Some elementary explicit bounds for two mollifications of the Moebius function

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Abstract

We prove that the sum $\sum_{\substack{d\le x,\\(d,r)=1}}\mu(d)/d^{1+\varepsilon}$ is bounded by $1+\varepsilon$, uniformly in $x\ge 1$, r and $\varepsilon>0$. We prove a similar estimate for the quantity $\sum_{\substack{d\le x,\\(d,r)=1}}\mu(d)\log(x/d)/d^{1+\varepsilon}$. When $\varepsilon=0$, r varies between 1 and a hundred, and x is below a million, this sum is non-negative and this raises the question as to whether it is non-negative *for every* x.

1 Introduction and results

Our first result is the following:

Theorem 1.1. *When $r\ge 1$ and $\varepsilon\ge 0$, we have*

$$\Bigl|\sum_{\substack{d\le x,\\(d,r)=1}}\frac{\mu(d)}{d^{1+\varepsilon}}\Bigr|\le 1+\varepsilon.$$

This Lemma generalizes the estimate of [5, Lemme 10.2] which corresponds to the case $\varepsilon=0$. This generalization is *not* straightforward at all and requires a change of proof. The case $\varepsilon=0$ and $r=1$ is classical. The parameter ε that is being introduced induces some flexibility useful when applying Rankin's method (devised in [8]). As it turns out, we can do somewhat better concerning the lower bound, and we prove that

$$-\tfrac{11}{15}(1+\varepsilon)\le\sum_{\substack{d\le x,\\(d,r)=1}}\frac{\mu(d)}{d^{1+\varepsilon}}.$$

We ran computations covering the range $1\le x\le 10^6$ and $1\le r\le 100$ with $\varepsilon=0$; we found that the lowest lower bound was met at $x=13$ and $r=1$. This raises the following question:

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Question 1. *It is true that*

$$\sum_{\substack{d\le x,\\ (d,r)=1}} \frac{\mu(d)}{d} \ge -2323/30030 \quad ?$$

See section 2 for a very preliminary result in this direction.

We proceed by proving the following more involved form:

Theorem 1.2. *When $r \ge 1$ and $1.38 \ge \varepsilon \ge 0$, we have*

$$\left|\sum_{\substack{d\le x,\\ (d,r)=1}} \frac{\mu(d)}{d^{1+\varepsilon}} \log\frac{x}{d}\right| \le 1.4 + 4.7\varepsilon + 3.3\varepsilon^2 + (1+\varepsilon)\frac{r^{1+\varepsilon}}{\phi_{1+\varepsilon}(r)} x^{\varepsilon}$$

where

$$\frac{r^{1+\varepsilon}}{\phi_{1+\varepsilon}(r)} = \prod_{p|r} \frac{p^{1+\varepsilon}}{p^{1+\varepsilon}-1}. \tag{1}$$

The dependence in r is optimal as seen by taking for r the product of every primes not more than $\sqrt{x}$. The proof is again unbalanced with respect to the upper and the lower bound, and we prove a somewhat better lower bound:

$$-(1.434 + 4.992\varepsilon + 3.558\varepsilon^2) \le \sum_{\substack{d\le x,\\ (d,r)=1}} \frac{\mu(d)}{d^{1+\varepsilon}} \log\frac{x}{d}.$$

I expect the factor x^{ε} in the upper bound to be a blemish; however, the (limited) numerical verifications we ran suggest that the factor $r^{1+\varepsilon}/\phi_{1+\varepsilon}(r)$ *cannot* be omitted even if the condition $r \le x$ is added (this condition often appears in practice). It should be added that it is not difficult to prove that

$$\sum_{d\le x} \frac{\mu(d)}{d} \log\frac{x}{d} \sim 1 \quad (x\to\infty)$$

which means that one cannot expect an arbitary small constant in the right hand side of the inequality given in Theorem 1.2. We have checked that

$$0 \le \sum_{\substack{d\le x,\\ (d,r)=1}} \frac{\mu(d)}{d} \log\frac{x}{d} \le \frac{r}{\phi(r)} + 0.007 \quad (x \le 10^6, 1\le r\le 100)$$

