Planik and Square Dance, two planar permutation games

Olivier Ramaré

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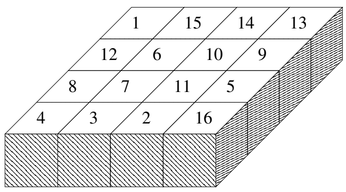
Abstract

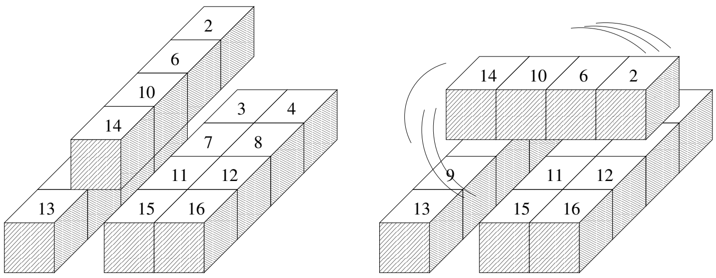
`File Planik-04.tex` The Planik is a two dimensional version of a game close to the Rubik's cube. In this paper, we present this game and the group theoretical background required to solve it. We adopt a low profile style for students to be able to follow, and add comments aimed at a more experienced audience. We end this study with notes concerning a second family of games that shares similarities with the Planik.

1 An introduction for the players

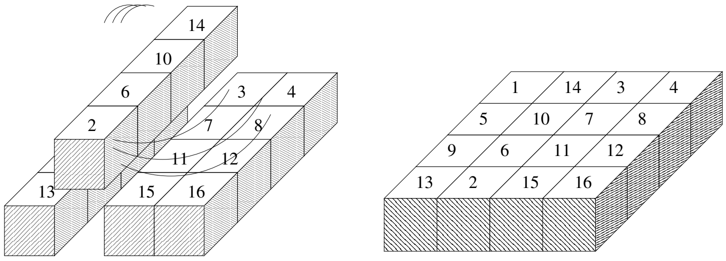
The focus rests on a game we call "Planik". We present later some generalizations as well as a second family of games generically called "Square Dance". The Planik is a single player permutation game on the 4×4 square:



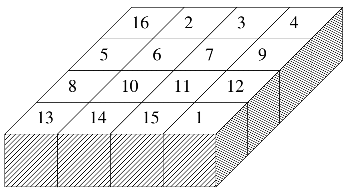
The aim is to sort the inscribed numbers by using only some given moves. These can be described swiftly: a player can invert any column and any row. In more details, here is a move on a column:



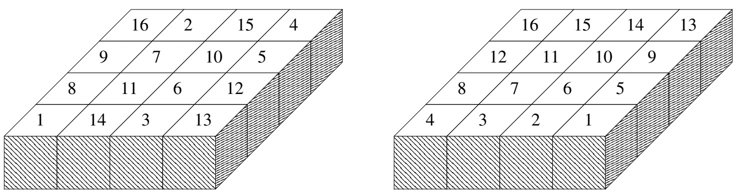
An elementary move



There are only eight elementary moves: four on the rows and four on the columns. In general, we start from a given distribution of the little cubes and seek to sort the Planik out by using a sequence of elementary moves. Here is a further problem:



After some efforts, the readers will understand that the new challenge is to prove that sorting this Planik out is *not* possible. Here are two more Planiks to sort:



Two additional problems

As the readers have now understood, we use the word *Planik* to denote two distinct objects: the game in itself, or a position of the cubes on this board.

The moves we shew are termed *elementary moves*; they are used in sequences, and such a sequence is generically called a *move*. When we need to recall a move we did, we use a, b, c, d for the elementary moves on the rows and A, B, C, and D for the ones on the columns.

2 An introduction for the teachers

Group theoretical courses can be very abstract, and this game may be used to introduce essential notions. I would advise to present the game at first and set as a goal the description of all the reachable positions, as well as a clear answer to the question:

Fundamental Question. *Given any two Planiks, is it possible to go from one to the other by elementary moves?*

The game just described is a *permutation* game, in which some pieces are shifted around, none are destroyed or created, and the board on which to play remains fixed.

In shorthand: executing two moves is the same as *composing two permutations*, so the collection of positions accessible from the initial one is the set of permutations generated by the elementary moves. As such, it is a subgroup, say G, of the full permutation group, which we now have to describe. It is given a priori by a *representation.* We shall show this representation has four *orbits* of four elements, and that the subgroup G is isomorphic to the subgroup of the product of the four permutation groups on these orbits restricted to the permutations for which the product of the *signatures* is 1.

