ON THE MISSING LOG FACTOR

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Abstract. This paper is the detailled written account of a talk with the same title given during the conference. Its guiding line is the elementarily proven bound $|\sum_{n\le x}\mu(n)/n| \le 1$. The trivial bound for the implied summation is $\log x + \mathcal{O}(1)$, while the Prime Number Theorem tells us that it is $o(1)$. Our starting estimate thus lies in-between, a fact that we explore under different lights.

1. Introduction

The Moebius function has attracted lots of attention in the last few years. As is classical in Analytic Number Theory, we are trying to estimate sums of the shape $\sum_{n\le x}\mu(n)g(x,n)$ for various and usually regular functions $g(x,n)$.

There are essentially three definition of the Moebius function:

- It is the multiplicative fonction with $\mu(p) = -1$ and $\mu(p^k) = 0$ $(k \ge 2)$,
- It is the convolution inverse of $\mathbb{1}$,
- It appears as the coefficients of the Dirichlet series of $1/\zeta(s)$.

All three are of course linked[1], but this list enables a rough and empirical classification of proofs. In this talk, we concentrate on the *second definition*, and we shall often add an explicit angle to our looking glass. We will in particular see that this combinatorial definition leads to functional analysis problems.

2. Meissel & Gram

Let us start our journey by an identity due the german mathematician Ernst Meissel[2] in 1854 which is equation (6) of [26]. Thanks to the DigiZeitschriften project hosted by the university of Göttingen, we can have access to this text online, though some knowledge of latin is required. The classical reference book [9] on history of numbers of L.E. Dickson may serve as a first guide, and for instance, the paper [26] is mentionned in Chapter XIX of this series of three books. In modern notation, the identity in

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[1] If only by the fact that they define the same function!

[2] His full name is Daniel Friedrich Ernst Meissel. This student of Carl Gustav Jacob Jacobi and Johann Peter Gustav Lejeune Dirichlet is born in 1826 and passed away in 1895. His full biography can be found in [31].

question reads

$$\sum_{n\le x}\mu(n)[x/n]=1 \tag{2.1}$$

where $[y]$ denotes the integer part of the real number y, while $\{y\}$ denotes its fractional part. This is established by noticing that $[y]=\sum_{1\le m\le y}1$ when y is non-negative and on using the property that $\sum_{mn=\ell}\mu(n)=\mathbb{1}_{\ell=1}$. Now let us replace $[y]$ by $y-\{y\}$ in the above; we get

$$\sum_{n\le x}\mu(n)\{x/n\}=-1+x\sum_{n\le x}\frac{\mu(n)}{n}. \tag{2.2}$$

We stop to emphasize three surprising aspects of this equation:

(1) *Error term treatment:* On the left-hand side, the summand $\mu(n)$ is contaminated by the error term $\{x/n\}$ while the contamination disappears on the right-hand side! The Prime Number Theorem thus implies that the left-hand side is indeed $o(x)$.

(2) *Identity:* We have used an identity, and the question arises is naturally to know whether it is an accident or a feature.

(3) *Log-factor:* When we bound $|\mu(n)|$ and $\{x/n\}$ by 1, we see that the trivial bound for the left-hand side is x, while the trivial bound for the right-hand side is ... $O(x\log x)$! As a consequence, the danish mathematician Jørgen Pedersen Gram showed in [16, p 196-197][3] that

$$\Big|\sum_{n\le x}\mu(n)/n\Big|\le 1 \tag{2.3}$$

for every positive x. It is of course a consequence of the Prime Number Theorem that this sum goes to zero, but this partial result is striking.

The identity angle leads to more curious identities. Here is another one obtained much later by the canadian mathematician Robert Allister MacLeod in [25]:

$$\sum_{n\le x}\mu(n)\frac{\{x/n\}^2-\{x/n\}}{x/n}=x\sum_{n\le x}\frac{\mu(n)}{n}-\sum_{n\le x}\mu(n)-2+\frac{2}{x}.$$

In fact, MacLeod exhibits a full family of similar identities, all valid for any $x\ge 1$. Yet again, the reader can see that the left-hand side is contaminated by an "error term"-like function, while this contamination is absent from the right-hand side.

[3]This reference has been kindly provided to us by M. Balazard. The reader is refered to the MacTutor archive maintained by the University of Saint Andrew, in Scotland for the biography of J.-P. Gram. We just mention here that Meissel travelled to Denmark in 1885 to meet the 23 years old Gram who had just won the Gold Medal of the Royal Danish Academy of Sciences for the memoir we refer to. The inequality we extract from this memoir is not its main matter but rather a pleasant sidedish.

I am showing you this identity to insist on the strange aspect these relations may take. Are these identities just curiosities or is a better understanding possible? Can we give some order to these facts?

3. Generalizing Meissel's proof, I

While trying to shed some light on Meissel's identity I devised the next theorem that shows that, under rather general conditions, we *always* save a log factor. More refined version are possible, but this simplistic one captures the main power of Gram's statement. We first need two general lemmas.

Lemma 3.1. *When $Q \ge 0$, we have*

$$\sum_{p^\nu \le Q} \nu^2 \log p \le 3Q,$$

the sum being over every prime powers p^ν.

Proof. We first use GP/Pari [30] to establish the claimed inequality when Q is below 10^6. Then we express our sum, say S, in the following manner:

$$S = \sum_{p^\nu \le Q} \log p + \sum_{p^\nu \le Q} (\nu^2 - 1)\log p \le \psi(Q) + \sum_{p \le \sqrt{Q}} \left(\frac{\log Q}{\log p}\right)^2 \log Q$$
$$\le \psi(Q) + \pi(\sqrt{Q})\frac{\log^3 Q}{\log^2 2}$$

with the usual Tchebyshev function ψ and π. We recall that $\psi(x) \le 1.04\,x$ for every $x > 0$ by [41, (3.35)] and that $\pi(Q) \le 1.26x/\log x$ by [41, (3.6)]. A numerical application ends the proof of the lemma. □

The next lemma follows the path initiated by Levin & Fainleib in [24], and trodden by several authors, like in [18].