(where x is a real number and not especially an integer) and all these maxima were in fact very close to $r/\phi(r)$. These computations raise two questions:

Question 2. *Is it true that*

$$\sum_{\substack{d\le x,\\ (d,r)=1}} \frac{\mu(d)}{d} \log\frac{x}{d} \ge 0, \quad (x\ge 1,\ r\ge 1) \quad ?$$

Question 3. *Is it true that*

$$\sum_{\substack{d\le x,\\ (d,r)=1}} \frac{\mu(d)}{d}\log\frac{x}{d} \le \frac{r}{\phi(r)}+1, \quad (x\ge 1,\ r\ge 1) \quad ?$$

In both these questions, x is only assumed to be a positive real number. On recalling what happens in the case of Turán's conjecture on the summatory function of the Liouville function divided by its argument, see [2], we believe that the answer to the first question is no. The sum is however less likely to be very erratical because of the smoothing factor, a factor that is absent in Turán's problem. In direction of these conjecture, we note the following formula

$$\int_1^\infty \sum_{\substack{d\le x,\\ (d,r)=1}} \frac{\mu(d)}{d}\log\frac{x}{d}\frac{dx}{x^{s+1}} = \frac{r^{1+s}}{\phi_{1+s}(r)}\frac{1}{s^2\zeta(1+s)}$$

from which we easily deduce (on taking $s=\varepsilon>0$ and letting ε go to infinity) that

$$\limsup_x \sum_{\substack{d\le x,\\ (d,r)=1}} \frac{\mu(d)}{d}\log\frac{x}{d} \ge \frac{r}{\phi(r)}.$$

We discuss some related points in the last section.

Notation

We use here the notation $h=\mathcal{O}^*(k)$ to mean that $|h|\le k$. We denote by $\tau(m)$ the number of (positive) divisors of m, and by (a,b) the gcd of a and b. For $\varepsilon\ge 0$ and $r\ge 1$ any natural squarefree number, we define two functions. The first one is alternatively defined by

$$f_{r,\varepsilon}(n) = \sum_{\substack{\ell|n,\\ (\ell,r)=1}} \frac{\mu(\ell)}{\ell^\varepsilon}\tau(n/\ell) \tag{2}$$

or, in multiplicative form, by:

$$f_{r,\varepsilon}(n) = \prod_{\substack{p^\nu\|n,\\ p\nmid r}}\Big(\nu+1-\frac{\nu}{p^\varepsilon}\Big)\prod_{\substack{p^\nu\|n,\\ p|r}}(\nu+1). \tag{3}$$

We easily determine its Dirichlet series: $\sum_{n\ge1} f_{r,\varepsilon}(n)/n^s = \zeta(s)^2/\zeta(s+\varepsilon)$. We shall further write

$$f_{r,\varepsilon}(n) = \mathbb{1}\star g_{r,\varepsilon}(n) \tag{4}$$

where the function $g_{r,\varepsilon}$ has the essential property of being non-negative and is being defined by:

$$g_{r,\varepsilon}(n) = \sum_{\substack{\ell|n,\\ (\ell,r)=1}} \frac{\mu(\ell)}{\ell^\varepsilon} \geq 0. \tag{5}$$

Thanks

Sincere thanks are due to the careful referee who has checked our computations and indeed has rooted out several mistakes.

2 Verifying Theorem 1.1 for small values

We study what happens for small values here. The proof is pedestrian and painful, but I have not seen any way to avoid it, or to present it in a more general frame.