We now need to define all the words we have employed and then proceed to the proof. As the readers will see, the full description of the possible Planiks requires only little background. Attention and energy are all that will be asked on the students. I have done this study with 17-year-old kids, leaving them lots of room for experiments and private investigations.

On the writing style

We present in the sequel an answer to the Fundamental Question above by trying to stick to rudimentary vocabulary and notions. Comments that may sound obscure to beginners are sometimes added, but on the whole, this paper should be accessible to both an advanced or a less specialized audience. As a consequence, our notation is more explicit than usual. For instance, the composition of two permutations, say σ_1 and σ_2, is denoted by $\sigma_1 \circ \sigma_2$ rather then the by the short form $\sigma_1\sigma_2$. We will however often mix the settings and talk of the "product" of two permutations and not of their "composition". The transposition of a and b is denoted by $\tau_{a,b}$ and not by (a, b); the same holds for cycles.

3 More on the surrounding

The Planik is of course inspired from the Rubik's cube. Our main idea has been to produce a simpler version which would in particular avoid the problem of the orientation of the corners. Concerning the Rubik's cube, the readers may have a look at the book [3] by Frey, Alexander & Singmaster. The *diameter* of the Rubik's cube group has been shown to be equal to 20 in [7] by Rokicki, Kociemba, Davidson and Dethridge. The diameter (with respect to a given set of generators) is defined as the minimal number of elementary moves that one needs to go from one position to the next one. In the Planik's case, I have the feeling that the diameter is 8, but a proof is as yet missing. The readers may also read to very complete notes [5] by Joyner that have been floating on the web, or the book [4].

The readers will find in [6] by Larsen a modification of the Rubik's cube presented in a group theoretical viewpoint. This modification goes in the direction of heightening the complexity of the problem, while our aim is reverse. The paper [1] by Alm, Gramelspacher & Rice is also a good read.

Just as the Rubik's cube was enlarged to more than 3 cubes on one edge, see for instance [2] by Bonzio, Loi and Peruzzi, the Planik may be extended to an $n \times n$ square. The 3×3 one is interesting enough for kids, and I leave the general study to the enthusiast readers. We stick here to the 4×4 case.

4 From moves to permutation

We use the langage of *permutations* to describe our game. We identify the board with the set $\mathscr{E} = \{1, 2, \cdots, 15, 16\}$. A move can be described as a permutation of this set, and for instance the elementary move A, i.e. the inversion of the first column is also the permutation defined by

{longdesc}

$$\begin{cases} A(1) = 13, \\ A(5) = 9, \\ A(9) = 5, \\ A(13) = 1, \end{cases} \begin{cases} A(2) = 2, \\ A(6) = 6, \\ A(10) = 10, \\ A(14) = 14, \end{cases} \begin{cases} A(3) = 3, \\ A(7) = 7, \\ A(11) = 11, \\ A(10) = 10, \end{cases} \begin{cases} A(4) = 4, \\ A(8) = 8, \\ A(12) = 12, \\ A(16) = 16. \end{cases} \tag{1}$$

This is to be read as "We put the cube that was at the position *initially* numbered 1 to the position that was *initially* numbered 13". The long description given in (1) may be shortened in

$$\left(\begin{array}{cccc|cccc|cccc|cccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\ 13 & 2 & 3 & 4 & 9 & 6 & 7 & 8 & 5 & 10 & 11 & 12 & 1 & 14 & 15 & 16 \end{array}\right)$$

The fundamental property of the interpretation of a move as a map is that, when we apply successively two moves, say A followed by a, the result may be described by the composition $a \circ A$ of the permutations. Please notice that the move A is played *before* the move a, but the writing as a composition is reversed. We may also write this combined move in the more natural (but reversed!) way Aa. The set of all the permutations of $\mathscr{E}$ is denoted by $\mathfrak{S}(\mathscr{E})$ (the

[3] A. H. Frey Jr. and D. Singmaster, 1982, *Handbook of cubik math.*
[7] T. Rokicki et al., 2013, "The diameter of the Rubik's cube group is twenty".
[5] W. Joyner, 1996, *Mathematics of the Rubik's cube.*
[4] D. Joyner, 2008, *Adventures in group theory.*
[6] M. E. Larsen, 1985, "Rubik's revenge: the group theoretical solution".
[1] J. Alm, M. Gramelspacher, and T. Rice, 2013, "Rubik's on the torus".
[2] S. Bonzio, A. Loi, and L. Peruzzi, 2018, "On the $n \times n \times n$ Rubik's cube".

letter $\mathfrak{S}$ is the uppercase gothic letter S) which is a *group* under composition. As the symbols used to describe the elements of set $\mathscr{E}$ are irrelevant, we shorten $\mathfrak{S}(\mathscr{E})$ in $\mathfrak{S}(16)$.