Lemma 3.2. *Let h be a non-negative multiplicative function for which there exists a parameter H such that $\big|h(p^\nu)\big| \le H\nu$ for every prime power p^ν. Then we have*

$$\sum_{n \le x} h(n) \le \frac{3H\,x}{\log x} \sum_{n \le x} \frac{h(n)}{n}.$$

Proof. We start by

$$\sum_{n \le x} h(n)\log x = \sum_{n \le x} h(n)\log n + \sum_{n \le x} h(n)\log\frac{x}{n}$$
$$\le \sum_{n \le x} h(n)\log n + x\sum_{n \le x}\frac{h(n)}{n}.$$

Concerning the sum with $h(n)\log n$, we write

$$\log n = \sum_{p^\nu \| n} \log\big(p^\nu\big)$$

where the summation ranges over every prime power p^ν dividing n and such that $p^{\nu+1}$ does not divides n. In other words, ν is the proper power of p that divides n. We infer from this identity that:

$$\begin{aligned}\sum_{n\le x} h(n)\log n &= \sum_{p^\nu\le x}\log\bigl(p^\nu\bigr)\sum_{p^\nu\| n\le x} h(n) \le \sum_{p^\nu\le x}\log\bigl(p^\nu\bigr)h\bigl(p^\nu\bigr)\sum_{\substack{n\le x/p^\nu,\\ (n,p)=1}} h(n)\\ &\le H\sum_{p^\nu\le x}\nu\log\bigl(p^\nu\bigr)\sum_{n\le x/p^\nu} h(n)\\ &\le H\sum_{n\le x} h(n)\sum_{p^\nu\le x/n}\nu\log\bigl(p^\nu\bigr).\end{aligned}$$

To conclude, we employ Lemma 3.1 above. □

Theorem 3.3. *Let K be some real parameter and let g be a multiplicative function such that $\bigl|g\bigl(p^\nu\bigr)\bigr|\le K$ for every prime power p^ν. Then we have*

$$\Bigl|\sum_{n\le x}\frac{g(n)}{n}\Bigr| \le \frac{9K}{\log x}\sum_{n\le x}\frac{|g(n)|+|(\mathbb{1}\star g)(n)|}{n}$$

On taking $g=\mu$, and $K=1$, we recover the fact that the partial sum $\sum_{n\le x}\mu(n)/n$ is bounded.

Proof. We consider the sum $S=\sum_{n\le x}(\mathbb{1}\star g)(n)$ which we write in the form

$$S=\sum_{m\le x} g(m)[x/m] = x\sum_{m\le x}\frac{g(m)}{m} - \sum_{m\le x} g(m)\{x/m\}.$$

We deduce from the above that

$$\Bigl|\sum_{n\le x}\frac{g(n)}{n}\Bigr| \le \frac{1}{x}\sum_{n\le x}(|g|+|\mathbb{1}\star g|)(n).$$

We next notice that both functions $|g|$ and $|\mathbb{1}\star g|$ are multiplicative and non-negative. Furthermore $\bigl|g\bigl(p^\nu\bigr)\bigr|\le K\le K\nu$ by hypothesis, while the reader will readily check that $\bigl|(\mathbb{1}\star g)\bigl(p^\nu\bigr)\bigr|\le K(\nu+1)\le 2K\nu$. We are thus in a position to apply Lemma 3.2 twice, namely to the two multiplicative functions $|g|$ and $|\mathbb{1}\star g|$. Concluding the proof of the theorem is then straightforward. □

An intriguing example. Seeing the appearance of $|g|$ and $|\mathbb{1}\star g|$, one may want to balance the effect of both factors; this almost happens when one selects $g(d)=\mu(d)/2^{\omega(d)}$. This case has in fact been considered long ago by Sigmund Selberg, a mathematician like his more famous brother Atle Selberg, in his 1954 paper [43] where he used Meissel's approach in a very careful manner to show the next theorem.

Theorem 3.4 (S. Selberg, 1954)**.** *We have, for every $x>0$,*

$$0\le\sum_{n\le x}\frac{\mu(n)}{2^{\omega(n)}n}\le 1.$$

Proof. Let us denote by f the function that associates $\mu(n)/2^{\omega(n)}$ to the integer n. The reader will readily check that $(\mathbb{1}\star f)(n)=1/2^{\omega(n)}$. We thus get

$$\sum_{n\le x}\frac{1}{2^{\omega(n)}}=\sum_{\ell\le x}\frac{\mu(\ell)}{2^{\omega(\ell)}}\left[\frac{x}{\ell}\right]=x\sum_{\ell\le x}\frac{\mu(\ell)}{\ell 2^{\omega(\ell)}}-\sum_{\ell\le x}\frac{\mu(\ell)}{2^{\omega(\ell)}}\left\{\frac{x}{\ell}\right\}$$

from which we deduce that

$$x\sum_{\ell\le x}\frac{\mu(\ell)}{\ell 2^{\omega(\ell)}}=\sum_{n\le x}\frac{1}{2^{\omega(n)}}\biggl(1+\mu(n)\left\{\frac{x}{n}\right\}\biggr).$$

This astounding equation immediately implies that the left hand side is non-negative and bounded above. To prove the more precise bound 1, we first notice that it is enough to prove it for positive integers x, in which case we first find that

$$\frac{1}{2^{\omega(1)}}\biggl(1+\mu(1)\left\{\frac{x}{1}\right\}\biggr)=1$$

and then that, as soon as $n\ge 2$, we have $2^{\omega(n)}\ge 2$, hence

$$\frac{1}{2^{\omega(n)}}\biggl(1+\mu(n)\left\{\frac{x}{n}\right\}\biggr)\le\frac{2}{2}=1.$$

It is straightforward to conclude from these two inequalities. □

A consequence of Theorem 3.3 is also that, when $x\ge 2$, we have

$$\sum_{n\le x}\frac{\mu(n)}{2^{\omega(n)}n}\ll 1/\sqrt{\log x}. \tag{3.1}$$

This is for instance a consequence of the following theorem that we infer from the more precise [35, Theorem 21.1]. This theorem is in essence the one of Levin & Fainleib [24] we refered to above.