We study the following quantity:

$$m_0(r,x) = \sum_{\substack{d\leq x,\\ (d,r)=1}} \frac{\mu(d)}{d^{1+\varepsilon}}. \tag{6}$$

Lemma 2.1. *When $x < 10$ and $\varepsilon \geq 0$, we have $-1/30 \leq m_0(r,x) \leq 1$.*

Proof. The sum we consider reads

$$1 - \frac{h(2)}{2^{1+\varepsilon}} - \frac{h(3)}{3^{1+\varepsilon}} - \frac{h(5)}{5^{1+\varepsilon}} + \frac{h(6)}{6^{1+\varepsilon}} - \frac{h(7)}{7^{1+\varepsilon}}$$

where h is the characteristic function of the integers $\leq x$ that are coprime with r. The minimum is clearly

$$1 - \frac{1}{2^{1+\varepsilon}} - \frac{1}{3^{1+\varepsilon}} - \frac{1}{5^{1+\varepsilon}}$$

which is minimal when $\varepsilon = 0$. This is the $-1/30$. The maximum contains the summand 1. If the summand $1/6^{1+\varepsilon}$ is present, then so is the summand $-1/2^{1+\varepsilon}$. This concludes the proof. □

3 Auxiliaries

Lemma 3.1. *When $\varepsilon \geq 0$, we have*

$$\sum_{h\leq H} h^\varepsilon = \frac{H^{1+\varepsilon}}{1+\varepsilon} + \mathcal{O}^*\big(H^\varepsilon\big).$$

This is also $\leq H^{1+\varepsilon}$. When H is an integer, we have $\sum_{h\leq H} h^\varepsilon \geq \frac{H^{1+\varepsilon}}{1+\varepsilon}$.

Proof. Indeed, when $\varepsilon > 0$, a summation by parts gives us directly

$$\begin{aligned}\sum_{h\le H} h^\varepsilon &= \sum_{h\le H}\varepsilon\int_0^h dt/t^{1-\varepsilon} = \varepsilon\int_0^H \sum_{t<h\le H} 1\,dt/t^{1-\varepsilon}\\ &= \varepsilon\int_0^H (H-t)\,dt/t^{1-\varepsilon} + \mathcal{O}^*(H^\varepsilon).\end{aligned}$$

We proceed by continuity to cover the case $\varepsilon = 0$. When H is an integer, a comparison to an integral gives the result. □

Lemma 3.2. *For $L > 1$, we have*

$$\sum_{n\le L} f_{r,\varepsilon}(n) \le L\sum_{\ell\le L} g_{r,\varepsilon}(\ell)/\ell. \tag{7}$$

Proof. We recall (4) and write, since $g_{r,\varepsilon} \ge 0$

$$\sum_{n\le L} f_{r,\varepsilon}(n) = \sum_{km\le L} g_{r,\varepsilon}(m) \le L\sum_{m\le L} g_{r,\varepsilon}(m)/m.$$

The Lemma follows readily. □

Lemma 3.3. *For every integer n and any $\varepsilon \ge 0$, we have*

$$g_{1,\varepsilon}(\ell) \le \sum_{mn=\ell} g_{1,\varepsilon/2}(m)g_{1,\varepsilon/2}(n).$$

Proof. We check that, when $\alpha \ge 1$ is an integer and p a prime number,

$$g_{1,\varepsilon}(p^\alpha) = 1 - \frac{1}{p^\varepsilon} = 1 - \frac{1}{p^{\varepsilon/2}} + \frac{1}{p^{\varepsilon/2}}\Bigl(1 - \frac{1}{p^{\varepsilon/2}}\Bigr)$$

$$\le\le g_{1,\varepsilon/2}(p^\alpha)g_{1,\varepsilon/2}(1) + g_{1,\varepsilon/2}(1)g_{1,\varepsilon/2}(p^\alpha)$$

$$\le \sum_{0\le\beta\le\alpha} g_{1,\varepsilon/2}(p^{\alpha-\beta})g_{1,\varepsilon/2}(p^\beta).$$

We conclude by invoking the multiplicativity of $g_{1,\varepsilon/2}$. □

Lemma 3.4. *We have when $L \ge 7.2$,*

$$\sum_{p\le L}\frac{\log p}{p-1} \le \log L.$$

Proof. We cite [9, (2.8)]:

$$\sum_{p\le L}\frac{\log p}{p} \le \log L - \gamma - \sum_{p\ge 2}\frac{\log p}{p(p-1)} + \frac{1}{2\log L}, \quad (L \ge 319)$$

from which we deduce, for $L \geq 319$,

$$\sum_{p \leq L} \frac{\log p}{p-1} \leq \log L - \gamma + \frac{1}{2\log L}.$$

A simple GP script shows that

$$\sum_{p \leq L} \frac{\log p}{p-1} \leq \log L$$

when $1000 \geq L \geq 7.2$, and the reader will conclude readily. $\square$

Lemma 3.5. *We have, when $L \geq 1$ and $\varepsilon \geq 0$,*

$$\sum_{\ell \leq L} g_{1,\varepsilon}(\ell)/\ell \leq L^{\varepsilon}. \tag{8}$$