When composing an arbirary number of elementary moves, we build a subset G of $\mathfrak{S}(\mathscr{E})$. This subset happens to also be a group: it contains the identity as for instance $A \circ A = \mathrm{Id}$, and every element has an inverse. This last property always holds in such a setting for whom knows enough of group theory[1], but it may also be easily established in our setting: take for instance the move $AaBCab$. The move $baCBaA$ will invert it, as the readers will readily check, and put the Planik back in the initial shape.

5 Orbits

Our group G acts on the Planik $\mathscr{E}$ and this action has some geometrical properties. One of them is that the cubes of the corners $\mathscr{C} = \{1, 4, 13, 16\}$ remain in the corners, whatever move we do. The same holds for the four middle cubes from $\mathscr{M} = \{6, 7, 10, 11\}$, for the set of the four cubes on the inner horizontal sides, namely $\mathscr{H} = \{2, 3, 14, 15\}$ and for the set of the four cubes on the inner vertical sides $\mathscr{V} = \{5, 8, 9, 12\}$. The classical vocabulary says that these subsets are *globally invariant*. The readers will check that these are the smallest subsets of $\mathscr{E}$ globally invariant under the action of G. They are called the *orbits*.

The fact that these subsets are invariant reduces enormously the number of Planiks one may reach from the initial one. Indeed, the full group of permutations $\mathfrak{S}(\mathscr{E})$ contains

$$16! = 20\,922\,789\,888\,000$$

elements, while there are only $4! = 24$ possible permutations of the corners, and also 24 for each of the central block $\mathscr{M}$, the inner horizontal sides block $\mathscr{H}$ and the innert vertical sides one $\mathscr{V}$. This amounts in all to at most $24^4 = 331\,776$ permutations. This is already good, but the fundamental question is still not answered. Let us formalize this argument and consider the *structural map*

$$\begin{aligned} \mathfrak{S}(\mathscr{E}) &\to \mathfrak{S}(\mathscr{C}) \times \mathfrak{S}(\mathscr{M}) \times \mathfrak{S}(\mathscr{H}) \times \mathfrak{S}(\mathscr{V}) \\ \sigma &\mapsto (\sigma_{\mathscr{C}}, \sigma_{\mathscr{M}}, \sigma_{\mathscr{H}}, \sigma_{\mathscr{V}}) \end{aligned} \qquad (2) \quad \{\texttt{StructuralMap}\}$$

which associates to a move σ on $\mathscr{E}$ the four induced permutations on the orbits. To go further, we need the notion of *signature* of a permutation. Defining that may be a demanding task in general, as we need a definition that is independant of the writing to show that the signature of a product of two permutations is the product of the two individual signatures. However, we need only the signature on $\mathfrak{S}(4)$, and the work is easier. We present several versions, according to the background of the reader.

6 Signature on $\mathfrak{S}(4)$

We present three definitions of the signature on $\mathfrak{S}(4)$. an interesting exercise for the reader is to show that that indeed amount to the same thing!

[1] Let x be an element of G. The collection $x, x^2, x^3, \cdots$ is finite, from which we deduce that we must have $x^a = x^b$ for some $a < b$. As $\mathfrak{S}(\mathscr{E})$ is a group to which these elements belong, we may simplify by x^a and get $x^{b-a} = \mathrm{Id}$. Whence x^{b-a-1} in the inverse of x. This element x^{b-a-1} indeed belongs to G, concluding the proof.

6.1 A definition through spatial geometry

Let us start with a tetrahedron on the four vertices $\{A, B, C, D\}$. These vertices may be permuted by isometries that preserve the orientation. Such isometries that keep our tetrahedron globally invariant induce a group of permutations that is a subgroup of $\mathfrak{S}(4)$: that much is clear. The reader will ready show that all of them may be described in this manner[2]. Our next task is to describe "orientation preserving". The definition being that the determinant is positive, we somehow need this notion on 3×3 matrices. The matrix of each of our isometries in the basis $(\vec{i}, \vec{j}, \vec{k})$ has an (integer) determinant. One of them has a negative determinant, for instance when we fix $\vec{i}$ and exchange $\vec{j}$ and $\vec{k}$, and so half of them have a positive determinant. The sign of this determinant is what we call the *signature* $\varepsilon(\sigma)$ of the permutation σ. It borrows from determinants the properties that $\varepsilon(\sigma_1 \circ \sigma_2) = \varepsilon(\sigma_1)\varepsilon(\sigma_2)$.