Theorem 3.5. *Let g be a non-negative multiplicative function. Let κ be a non-negative real parameter such that*

$$\begin{cases}\displaystyle\sum_{\substack{p\ge 2,\nu\ge 1\\ p^\nu\le Q}} g\bigl(p^\nu\bigr)\log\bigl(p^\nu\bigr)=\kappa\log Q+O(1) \qquad (Q\ge 1),\\ \displaystyle\sum_{p\ge 2}\sum_{\nu,k\ge 1} g\bigl(p^k\bigr)g\bigl(p^\nu\bigr)\log\bigl(p^\nu\bigr)\ll 1.\end{cases}$$

Then, we have

$$\sum_{d\le D} g(d)=\frac{(\log D)^\kappa}{\Gamma(\kappa+1)}\prod_{p\ge 2}\biggl\{\Bigl(1-\frac{1}{p}\Bigr)^\kappa\sum_{\nu\ge 0} g\bigl(p^\nu\bigr)\biggr\}\left(1+O(1/\log D)\right).$$

To infer (3.1) from Theorem 3.3, we use Theorem 3.5 twice with $\kappa=1/2$. We leave the details to the reader.

We are thus in a position to prove elementarity and with no use of the Prime Number Theorem that the sum $\sum_{n\le x}\frac{\mu(n)}{2^{\omega(n)}n}$ goes to 0! So why not try to reconstruct the Moebius function from this? This is easily achieved by employing Dirichlet's series. We first define the multiplicative function f_0 by $f_0(p^\nu) = -(\nu-1)/2$ and then find that

$$\sum_{n\geq 1}\frac{\mu(n)}{n^s} = \biggl(\sum_{n\geq 1}\frac{\mu(n)}{2^{\omega(n)}n^s}\biggr)^2\sum_{n\geq 1}\frac{f_0(n)}{n^s}. \tag{3.2}$$

The abscissa of absolute convergence of $D(f_0,s)=\sum_{n\geq 1} f_0(n)/n^s$ is $1/2$.

Proof. All the implied functions being multiplicative, it is enough to check this identity on each local p-factor, i.e that

$$1-\frac{1}{p^s} = \Bigl(1-\frac{1}{2p^s}\Bigr)^2\Bigl(1-\sum_{\nu\geq 1}\frac{\nu-1}{2^\nu p^{\nu s}}\Bigr).$$

This comes from the following formal identity, with $Y = X/2$:

$$\frac{1-X}{(1-\frac{X}{2})^2} = \frac{1}{1-Y} - \frac{Y}{(1-Y)^2} = \sum_{k\geq 0} Y^k - \sum_{k\geq 1} kY^k.$$

To get the abscissa of absolute convergence we consider, with $\sigma = \Re s$,

$$\begin{aligned}\Delta = \sum_{p\geq 2}\Bigl|\sum_{\nu\geq 2}\frac{\nu-1}{2^\nu p^{\nu s}}\Bigr| &\leq \sum_{p\geq 2}\frac{1}{4p^{2\sigma}}\Bigl|\sum_{k\geq 0}\frac{k+1}{(2p^\sigma)^k}\Bigr| \\ &\leq \sum_{p\geq 2}\frac{1}{4p^{2\sigma}}\frac{1}{(1-1/(2p^\sigma))^2} \leq \sum_{p\geq 2}\frac{1}{p^{2\sigma}}.\end{aligned}$$

This is bounded when $\sigma > 1/2$, showing that the product

$$\prod_{p\geq 2}\sum_{\nu\geq 0}\frac{f_0(p^\nu)}{p^{\nu s}}$$

is absolutely convergent when $\Re s > 1/2$. An immediate consequence is that the series is absolutely convergent in the same half-plane at least. The reader will readily see that the series of $|f_0(n)|/n^s$ diverges when $s = 1/2$, thus establishing that the half-plane $\Re s = 1/2$ is the actual half-plane of absolute convergence of $D(f_0, s)$. □

The function $f_0(n)/n$ being much smaller than the function $\mu(n)2^{-\omega(n)}/n$, a first goal before finding bounds for $\sum_{n\leq x}\mu(n)/n$ from bounds on $\mu(n)2^{-\omega(n)}/n$ is to estimate the quantity

$$\sum_{\ell m\leq x}\frac{\mu(\ell)\mu(m)}{2^{\omega(\ell)+\omega(m)}\ell m}.$$

The Dirichlet hyperbola formula is made for that, i.e. we write

$$\sum_{\ell m\leq x}\frac{\mu(\ell)\mu(m)}{2^{\omega(\ell)+\omega(m)}\ell m} = 2\sum_{\ell\leq\sqrt{x}}\frac{\mu(\ell)}{2^{\omega(\ell)}\ell}\sum_{m\leq x/\ell}\frac{\mu(m)}{2^{\omega(m)}m} - \biggl(\sum_{\ell\leq\sqrt{x}}\frac{\mu(\ell)}{2^{\omega(\ell)}\ell}\biggr)^2.$$

The second term is $O(1/\log x)$ while the first one is

$$\ll \sum_{\ell\leq\sqrt{x}}\frac{|\mu(\ell)|}{2^{\omega(\ell)}\ell}\frac{1}{\sqrt{\log x}} \ll 1$$

because we are loosing the sign of the Moebius factor $\mu(\ell)$. The bound (3.1) fails to improve on (2.3)! The reader may want to use the non-negativity bound and distinguish as to whether ℓ has an even or an odd number of prime factors... And for instance aim at a lower estimate: when $\mu(\ell) = 1$, we use the fact that the summand is non-negative, and otherwise that it is $O(1/\sqrt{\log x})$. We have then to estimate

$$\begin{aligned}\sum_{\ell \le \sqrt{x}} \frac{1+\mu(\ell)}{2} \frac{|\mu(\ell)|}{2^{\omega(\ell)}\ell} &= \frac{1}{2}\sum_{\ell \le \sqrt{x}} \frac{|\mu(\ell)|}{2^{\omega(\ell)}\ell} + \frac{1}{2}\sum_{\ell \le \sqrt{x}} \frac{\mu(\ell)}{2^{\omega(\ell)}\ell} \\ &= \frac{1}{2}\sum_{\ell \le \sqrt{x}} \frac{|\mu(\ell)|}{2^{\omega(\ell)}\ell} + O\Big(\frac{1}{\sqrt{\log x}}\Big)\end{aligned}$$

and so, we have only saved a factor 1/2—

Yet a third path opens before us: we may want to use the non-negativity of the sum $\sum_{n \le x} \frac{\mu(n)}{2^{\omega(n)} n}$ in a stronger manner via Landau's Theorem on Mellin transform of non-negative functions, and maybe derive a stronger estimate! Indeed, the integral

