q_1, q_2)$. Then, following the same method as above but using the estimate for $T_1(n/q_2; q_1, q_2)$ from Proposition 4.1, we get $\Delta(Q_n - 1) \ll n^2/\log n$. Thus $\Delta(Q_n - 1) \ll |Q_n|^2$ and an application of Lemma 4.1 yields the required conclusion. $\square$

Note. Javier Cilleruelo was the moving force behind this article. Javier passed away on May 15, 2016. The themes considered here were very dear to him. In particular, the problems posed in Section 1 are due to him. Javier was an extraordinarily decent person, unfailingly kind and generous. When not absorbed in his mathematics or with his family, he was actively organizing for the Podemos, playing the Spanish guitar or throwing darts at a pub. His untimely death leaves an emptiness that will be felt forever.

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