Proof. Verifying the stated inequality for $1 \leq L < 8$ is (tedious but) easy, hence we can now assume that $L \geq 8$. We readily find that the sum in question is not more than

$$T = \prod_{p \leq L} \frac{1 - p^{-1-\varepsilon}}{1 - p^{-1}} = \exp \sum_{p \leq L} \log\left(1 + \frac{1 - p^{-\varepsilon}}{p-1}\right).$$

We apply $\log(1+x) \leq x$ for non-negative x and $1 - p^{-\varepsilon} \leq \varepsilon \log p$ to get, when $L \geq 8$,

$$T \leq \exp \varepsilon \sum_{p \leq L} \frac{\log p}{p-1} \leq L^{\varepsilon}$$

by invoking Lemma 3.4. $\square$

Lemma 3.6. *We have, when $L \geq 1$, $r \geq 1$ and $\varepsilon \geq 0$,*

$$\sum_{\ell \leq L} g_{r,\varepsilon}(\ell)/\ell \leq \frac{r^{1+\varepsilon}}{\phi_{1+\varepsilon}(r)} L^{\varepsilon}. \tag{9}$$

Proof. We use the notation $d|r^\infty$ to say that each prime factor of d divides r. We write

$$\begin{aligned}\sum_{\ell \leq L} \frac{g_{r,\varepsilon}(\ell)}{\ell} &= \sum_{\substack{d|r^\infty, \\ d \leq L}} \sum_{\substack{\ell \leq L/d, \\ (\ell,r)=1}} \frac{g_{r,\varepsilon}(\ell)}{\ell d} \\ &\leq L^{\varepsilon} \sum_{d|r^\infty} \frac{1}{d^{1+\varepsilon}} = L^{\varepsilon} \frac{r^{1+\varepsilon}}{\phi_{1+\varepsilon}(r)}\end{aligned}$$

by Lemma 3.5. The Lemma follows readily. $\square$

Lemma 3.7.

$$\sum_{m\le M} m^\varepsilon \tau(m) = \frac{M^{1+\varepsilon}}{1+\varepsilon}\left(\log M + 2\gamma - \frac{1}{1+\varepsilon}\right) + \mathcal{O}^*\left(0.961(1+2\varepsilon)M^{\frac{1}{2}+\varepsilon}\right)$$

Proof. We recall part of [1, Theorem 1.1]:

$$\sum_{m\le t}\tau(m) = t\log t + (2\gamma-1)t + \mathcal{O}^*(0.961\sqrt{t}), \quad (t\ge 1).$$

Since $(t\log t + (2\gamma-1)t)/\sqrt{t}$ is seen to vary between -0.681 and 0.155 when t varies between 0 and 1, this estimate is also valid for $t>0$. We use summation by parts and find that

$$\begin{aligned}\sum_{m\le M} m^\varepsilon\tau(m) &= M^\varepsilon\sum_{m\le M}\tau(m) - \varepsilon\int_0^M \sum_{m\le t}\tau(m)dt/t^{1-\varepsilon}\\ &= M^{1+\varepsilon}(\log M + 2\gamma - 1) + \mathcal{O}^*\left(0.961M^{\frac{1}{2}+\varepsilon}\right)\\ &- \varepsilon\int_0^M(\log t + 2\gamma - 1)t^\varepsilon dt + \mathcal{O}^*\left(0.961\varepsilon\int_0^M t^{\varepsilon-1/2}dt\right)\\ &= \frac{M^{1+\varepsilon}}{1+\varepsilon}\left(\log M + 2\gamma - \frac{1}{1+\varepsilon}\right) + \mathcal{O}^*\left(0.961(1+2\varepsilon)M^{\frac{1}{2}+\varepsilon}\right).\end{aligned}$$