$$\int_1^\infty \sum_{n \le x} \frac{\mu(n)}{2^{\omega(n)} n} \frac{dx}{x^{s+1}}$$

represents the function $(1/s) \sum_{n \ge 1} \frac{\mu(n)}{2^{\omega(n)} n^{s+1}}$. Hence the abscissa of convergence of the integral should be a pole of the function represented. Can we show in this fashion that the integral converges for $\Re s > 1/2$ hence improving on (3.1)? This is tempting, but does not work: the series $\sum_{n \ge 1} \frac{\mu(n)}{2^{\omega(n)} n^{s+1}}$ behaves like $1/\sqrt{\zeta(s+1)}$, i.e. like $\sqrt{s}$ next to $s = 0$, and the innocent looking factor $(1/s)$ in front of the series above shows that the integral has a polar contribution at $s = 0$. In fact, S. Selberg already showed in [43] that $\sum_{n \le x} \frac{\mu(n)}{2^{\omega(n)} n}$ is equivalent to $C/\sqrt{\log x}$, where C is some well-defined and non-zero constant.

The purpose of this digression was to show the reader that the results we are looking at are tight. Any improvement would have acute consequences.

4. The Axer-Landau Equivalence Theorem

We have studied the situation from the angle of general multiplicative functions; let us now restrict more closely our attention to the case of the Moebius function. Here is an enlightening result in this direction. We first recall how the van Mangoldt function Λ is defined:

$$\Lambda(n) = \begin{cases} \log p & \text{when } n = p^\nu \\ 0 & \text{else.} \end{cases} \tag{4.1}$$

Theorem 4.1 (Axer-Landau, 1899-1911)**.** *The five following statements are equivalent:*

- *(S_1) The number of primes up to x is asymptotic to $x/\log x$.*
- *(S_2) $M(x) = \sum_{n \le x} \mu(n)$ is $o(x)$.*

(S_3) $m(x) = \sum_{n\le x} \mu(n)/n$ *is* $o(1)$.
(S_4) $\psi(x) = \sum_{n\le x} \Lambda(n)$ *is asymptotic to* x.
(S_5) $\tilde\psi(x) = \sum_{n\le x} \Lambda(n)/n$ *is* $\log x - \gamma + o(1)$.

In fact, proving that (S_3) implies (S_2) or that (S_5) implies (S_4) is a simple matter of summation by parts, as is the equivalence of (S_1) and (S_4). Edmund Landau in 1899 in [22] was the first to investigate this kind of result: he showed that (S_1) implies (S_3). The viennese mathematician A. Axer continued in 1910 in [1] by establishing that (S_2) implies (S_3). Landau immediately applied Axer's method to prove that (S_4) and (S_5) are equivalent and concluded in [21] essentially by showing that (S_3) implies (S_4). See also [2] and [23].

Concerning our question, this theorem shows that we clearly need to save the logarithm factor over the trivial estimate for $m(x)$, as well as for $\tilde\psi$. A second aspect arises from this theorem: the call for an quantitative version of it. If one follows the proofs of Axer and Landau, the saving is essentially limited at $O(1/\sqrt{\log x})$, though some later authors, like the swiss mathematician Alfred Kienast in [20], went further.

A related problem. In [36] and more fully in [33] with David Platt from Bristol, I investigated the problem of deriving quantitatively (near) optimal results on $\tilde\psi(x)$ once one supposes results for $\psi(x)$. This implication has been shown to be false in the general context of Beurling integers[4] by Harold Diamond & Wen-Bin Zhang in [8]. They even exhibit a Beurling system $\mathcal{B}$ where one has $\psi_{\mathcal{B}}(x) \sim x$ while $\tilde\psi_{\mathcal{B}}(x) - \log x \gg \log\log x$, with obvious notation. This means in particular that something special linked with the nature of the integers is required. It took us quite a while to understand what was happening, though I had essentially settled the problem in the q-aspect several years ago in [34]: instead of looking at primes, I was looking at primes in some arithmetic progression, say modulo some q; the error term has then a dependence in q and in x. In the mentionned paper, I resolved this question *provided* the question for $q = 1$ was solved! I thought that I had reduced the problem to a simpler one, but it is more correct to say that the x-aspect is the one that leads to real difficulties.

The first idea is of course to use a summation by parts, i.e. to write

$$\tilde\psi(x) - \log x = \frac{\psi(x) - x}{x} + 1 + \int_1^x \frac{\psi(t) - t}{t^2} dt. \tag{4.2}$$

A careful look at this equation will in fact be enough to solve the question. We can understand on it the idea of Diamond & Zhang: they built a Beurling system where the integral above does not converge. A different approach from this same starting point leads to the next theorem we proved with D.Platt.

[4] The Beurling integers are the multiplicative semi-group built on a family of "primes" to be chosen real numbers from $(1, \infty)$.

Theorem 4.2 (D. Platt + O.R., 2016)**.** *There exists $c > 0$ such that, when $x \geq 10$, we have:*

$$\tilde{\psi}(x) - \log x + \gamma \ll \max_{x \leq y \leq 2x} \frac{|\psi(y) - y|}{y} + \exp\biggl(-c\,\frac{\log x}{\log\log x}\biggr).$$

A similar statement for primes in arithmetic progressions holds true. This theorem is very efficient to compare $\tilde{\psi}$ together with ψ, and is in fact nearly optimal from a quantitative viewpoint. We are almost saving a power of x; a look at the proof discloses that the zero-free region for the Riemann-zeta function is used only up to the height $\log x$. This has the consequence that numerically, verifying the Riemann Hypothesis up to the height H gives control for x roughly up to e^H! And since X. Gourdon & P. Demichel [15] have checked this Riemann Hypothesis[5] up to height $2.445 \cdot 10^{12}$, we can assume the Riemann Hypothesis is available when $x \leq e^{10^{12}}$, which is enormous! Practically, this discussion shows that the factor $\exp(-c(\log x)/\log\log x)$ can be replaced by a very small quantity. We shall see below some very explicit consequences of this fact, but let us start by a rough explanation of the proof. This is not the manner the proof appeared at first, but how I now understand it. We first note that

$$1 + \int_1^\infty \frac{\psi(t) - t}{t^2}dt = -\gamma. \tag{4.3}$$

This is highly non-obvious if seen like that. The Prime Number Theorem with a remainder term ensures that the integral converges, but the full proof requires the limited development $\zeta(s) = (s-1)^{-1} + \gamma + O(s-1)$ around $s = 1$ which implies that $-(\zeta'/\zeta)(s) = (s-1)^{-1} - \gamma + O(s-1)$. We leave the details to the reader. What is really important for us is that this quantity is indeed a constant so that we can rewrite (4.2) in

$$\tilde{\psi}(x) - \log x = \frac{\psi(x) - x}{x} - \gamma - \int_x^\infty \frac{\psi(t) - t}{t^2}dt. \tag{4.4}$$