It is enough to talk about the sign of our determinants, but in fact, the signature *is* this determinant. Indeed the isometries we consider are of finite order, meaning that when we iterate them so many times we reach the identity. Hence their determinant is a root of unity that is a also real number: its value must belong to $\{\pm 1\}$.

6.2 Definition through linear algebra

If the reader is confortable with linear algebra, we may define the signature through permutation matrices. Here is how it goes: consider the vector space $\mathbb{R}^4$ and the basis made of $e_1 = (1, 0, 0, 0)$, $e_2 = (0, 1, 0, 0)$, $e_3 = (0, 0, 1, 0)$ and $e_4 = (0, 0, 0, 1)$. Given a permutation σ of $\{1, 2, 3, 4\}$, we may consider the linear map φ_σ that swaps the e_i's accordingly, i.e. that is such that $\varphi_\sigma(e_i) = e_{\sigma(i)}$, for $i \in \{1, 2, 3, 4\}$. Composing the permutations or composing these maps is a same thing, i.e. $\varphi_{\sigma_1 \circ \sigma_2} = \varphi_{\sigma_1} \circ \varphi_{\sigma_2}$. The matrix of each φ_σ in the basis (e_1, e_2, e_3, e_4) has a determinant that belongs forcibly to $\{\pm 1\}$, thanks to the same argument as above. This we call the signature.

6.3 Definition through combinatorics

Here a third way to present the signature which requires less geometry but more abstract strength. Given a permutation σ on $\{1, 2, 3, 4\}$, both non-zero integers

$$\prod_{1 \leqslant u \neq v \leqslant 4} (\sigma(v) - \sigma(u)) = (-1)^6 \prod_{1 \leqslant u < v \leqslant 4} (\sigma(v) - \sigma(u))^2$$

[2] To build *all* the permutations of the vertices of our tetrahedron induced by isometries, we set $\vec{i} = \overrightarrow{AB}$, $\vec{j} = \overrightarrow{AC}$ and $\vec{k} = \overrightarrow{AD}$, so that $\overrightarrow{BC} = \vec{j} - \vec{i}$, $\overrightarrow{CD} = \vec{k} - \vec{j}$ and $\overrightarrow{DB} = \vec{i} - \vec{k}$. Any of the isometries we seek preserves globally[3] the set of edges:

$$\{\pm\vec{i}, \pm\vec{j}, \pm\vec{k}, \pm(\vec{j} - \vec{i}), \pm(\vec{k} - \vec{j}), \pm(\vec{i} - \vec{k})\}.$$

This a finite though rather large set, but of course, the choice of the image of $\vec{i}$, $\vec{j}$ and $\vec{k}$ determines our permutation. The readers will check that fixing the image of $\vec{i}$ as any one element in this sets restricts the images of $(\vec{j}, \vec{k})$ to only two choices. For instance, if $\vec{i}$ is transformed in $-\vec{k}$, then A goes to D and B goes to A, leaving only two choices for the images of C and D. Therefore we generate 24 permutations and, hence, we may represent any permutation of $\mathfrak{S}(4)$ in this manner!

and

$$\prod_{1\leqslant u\neq v\leqslant 4}(v-u) = (-1)^6 \prod_{1\leqslant u<v\leqslant 4}(v-u)^2$$

are equal. Hence the number

$$\varepsilon(\sigma) = \prod_{1\leqslant u<v\leqslant 4}\frac{\sigma(v)-\sigma(u)}{v-u} \tag{3}$$ {defsig}

is either 1 or -1. This is another definition of the signature. We now have to prove that this is a group morphism, i.e. that $\varepsilon(\sigma_1\circ\sigma_2)=\varepsilon(\sigma_1)\varepsilon(\sigma_2)$. To do so, we rewrite the definition above. Let $\mathcal{P}_4$ the set of (non-ordered) *pairs* of distinct integers from $\{1,2,3,4\}$. A moment's thought is enough to see that (3) may be written in the form