□

Lemma 3.8. *We have, when* $n\ge 2$,

$$g_{r,\varepsilon}(n) \le 1 - \frac{\mathbb{1}_{(n,r)=1}\mu^2(n)}{n^\varepsilon}.$$

Proof. Indeed, we verify that $(1-a)(1-b)\le(1-ab)$ when $0\le a,b\le 1$. The Lemma readily follows by recursion on the number of prime factors of n. □

4 Some lemmas on squarefree numbers

Here is a Lemma from [4]:

Lemma 4.1. *We have, for* $D\ge 1\,664$

$$\sum_{d\le D}\mu^2(d) = \frac{6D}{\pi^2} + \mathcal{O}^*\big(0.1333\sqrt{D}\big).$$

In particular, this is not more than $0.62D$ *when* $D\ge 1700$.

Lemma 4.2. *We have*

$$\sum_{d\le x} \mu^2(d)/\sqrt{d} \le 1.33\,\sqrt{x}, \quad (x \ge 1).$$

If we are ready to assume larger, we would not save much since the best constant one can get is $12/\pi^2 = 1.215 + \mathcal{O}^*(0.001)$.

Proof. We use PARI/GP see [7] and the following script:

```
{check(borne) =
   my(res = 0.0, coef = 0);
   for(d = 1, borne,
       res += moebius(d)^2/sqrt(d);
       coef = max(coef, res/sqrt(d)));
   return(coef)}
```

It is then almost immediate to check our result when $x \le 10^7$, despite the lack of refinement of the script proposed. For larger values, we use a summation by parts together with Lemma 4.1. □

Lemma 4.3. *We have*

$$\sum_{d\le x} \mu^2(d) \le \tfrac{11}{15}\,x, \quad (x \ge 9).$$

We note that $11/15 = 0.7333\ldots$ while the asymptotically best constant is rather lower, namely $6/\pi^2 = 0.607927\ldots$. Reaching $73/115 = 0.63478\ldots$ already requires to take $x \ge 75$, and this means we would have to handle the possible divisibility by 21 primes in section 2. This is out of reach of the simple-minded method we have at our disposal.

Proof. We use PARI/GP see [7] and the following script:

```
{check(borneinf, bornesup) =
   my(res = 0.0, coef = 0);
   res = sum(d = 1, borneinf-1, moebius(d)^2);
   for(d = borneinf, bornesup,
       res += moebius(d)^2;
       coef = max(coef, res/d));
   return(coef)}
```

It is then almost immediate to check our result when $x \le 10^7$, despite the lack of refinement of the script proposed. For larger values, the result is an immediate consequence of Lemma 4.1. □

5 Proof of Theorem 1.1

Lemma 2.1 establishes Theorem 1.1 when $x < 10$, so we may assume $x \geq 10$. We further may restrict our attention to integer values of x. We start with

$$S_0 = \sum_{n \leq x} n^\varepsilon g_{r,\varepsilon}(n) = \sum_{n \leq x} \sum_{\substack{d|n,\\(d,r)=1}} \mu(d)(n/d)^\varepsilon.$$

Using the first expression yields $0 \leq S_0$ as well as

$$S_0/x^\varepsilon \leq 1 + \sum_{2 \leq n \leq x} \left(g_{r,\varepsilon}(n) + \frac{\mathbb{1}_{(n,r)=1}\mu^2(n)}{n^\varepsilon} \right) - \sum_{\substack{2 \leq n \leq x,\\(n,r)=1}} \frac{\mu^2(n)}{n^\varepsilon}$$

Each summand in the second sum is bounded above by 1 by Lemma 3.8. We get

$$0 \leq S_0/x^\varepsilon \leq x - \sum_{\substack{2 \leq n \leq x,\\(n,r)=1}} \frac{\mu^2(n)}{n^\varepsilon}.$$

Let us write the second expression for S_0:

$$S_0 = \sum_{\substack{d \leq x,\\(d,r)=1}} \mu(d) \sum_{m \leq x/d} m^\varepsilon.$$