This formula is not enough to conclude but a very small modification of it will suffice: let $F : [1, \infty) \to \mathbb{R}$ be a smooth function such that $F(y) = 1$ when $y \geq 2$. We have

$$\begin{aligned}\tilde{\psi}(x) - \log x = \frac{\psi(x) - x}{x} - \gamma - \int_x^{2x} (1 - F(t/x))\frac{\psi(t) - t}{t^2}dt \\ + \int_x^\infty F(t/x)\frac{\psi(t) - t}{t^2}dt.\end{aligned} \tag{4.5}$$

The integral over $[x, 2x]$ can be controlled by $\max_{x<y<2x} |\psi(y) - y|/x$, but what about the last integral? In short: we express it in terms of the zeros of the Riemann zeta-function and get in this manner a fastly convergent sum. Why is that so? The reader may think it is because of the smoothing and the involvement of Mellin transforms... And would be right! A fact

[5] This computation has not been the subject of any published paper. D. Platt in [32] has checked this hypothesis up to height 10^9 by with a very precise program using interval arithmetic.

that had escaped my attention so long, and not only mine, is that this argument works for the point at infinity. Repeated integrations by parts for instance, when assuming F smooth enough, show that the corresponding Mellin transform decays vary rapidly in vertical strips.

Here are two very explicit consequences that we promised earlier.

Theorem 4.3. *We have*

$$\Big|\sum_{n\le x}\Lambda(n)/n-\log x+\gamma\Big|\le\frac{1}{149\log x}\quad(x\ge 23).$$

The previous result is due to J. Rosser & L. Schoenfeld in [41] and had a 2 instead of 149.

Theorem 4.4. *We have*

$$\Big|\sum_{n\le x}\Lambda(n)/n-\log x+\gamma\Big|\le\frac{2}{(\log x)^2}\quad(x>1).$$

This result has no ancestor that I know of. There are related work by P. Dusart [12], [11], L. Faber & H. Kadiri, H. Kadiri & A. Lumley [19] (and more to come), C. Axler [3], L. Panaitopol [29], R. Vanlalnagaia [46], ...

The sketch I propose above is the manner I now explain the proof, but the two initial papers, [36] and [33] proceeded in a very different manner: the integral $\int_x^\infty(\psi(t)-t)dt/t^2$ was expressed in terms of the zeros of zeta and the relevant expression was compared with another one more convergent. The better understood scheme above will have an interesting consequence we shall see later.

The horizon. It is time to set the horizon! Here are three conjectures.

Conjecture C. *There exists a constant $A>0$ such that*

$$m(x)\overset{?}{\ll}\max_{x/A<y\le xA}|M(y)|/y+x^{-1/4}.$$

And since we would like to have control of $M(x)$ via[6] $\psi(x)$, I also believe the following.

Conjecture D. *There exists a constant $A>0$ such that*

$$m(x)\overset{?}{\ll}\max_{x/A<y\le xA}|\psi(y)-y|/y+x^{-1/4}.$$

And we recall the conjecture of [36].

Conjecture A. *There exists a constant $A>0$ such that*

$$\tilde\psi(x)\overset{?}{\ll}\max_{x/A<y\le xA}|\psi(y)-y|/y+x^{-1/4}.$$

These three conjectures are trivially true under the Riemann Hypothesis, even with the $x^{-1/4}$ replaced by $x^{-1/2+\varepsilon}$. This exponent $1/4$ is not particularly relevant, the saving of any power of x would be a true achievement.

[6]I formulated such a more precise conjecture, say Conjecture B, in [37].

These three conjectures are obvious if we allow a factor $\log x$ in front of the maxima, simply by using integration by parts, but even if we allow a factor between 1 and $\log x$, like $\sqrt{\log x}$ for instance, the answer is not known.

5. From M to m

The proof we presented of Theorem 4.2 allows one to dispense with the notion of zeros, though introducing them is numerically much more efficient. We can however express the function F in (4.5) in terms of its Mellin transform. This Mellin transform decreases fast in vertical strips[7] and this is enough to get the result! We provide a full proof appear in [40].

Theorem 5.1. *There exists $c > 0$ such that, when $x \geq 10$, we have:*

$$m(x) \ll \max_{x\leq y\leq 2x} \frac{|M(y)|}{y} + \exp\left(-c\,\frac{\log x}{\log\log x}\right).$$

The difference with the case of $\tilde{\psi}$ is that we do not have any efficient version of this theorem.

6. Generalizing Meissel's proof, II

M. Balazard took another path to understand Meissel's formula. He rewrote this identity in the form:

$$\frac{1}{x}\int_1^x M(x/t)\frac{\{t\}}{t}dt = m(x) - \frac{M(x)}{x} - \frac{\log x}{x} \tag{6.1}$$

and did the same for the MacLeod identity:

$$\frac{1}{x}\int_1^x M(x/t)\frac{(2\{t\}-1)t+\{t\}-\{t\}^2}{t^2}dt = m(x) - \frac{M(x)}{x} - \frac{2}{x} + \frac{2}{x^2}.$$

Some order emerges in this manner, but the question remains as to whether these identities are oldies to be thrown in the wastebasket or not. The situation has been further cleared by F. Daval[8] [6] in the next theorem.