$$\varepsilon(\sigma) = \prod_{\{u,v\}\in\mathcal{P}_4}\frac{\sigma(v)-\sigma(u)}{v-u}$$

simply because the quotient $(\sigma(v)-\sigma(u))/(v-u)$ takes the same value on (u,v) and on (v,u). Once this crucial formula is acquired, we may write

$$\begin{aligned}\varepsilon(\sigma_1\circ\sigma_2) &= \prod_{\{u,v\}\in\mathcal{P}_4}\frac{\sigma_1(\sigma_2(v))-\sigma_1(\sigma_2(u))}{\sigma_2(v)-\sigma_2(u)}\prod_{\{u,v\}\in\mathcal{P}_4}\frac{\sigma_2(v)-\sigma_2(u)}{v-u}\\ &= \prod_{\{u,v\}\in\mathcal{P}_4}\frac{\sigma_1(\sigma_2(v))-\sigma_1(\sigma_2(u))}{\sigma_2(v)-\sigma_2(u)}\varepsilon(\sigma_2).\end{aligned}$$

But since σ_2 is one-to-one on $\{1,2,3,4\}$, we may write the elements of $\mathcal{P}_4$ in the form $\{\sigma_2(u),\sigma_2(v)\}$, from which we conclude that indeed $\varepsilon(\sigma_1\circ\sigma_2)=\varepsilon(\sigma_1)\varepsilon(\sigma_2)$.

7 Using the signature

Once the notion of signature on $\mathfrak{S}(4)$ is acquired, we may proceed. The first thing to notice is that the signature of any transposition is -1. In the geometrical context, we simply compute the required determinant, while in the combinatorical one, we preliminarily establish the (very useful) formula[4]

$$\sigma\circ\tau_{a,b}\circ\sigma^{-1} = \tau_{\sigma(a),\sigma(b)} \tag{4}$$ {conjugation}

where we have denoted the transposition of a and b by $\tau_{a,b}$.

This implies that the signature of every transposition is the same. Since every permutation is a product of transpositions, if we had $\varepsilon(\tau_{a,b})=1$, then the signature would be constant on $\mathfrak{S}(4)$. This is readily disproved. We may of course prove that the signature of $\tau_{1,2}$ is -1 by investigating the sign of the formula (3) in this case. Additionally, the fact that every permutation is a product of transpositions implies that formula (4) holds not only for transpositions, but also for also for any cycle, i.e.

$$\sigma\circ\tau_{a_1,a_2,\ldots,a_r}\circ\sigma^{-1} = \tau_{\sigma(a_1),\sigma(a_2),\ldots,\sigma(a_r)}$$

[4]This formula is readily proved: on denoting $u=\sigma(a)$ and $v=\sigma(b)$, then u (resp. v) is sent to a (resp. b), then swapped and send to v (resp. u). The other points are first moved then put back on their initial place.

where we have denoted by $\tau_{a_1,a_2,\ldots,a_r}$ the cycle on $a_1, a_2, \ldots, a_r$. We also check that $\tau_{a_1,a_2}\tau_{a_1,a_2,\ldots,a_r} = \tau_{a_2,\ldots,a_r}$, from which we deduce that

$$\tau_{a_1,a_2,\ldots,a_r} = \tau_{a_1,a_2}\tau_{a_2,a_3}\cdots\tau_{a_{r-1},a_r}.$$

This expression implies that a cycle of length r has signature $(-1)^{r-1}$.

Let us go back to the structural map defined in (2). The fundamental remark is that

{tto} $$\forall\sigma\in G,\quad \varepsilon(\sigma_{\mathscr{C}})\varepsilon(\sigma_{\mathscr{M}})\varepsilon(\sigma_{\mathscr{H}})\varepsilon(\sigma_{\mathscr{V}}) = 1. \tag{5}$$

Indeed, this relation is verified on the elementary moves, and thus holds for every permutation of G.

8 Understanding G

The next result determines G fully.

{StructPlanik}

Theorem 8.1. *A permutation $\sigma \in \mathfrak{S}(16)$ belongs to G if and only if the two next two conditions are met:*

- *The permutation σ preserves globally the corners, the middle block, and both the inner vertical sides and the horizontal ones,*
- *The permutation σ satisfies (5).*

We owe the proof that follows to Joseph Oesterlé.

Proof. We have shown that these two conditions are necessary, and our task is to show that there are indeed sufficient. Let σ be a permutation that verifies these conditions. We need to represent it as a product of elementary moves, and we shall do so by finding such a product, say π, such that $\pi\circ\sigma = \mathrm{Id}$. We separate the proof in two steps. From a player's viewpoint, we start from a shuffled Planik and need to reorder it.