We employ Lemma 3.1; we treat the case $d = 1$ separately for the lower bound to reach

$$\frac{x^{1+\varepsilon}}{1+\varepsilon} \sum_{\substack{d \leq x,\\(d,r)=1}} \frac{\mu(d)}{d^{1+\varepsilon}} - x^\varepsilon \sum_{\substack{2 \leq d \leq x,\\(d,r)=1}} \mu^2(d) d^{-\varepsilon} \leq S_0$$

$$\leq \frac{x^{1+\varepsilon}}{1+\varepsilon} \sum_{\substack{d \leq x,\\(d,r)=1}} \frac{\mu(d)}{d^{1+\varepsilon}} + x^\varepsilon \sum_{\substack{d \leq x,\\(d,r)=1}} \mu^2(d) d^{-\varepsilon}.$$

The lower bound requires x to be an integer, but not the upper bound. We rewite the above as

$$S_0 - x^\varepsilon \sum_{\substack{d \leq x,\\(d,r)=1}} \mu^2(d) d^{-\varepsilon} \leq \frac{x^{1+\varepsilon}}{1+\varepsilon} \sum_{\substack{d \leq x,\\(d,r)=1}} \frac{\mu(d)}{d^{1+\varepsilon}} \leq S_0 + x^\varepsilon \sum_{\substack{2 \leq d \leq x,\\(d,r)=1}} \mu^2(d) d^{-\varepsilon}.$$

By conjugating both estimates, we get,

$$-x^\varepsilon \sum_{\substack{d \leq x,\\(d,r)=1}} \mu^2(d) d^{-\varepsilon} \leq \frac{x^{1+\varepsilon}}{1+\varepsilon} \sum_{\substack{d \leq x,\\(d,r)=1}} \frac{\mu(d)}{d^{1+\varepsilon}} \leq x^{1+\varepsilon}.$$

The right hand side is easily handled. We use Lemma 4.3 for the left hand side via, when $x \geq 9$:

$$\sum_{\substack{d\leq x,\\(d,r)=1}} \mu^2(d)d^{-\varepsilon} \leq \sum_{d\leq x}\mu^2(d) \leq \tfrac{11}{15}x.$$

By conjugating both estimates, we get

$$-\tfrac{11}{15}(1+\varepsilon) \leq \sum_{\substack{d\leq x,\\(d,r)=1}} \frac{\mu(d)}{d^{1+\varepsilon}} \leq 1+\varepsilon. \quad (x\geq 9) \tag{10}$$

Theorem 1.1 is proved.

6 Proof of Theorem 1.2

The proof relies on two ways of writing the sum

$$S_1 = \sum_{n\leq x} n^\varepsilon f_{r,\varepsilon}(n) = \sum_{n\leq x}\sum_{\substack{d|n,\\(d,r)=1}} \mu(d)(n/d)^\varepsilon \tau(n/d).$$

The first form shows that $0 \leq S_1 \leq x^{1+2\varepsilon}r^{1+\varepsilon}/\phi_{1+\varepsilon}(r)$ by combining Lemma 3.2 together with Lemma 3.6. Let us write this sum differently:

$$S_1 = \sum_{\substack{d\leq x,\\(d,r)=1}} \mu(d) \sum_{m\leq x/d} m^\varepsilon \tau(m)$$

and we use Lemma 3.7 to reach

$$S_1 = \frac{x^{1+\varepsilon}}{1+\varepsilon} \sum_{\substack{d\leq x,\\(d,r)=1}} \frac{\mu(d)}{d^{1+\varepsilon}}\Bigl(\log\frac{x}{d} + 2\gamma - \frac{1}{1+\varepsilon}\Bigr) + \mathcal{O}^*\Bigl(0.961\times 1.33\,(1+2\varepsilon)x^{1+\varepsilon}\Bigr)$$

since $\sum_{d\leq x}\mu^2(d)/\sqrt{d} \leq 1.33\sqrt{x}$ by Lemma 4.2. We set

$$\alpha = 2\gamma - \frac{1}{1+\varepsilon} \in [0,1]. \tag{11}$$

All of that amounts to:

$$\begin{aligned} S_1 &= \frac{x^{1+\varepsilon}}{1+\varepsilon} \sum_{\substack{d\leq x,\\(d,r)=1}} \frac{\mu(d)}{d^{1+\varepsilon}}\Bigl(\log\frac{x}{d} + \alpha\Bigr) + \mathcal{O}^*\bigl(1.279(1+2\varepsilon)x^{1+\varepsilon}\bigr) \\ &= S_1^* + \alpha S_0 + \mathcal{O}^*\bigl(1.279(1+2\varepsilon)x^{1+\varepsilon}\bigr) \end{aligned}$$

say. We thus have

$$-1.279(1+2\varepsilon)x^{1+\varepsilon} \le S_1^* + \alpha S_0 \le 1.279(1+2\varepsilon)x^{1+\varepsilon} + x^{1+2\varepsilon}\frac{r^{1+\varepsilon}}{\phi_{1+\varepsilon}(r)}.$$

We use (10) and Lemma 2.1, and reach

$$-1.279(1+2\varepsilon) - \alpha \le x^{-1-\varepsilon}S_1^* \le 1.279(1+2\varepsilon) + \tfrac{11}{15}\alpha + x^{\varepsilon}\frac{r^{1+\varepsilon}}{\phi_{1+\varepsilon}(r)}.$$

We use $\alpha \le 2\gamma - 1 + \varepsilon$. This gives

$$\begin{aligned}-1.434 - 4.992\varepsilon - 3.558\varepsilon^2 \le \sum_{\substack{d\le x,\\(d,r)=1}} \frac{\mu(d)}{d^{1+\varepsilon}}\log\frac{x}{d} \\ \le 1.393 + 4.684\varepsilon + 3.292\varepsilon^2 + (1+\varepsilon)\frac{x^{\varepsilon}r^{1+\varepsilon}}{\phi_{1+\varepsilon}(r)}.\end{aligned}$$

Since $x^{\varepsilon}r^{1+\varepsilon}/\phi_{1+\varepsilon}(r) \ge 1$, we check that the right hand side is larger than minus times the left hand side. Theorem 1.2 follows.

7 A generalization and a remark

It is not difficult to get along these lines the following Lemma:

Lemma 7.1. *When $r \ge 1$ and $k \ge 1$, we have*

$$\sum_{\substack{d\le x,\\(d,r)=1}} \frac{\mu(d)}{d^{1+\varepsilon}}\log^k\frac{x}{d} \ll_k \left(\frac{r}{\phi(r)}\right)^k (\log x)^{k-1}.$$

Such quantities appear for instance in [10] where cases $k=0$ and $k=1$ are used, while case $k=2$ is evaluated (there is a main term), but all with no coprimality conditions (i.e. $r=1$) and no ε. The reader will find in [3, Chapter 1] the evaluation of case $k=3$, $r=1$ and $\varepsilon=0$. [6] also pertains to these quantities.

Proof. Indeed, we first prove that

$$\sum_{n\le x}\sum_{\substack{d|n,\\(d,r)=1}} \mu(d)(n/d)^{\varepsilon}\tau_{k+1}(n/d) \ll \left(\frac{r}{\phi(r)}\right)^k x(\log x)^{k-1}.$$

We then continue as in section 6. □

Here is a surprising elementary consequence.

Lemma 7.2. *For any $c > 0$, we have*

$$\sum_{\substack{d \le x, \\ (d,r)=1}} \frac{\mu(d)}{d} - x^{\varepsilon} \sum_{\substack{d \le x, \\ (d,r)=1}} \frac{\mu(d)}{d^{1+\varepsilon}} \ll_c \varepsilon \frac{r}{\phi(r)}$$

provided that $0 \le \varepsilon \le c(\log x)^{-1}$.

Proof. It is enough to consider

$$\int_0^{\varepsilon} \sum_{\substack{d \le x, \\ (d,r)=1}} \frac{\mu(d) x^{\eta}}{d^{1+\eta}} \log(x/d) d\eta \ll \varepsilon \frac{r}{\phi(r)}.$$

□

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