Theorem 6.1 (Daval, 2016)**.** *Let $h : [0,1] \to \mathbb{C}$ be a continuous function normalized by $\int_0^1 h(u)du = 1$. When $x \geq 1$, we have*

$$\frac{1}{x}\int_1^x M(x/t)\left(1-\frac{1}{t}\sum_{n\leq t} h(n/t)\right)dt = m(x) - \frac{M(x)}{x} - \frac{1}{x}\int_{1/x}^1 \frac{h(y)}{y}dy.$$

Like many identities, once it is written, it is not very difficult to establish. On selecting $h = 1$, we recover the Meissel identity, and on selecting $h(t) = 2t$, the MacLeod identity I showed is being recovered. We thus see that a "Riemann integral-remainder" appears; functional analysis is coming in! Among the natural questions, let us mention this one: given a function f over $[0,1]$, can it be approximated by such a Riemann-remainder term? If not what is the best approximation? Before continuing, let us mention that there are some other identities in this area, and for instance, following J.-P.

[7] As already stated, we show that by classically repeated integrations by parts.

[8] F. Daval was at the time a PhD student of mine.

Gram [16], R. MacLeod [25] and M. Balazard [4], here is, in Balazard's form, a typical identity I obtained in [39, Lemma 3.2]:

$$\sum_{n\le x}\frac{\mu(n)}{n}\log\Big(\frac{x}{n}\Big)-1=\frac{6-8\gamma}{3x}-\frac{5-4\gamma}{x^2}+\frac{6-4\gamma}{3x^4}-\frac{1}{x}\int_1^x M(x/t)h'(t)dt$$

where[9] the function h is differentiable except at integer points where it has left and right-derivative, and satisfies $0\le t^2|h'(t)|\le\frac{7}{4}-\gamma$. The function h is this time linked with the error term $\sum_{n\le t}1/n-\log t$. Similar identities have been proved with $\log^k(x/n)$ instead of $\log(x/n)$, for any positive integer k. The theory of F. Daval can most probably be adapted to these cases. One striking consequence is the next result.

Theorem 6.2 ([39, Theorem 1.5]). *When* $x\ge 3155$, *we have*

$$\Big|\sum_{n\le x}\frac{\mu(n)}{n}\log(x/n)-1\Big|\le\frac{1}{389\log x}.$$

Note that one could try to derive such an estimate by writing

$$\sum_{n\le x}\frac{\mu(n)}{n}\log(x/n)=(\log x)\sum_{n\le x}\frac{\mu(n)}{n}-\sum_{n\le x}\frac{\mu(n)}{n}\log n$$

and using estimates for both. But to attain the accuracy level of our theorem, one would need to prove at least that $\sum_{n\le x}\mu(n)/n=\mathcal{O}^*(1/(389\log^2 x))$, and we are rather far from having this kind of results!

The problem at large. Let us try to formalize the problem. We start from a regular function $F:[1,\infty)\to\mathbb{C}$, for instance $F(t)=1$ or $F(t)=\log t$. The question is to find two functions H and G and a constant C such that

$$\sum_{n\le x}\frac{\mu(n)}{n}F(x/n)-C\frac{M(x)}{x}=\frac{1}{x}\int_1^x M(x/t)G(t)dt+H(x).$$

To avoid trivial solutions, we assume that

$$\int_1^\infty |F(t)|dt/t=\infty,\qquad \int_1^\infty |G(t)|dt/t<\infty,$$

and that H is smooth and "small". This looks like a functional transform from F to G, but there is a lot of slack! Indeed, when $F=1$ or when $F(t)=\log t$, there are several solutions.

Beginning of a theory when $F=1$. We start from Theorem 6.1 and, remembering the identities of MacLeod in Balazard's form, we aim at writing the integral with M in the form $\int M(x/t)f'(t)dt$. With this goal in sight we note that

$$\int_0^x\Big(1-\frac{1}{t}\sum_{n\le t}h(n/t)\Big)dt=\int_0^1\{ux\}\frac{h(u)}{u}du.$$

[9] As a matter of fact, the mentionned lemma is slightly different, but a corrigendum is on its way.

So, given $f : [1,\infty) \to \mathbb{C}$, we want to solve $f(x) = \int_0^1 \{ux\}\frac{h(u)}{u}du$. The change of variable $y = 1/x$ leads to the problem: given $g : [0,1] \to \mathbb{C}$, solve $g(y) = \int_0^1 \frac{\{u/y\}}{u/y}h(u)du$. We see another appearance of functional analysis! The operator T over the Hilbert space $L^2([0,1])$ which associates $\int_0^1 \frac{\{u/y\}}{u/y}h(u)du$ to h is a Hilbert-Schmidt, compact and contracting operator. Indeed, we readily check that the kernel $(u,y) \mapsto \frac{\{u/y\}}{u/y}$ belongs to $L^2([0,1]^2)$ and then, we for instance use [14] (around equations (9.6) − (9.8)). Since

$$\begin{aligned}\int_0^1\int_0^1\left|\frac{\{u/y\}}{u/y}\right|^2 dudy &= \int_0^1\int_0^{1/y}\left|\frac{\{z\}}{z}\right|^2 dz\, ydy \\ &\le \int_0^1\left(1+\int_1^\infty \frac{\{z\}}{z^2}dz\right)ydy = 2(1-\gamma) < 1\end{aligned}$$

we readily see by invoking the Cauchy-Schwarz inequality that T is strictly contracting. The general theory tells us that there exist a sequence of complex numbers $(\lambda_n)_n$ and two orthonormal sequences of functions $(\psi_n)_n$ and $(\varphi_n)_n$ such that

$$\int_0^1 \{u/y\}\frac{h(u)}{u}du = \sum_{n\ge1}\lambda_n\int_0^1 h(u)\overline{\psi_n(u)}du\varphi_n(y)$$

for every $y \in [0,1]$. By [44], this operator to be of Shatten class p for every $p > 1$, and I suspect it is *not* of trace class. The above decomposition is a consequence of the general theory of integral operator and a more specific study should be able to disclose arithmetical properties. For instance, the presence of the fractional part is not without recalling the Nyman-Beurling criteria. This is work in progress.

The localization problem, case $F = 1$. We are here going back to what has been done rather than guessing what could be happening in the future! It is easier to first state a result and then describe the problem at hand from there. We start with a lemma.