First step

Let us first investigate the positions on the corners. The elementary move a transposes the cubes (initially marked) 1 and 4, D transposes 4 and 16 and d transposes 13 and 16. These three transpositions generate $\mathfrak{S}(4)$, so we can use a product of them to put the corners back in the initial position. Similarly, we may order properly the central block by using B, C and c without disturbing the distribution on the corners, as these moves do not change the ordering on the corners.

Second step

We may thus assume that the corners and the central cubes are properly set, i.e. $\sigma_{\mathscr{C}} = \mathrm{Id}$ and $\sigma_{\mathscr{M}} = \mathrm{Id}$. Our information is that $\varepsilon(\sigma_{\mathscr{H}})\varepsilon(\sigma_{\mathscr{V}}) = 1$, i.e. $\varepsilon(\sigma_{\mathscr{H}}) = \varepsilon(\sigma_{\mathscr{V}})$.

The sequel of the proof is based on three observations:

- The succession of moves $aBaB$ (that you get by successively applying the moves a, B, a and B) permutes circularly the cubes 2, 3 and 14 and fixes all the other cubes. We generate similarly any other circular permutation on three symbols from the inner vertical sides.

- Similarly, the succession of moves $AbAb$ permutes circularly the cubes 5, 8 and 9 and fixes all the other cubes, ang yet again, we may generate any other circular permutation on three symbols from the inner horizontal sides.

- The succession $aDaDaD$ operates as a product of two transpositions: it transposes the inner side horizontal cubes 2 and 3 and the inner side vertical ones 8 and 12.

If $\varepsilon(\sigma_{\mathscr{H}}) = \varepsilon(\sigma_{\mathscr{V}}) = -1$, we first apply the third move above and reach a position where $\varepsilon(\sigma_{\mathscr{H}}) = \varepsilon(\sigma_{\mathscr{V}}) = 1$. The readers will then readily check that any permutation of signature 1 on $\mathfrak{S}(4)$ may be written as a product of circular permutations of length 3: we use that to write both $\sigma_{\mathscr{H}}$ and $\sigma_{\mathscr{V}}$ as product of elements of G. This ends the proof. □

9 The Square Dance game

Let us complete the Planik and its extension to larger boards by another family of planar games. These are played on similar boards, but the moves are distinct, leading to different solutions. The moves are also row per row or column per column, but instead on inverting one of them, we apply a circular permutation. So for instance, on a 4×4 board, the elementary move a is the circular permutation $\tau_{1,2,3,4}$. We only mention some facts on these games.

The 3×3 case

This game has a single orbit. However, a circular permutation of three symbols has signature equal to 1, meaning that, on iterating the elementary moves, we may only generate even permutation. Of course, we need the notion of signature to read this sentence, so that this game is only suitable for higher level students. As it turns out, every even permutation on $\{1, 2, \cdots, 9\}$ is reachable by such moves. One way to prove it is to first sort out the lowest and rightmost 2×2 square. By using the first column as a stack, we may also sort out the upper row. The signature property implies that the last two cubes will be automatically properly set.

The 4×4 case

We also encounter only one orbit, but the elementary moves now have signature -1. Again by using the first row as a stack, the readers will sort out the lowest rightmost 3×3 square, as well as the upper row. We now show that we may build transpositions by elementary moves. Consider the sequel

$aAaaaAaAAaaaA$ (notice that $aaa = a^{-1}$). Here is what happens:

1	2	3	4
5			
9			
13			

$a \rightarrow$

4	1	2	3
5			
9			
13			

$A \rightarrow$

13	1	2	3
4			
5			
9			

$aaa \rightarrow$

1	2	3	13
4			
5			
9			

$A \rightarrow$

9	2	3	13
1			
4			
5			

$a \rightarrow$

13	9	2	3
1			
4			
5			

$AA \rightarrow$

4	9	2	3
5			
13			
1			

$aaa \rightarrow$

9	2	3	4
5			
13			
1			

$A \rightarrow$

1	2	3	4
9			
5			
13			

From this construction which we owe to Julien Cassaigne, and after some routine work, we deduce the group of permutations generated is $\mathfrak{S}(16)$.

The 5×5 case

It is again trivial to show that there is only one orbit, but the elementary moves now all have signature 1. We leave the determination of the group of permutations to the interested readers!

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