Lemma 6.3 (F. Daval, 2017)**.** *Let $h : [0,1] \mapsto \mathbb{C}$ be a C^k-function for some $k \ge 2$, normalised with $\int_0^1 h(u)du = 1$. We further assume that*

- $h(0) = h'(0) = 0$,
- *When* $3 \le 2i+1 \le k-1$, *we have* $h^{(2i+1)}(0) = 0$,
- *When* $0 \le \ell \le k-2$, *we have* $h^{(\ell)}(1) = 0$.

Then we have, for $t \ge 1$,

$$\left|1 - \frac{1}{t}\sum_{n\le t} h(n/t)\right| \ll 1/t^k.$$

Given an integer $k \ge 2$, let us call $\mathscr{H}_k$ the class of functions h described above. Then, for any $h \in \mathscr{H}_k$, there exists a constant $C_k(h)$ such that

$$\left|\int_1^x M(x/t)\left(1 - \frac{1}{t}\sum_{n\le t} h(n/t)\right)dt\right| \le \frac{C_k(h)}{x}\int_1^x M(t)(t/x)^{k-2}dt.$$

F. Daval [6] has obtained the following table:

$k=$	3	4	5	6	7
$\min_h C_k(h) \le$	1.05	1.44	2.52	5.9	13.2

This improves of earlier values of M. Balazard in [4]. It would be interesting to determine numerically these minima with more accuracy. The value for $k=5$ has been obtained with the highly non-obvious choice $h_5(t) = 2t^2(1-t)^4(120t^2+52t+13)$. As a consequence, one can get for instance the following inequality:

$$\left|m(x) - \frac{M(x)}{x}\right| \le \frac{33/13}{x^4}\int_1^x |M(t)|t^3 dt + \frac{19/7}{x}$$

and such an inequality should give improvements for many of the results I obtained in [39]. We call this problem the "localization problem" because a high power of t in the integral above means that the values of $M(t)$ for t close to x have more weight than the lower ones. We recall that conjecture C claims that one can use only the values of t that are a constant multiple of x.

All in all, a lot remains to be understood in this area. I for instance wonder whether functions like $\{t^2+1\}$ could appear in these identities rather than $\{t\}$... I thought at first that the answer should be no but I am not so sure anymore.

From Λ to μ / From ψ to M. Let us continue our journey around the Axer-Landau Equivalence Theorem. We first notice that Wen-Bin Zhang has exhibited in [47] a Beurling system of integers where one have $M_{\mathcal{P}}(x) = o(x)$ without $\psi_{\mathcal{P}}(x) \sim x$. Our final destination being numerical estimates, we are however more interested in the reverse implication, i.e. to derive bounds for M from bounds for the primes. This problem has been studied by A. Kienast in [20] and by L. Schoenfeld [42], and they proceeded as I later did in [37] by using some combinatorial identities. The family of identities I produced is simply more efficient. It is better to refer the reader to the cited paper but let us give the general flavour. The first interesting case reads

$$\sum_{\ell \le x} \mu(\ell)\log^2 \ell = \sum_{d\ell \le x} \mu(\ell)\bigl(\Lambda \star \Lambda(d) - \Lambda(d)\log d\bigr). \tag{6.2}$$

It is worth mentioning that the Selberg[10] identity that is used for proving elementarily the Prime Number Theorem is $\Lambda \star \Lambda(d) + \Lambda(d)\log d = (\mu \star \log^2)(d)$ and that, assuming this Prime Number Theorem, both factors $\Lambda \star \Lambda(d)$ and $\Lambda(d)\log d$ contribute equally to the average. In particular, the function $\Lambda \star \Lambda(d) - \Lambda(d)\log d$ should be looked upon as a remainder term. We get information of its average order by using the Dirichlet hyperbola formula; it would most probably be better to use an explicit expression in terms of the zeros directly, but this involves the residues of $(\zeta'/\zeta)^2$ and there lacks a control of those, while the residues of ζ'/ζ are well understood. Some more thought discloses that we need essentially the L^1-norm of such residues, and since they are non-negative integers for ζ'/ζ, we may as well compute

[10]This one is Atle Selberg!

their simple average, which is readily achieved by a contour integration that has most of its path outside the critical strip. No such phenomenom is known to occur for $(\zeta'/\zeta)^2$! The reader may be wary of the Moebius factor that appears on the right-hand side of (6.2), but only one such factor appears. It is maybe more apparent in the next identity of this series:

$$\sum_{\ell\le x}\mu(\ell)\log^3\ell=\sum_{d\ell\le x}\mu(\ell)\bigl(\Lambda\star\Lambda\star\Lambda(d)-3\Lambda\star(\Lambda\log)(d)+\Lambda(d)\log^2 d\bigr).$$

When starting with the last identity with $k=3$, one can expect to save a $\log^3 x$ on the trivial estimate x, but the presence of the Moebius factor on the right-hand side reduces that to a saving of one $\log x$ less, so $\log^2 x$. This is because the Dirichlet hyperbola method is not used, though one may employ a recursion process: indeed, L. Schoenfeld does that, followed by H. Cohen, F. Dress & M. El Marraki in [5], [10] and [13]. I did not introduce such a step as it is numerically costly, but a more careful treatment is here possible.

7. Generalizing Meissel's proof, III

We now turn towards the third aspect of Meissel's identity, which is to provide a simple proof of $\sum_{n\le x}\mu(n)/n\ll 1$. Here is a theorem I proved a long time back with Andrew Granville in [17, Lemma 10.2].

Theorem 7.1. *For $x\ge 1$ and $q\ge 1$, we have*

$$\Biggl|\sum_{\substack{n\le x,\\ \gcd(n,q)=1}}\frac{\mu(n)}{n}\Biggr|\le 1.$$

This result belongs to the family of the eternally-reproved lemmas! In fact, I discovered much lated that it appeared already in an early paper of Harold Davenport as [7, Lemma 1]! The precise upper bound by 1 is not given, but the proof is already there. And Terence Tao reproved this result in [45], in a larger context, but the proof is again the same! We cannot even say that Davenport's paper or the one I co-authored are forgotten: they are simply cited for other reasons.

The main theme is the handling of the coprimality condition. Since we mentioned the investigations of Sigmund Selberg, it is worthwhile stating a surprising lemma that one finds in [43, Satz 4].

Theorem 7.2. *For $x\ge 1$ and $d,q\ge 1$, with $d|q$, we have*

$$0\le\sum_{\substack{n\le x,\\ \gcd(n,d)=1}}\frac{\mu(n)}{2^{\omega(n)}n}\le\sum_{\substack{n\le x,\\ \gcd(n,q)=1}}\frac{\mu(n)}{2^{\omega(n)}n}\le 1.$$

We should make a stop here; indeed the reader may think that removing the coprimality condition is an easy task. The standard manner goes by using the Moebius function and the identity:

$$\sum_{\substack{d|n,\\ d|q}} \mu(d) = \sum_{d|\gcd(n,q)} \mu(d) = \begin{cases} 1 & \text{when } \gcd(n,q)=1 \\ 0 & \text{else.} \end{cases} \tag{7.1}$$

However, here is what happens in our case:

$$\begin{aligned} \sum_{\substack{n\le x,\\ \gcd(n,q)=1}} \frac{\mu(n)}{n} &= \sum_{n\le x}\sum_{d|\gcd(n,q)} \mu(d)\frac{\mu(n)}{n} = \sum_{d|q}\mu(d)\sum_{d|n\le x}\frac{\mu(n)}{n} \\ &= \sum_{d|q}\frac{\mu^2(d)}{d}\sum_{\substack{m\le x/d,\\ \gcd(m,d)=1}}\frac{\mu(m)}{m} \end{aligned} \tag{7.2}$$

and thus a coprimality condition comes back in play! E. Landau has devised a long time ago a manner to go around this problem: it consists in comparing the multiplicative function $f(n) = \mathbb{1}_{(n,q)=1}\mu(n)$ with the function μ, i.e. to find a function g such that $f = \mu\star g_q$, where $\star$ is the arithmetical convolution product. Determining g_q is an exercise resolved by comparing the Dirichlet series. Once the reader has found the expression for g_q, he or she will find that it is somewhat unwieldy. The foremost problem is that it has an infinite support and thus, when we write

$$\sum_{\substack{n\le x,\\ \gcd(n,q)=1}} \frac{\mu(n)}{n} = \sum_{\ell\ge1} g_q(\ell)\sum_{m\le x/\ell}\mu(m)/m$$

one has to handle the case when ℓ is large, i.e. when x/ℓ is small. This leads to difficulties, for instance when one wants explicit estimates. But even if one aims only at theoretical results, diffulties appear: for instance, if one wants to bound $\sum_{\substack{n\le x,\\ \gcd(n,q)=1}} \frac{\mu(n)}{n}$ from the estimate $|\sum_{n\le x}\frac{\mu(n)}{n}| \le 1$ and the function g_q, the resulting bound is $\mathcal{O}(q/\phi(q))$, which can be infinitely larger than $\mathcal{O}(1)$.

I devised in [38] and [39] a workaround to handle this question. The two remarks needed are first that the Liouville function[11] λ is rather close to the Moebius function, and second that the Liouville function being completely multiplicative, the proof above (leading to (7.2)) would this time succeed. This implies a process in three steps:

(1) Go from μ to λ.
(2) Get rid of the coprimality with the Moebius function.
(3) Study the resulting sum by comparing λ to μ and by using results on μ.

In the second paper, I noticed that it is possible to combine steps 1 and 3, hence gaining in efficiency. This process is however only half a cure: one

[11] The Liouville function is the completely multiplicative function defined by $\lambda(n) = (-1)^{\Omega(n)}$, where $\Omega(n)$ is the number of prime factors of n, counted with multiplicity, so that $\Omega(12) = 3$.

indeed avoids short sums, and this is numerically important, but the factor $q/\phi(q)$ we talked about earlier still arises! Here is a typical result I obtained in this fashion.

Theorem 7.3. *When* $1 \le q < x$, *we have*

$$\Bigl|\sum_{\substack{d\le x,\\(d,q)=1}}\mu(d)/d\Bigr| \le \frac{4q/5}{\phi(q)\log(x/q)}.$$

Similar results with $\mu(d)\log(x/d)/d$ and $\mu(d)\log(x/d)^2/d$ are also presented. Let me end this section with a methological remark: Theorem 7.1 does not contain in its statement a natural restriction of q with respect to x, and as such is hard to improve upon. Indeed q could be the product of all the primes below x, in which case the bound is optimal. The factor $1/\log(x/q)$ in Theorem 7.3 avoids this fact, which is why I believe it can be largely improved. The removal of the factor $q/\phi(q)$ would be a interesting step.

A related problem. Meissel's identity leads to an excellent handling of the coprimality condition, and we saw at the beginning of this section that it was not obvious to generalize. In another paper [27] with Akhilesh P. concerning the Selberg sieve density function, we encountered the problem of bounding the sum

$$\sum_{\substack{k>K,\\\gcd(k,q)=1}}\frac{\mu(k)}{k\phi(k)} \tag{7.3}$$

uniformly in q. We were only able at the time to get a better than trivial estimate, but recently, together with Akhilesh P. in [28], we proved the next result by again employing the Liouville trick described above to which we added a sieving argument.

Theorem 7.4.

$$\limsup_{K\to\infty}\ K\ \max_q\Bigl|\sum_{\substack{k>K,\\\gcd(k,q)=1}}\frac{\mu(k)}{k^2}\Bigr| = 0.$$

Our result is more general and encompasses the sum (7.3). In essence, the proof runs as follows: when q has many prime factors, use a sieve bound; when q has few prime factors, remove the coprimality condition with Moebius. This time coprimality with $\prod_{p\le K, p\nmid q} p$ comes into play. Both arguments take care of extremal ranges of q (i.e. when q as many or few prime factors). These ranges do not overlap: there is a middle zone where this time, the oscillation of μ comes into play, and it is where we use the λ-trick to get rid of the coprimality condition.

The rate of convergence is however unknown to us. Under the Riemann Hypothesis, our proof gives a rate of convergence in $1/(\log K)^{1/3-\varepsilon}$ for any

$\varepsilon > 0$ but the best we have been able to prove concerning an Omega-result is that

$$\limsup_{K\to\infty} K\log K \max_q \Bigl|\sum_{\substack{k>K,\\(k,q)=1}} \frac{\mu(k)}{k^2}\Bigr| \geq 1.$$

We have not even been able to improve on this last constant 1, which we got by considering $q = \prod_{p\leq K} p$. Our journey ends here!

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