A purely analytical lower bound for $L(1,\chi)$

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Abstract. We give a simple proof of $L(1,\chi)\sqrt{q} \gg 2^{\omega(q)}$ when χ is an odd primitiv quadratic Dirichlet character of conductor q. In particular we do not use the Dirichlet class-number formula.

Résumé. Nous donnons une preuve simple de l'inégalité $L(1,\chi)\sqrt{q} \gg 2^{\omega(q)}$ lorsque χ est un caractère quadratique primitif impair. En particulier, nous n'utilisons pas la formule de Dirichlet liant $L(1,\chi)$ et e nombre de classes.

1. Main results

For a Dirichlet quadratic character χ of conductor q, several techniques were devised to get a lower bound for $L(1,\chi)$. One of them consists in estimating

$$S(\alpha) = \sum_{n\geq 1} \bigl(\sum_{d|n} \chi(d)\bigr)e^{-\alpha n} \tag{1.1}$$

in two ways, where $\alpha > 0$ is a parameter to be chosen. We first notice that $(1\star\chi)(n) \geq 0$ and even ≥ 1 if n is a square, thus obtaining the lower bound $S(\alpha) \gg \alpha^{-1/2}$. On an other side, reversing the inner summation yields

$$S(\alpha) = L(1,\chi)\alpha^{-1} + \sum_{d\geq 1} \chi(d)\left(\frac{1}{e^{\alpha d}-1} - \frac{1}{\alpha d}\right). \tag{1.2}$$

Using partial summation and the Polya-Vinogradov inequality, the "remainder term" is $\mathcal{O}(\sqrt{q}\operatorname{Log} q)$. Taking $\alpha^{-1} = c\sqrt{q}\operatorname{Log}^2 q$ for a large enough constant c yields $L(1,\chi)\sqrt{q} \gg 1/\operatorname{Log} q$. It is fairly easy to remove this $\operatorname{Log} q$ by noticing that only a smoothed version of the Polya-Vinogradov inequality is required, thus getting $L(1,\chi)\sqrt{q} \geq c$ for some positive constant c.

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The next question is to evaluate c. Recalling the Dirichlet class number formula

$$L(1,\chi)\sqrt{q} = \begin{cases} \pi h(-q) & \text{if } \chi(-1) = -1, q \geq 5 \\ \operatorname{Log} \epsilon_q \; h(q) & \text{if } \chi(-1) = +1, \end{cases} \tag{1.3}$$

we see that $c > \pi$ for q larger than an explicit value would give another solution of the class number 1 problem. As it turns out the previous method can be made to yield the bound $c = \pi - o(1)$. Such a result is a priori surprising since the method being analytical, the smoothing e^{-x} can be modified. We have not been able to solve the resulting extremal problem in order to get the best possible smoothing, but we have tried numerically several of them (including all the smoothings of the form $P(x)e^{-x}$ where P is a polynomial) and found no better one, which suggests that this choice is (sadly) optimal. The general case being cumbersome, we have chosen to present a shorter proof which is well adapted to this particular choice and enables us to give a more complete description. We shall however restrict our attention to odd characters.

Using genus theory of quadratic forms, one can deduce from the Dirichlet class number formula that $L(1,\chi) \geq \frac{\pi}{2} 2^{\omega(q)}/\sqrt{q}$ and it turns out that the previous method can be modified to yield a bound of similar strength, without any appeal either to the theory of quadratic forms, nor to the Dirichlet class number formula.

Theorem 1.1. *When q goes to infinity and χ is a primitive quadratic Dirichlet character such that $\chi(-1) = -1$, we have*

$$L(1,\chi) \geq (1 + o(1))\pi/\sqrt{q}.$$

Moreover, uniformly for $q \geq 3$, we have

$$L(1,\chi) \geq \frac{2^{\omega(q)-1}\pi}{7.5\sqrt{q}}.$$

To do so we first rewrite the above proof. The function

$$\Phi(s) = (2\pi/\sqrt{q})^{-s}\,\Gamma(s)\zeta(s)L(s,\chi), \tag{1.4}$$

verifies the relation $\Phi(1-s) = \Phi(s)$. Consequently and following Hecke's theory on Dirichlet series having a functional equation, we deduce that the function of the variable $\tau = x + iy$ with $y > 0$

$$f(\tau) = \frac{\sqrt{q}}{2\pi} L(1,\chi) + \sum_{n \geq 1} (1 \star \chi)(n) e^{2i\pi n\tau/\sqrt{q}}, \tag{1.5}$$

verifies $f(-1/\tau) = (\tau/i)f(\tau)$. To get these relations, we simply express f in term of Φ by the Mellin formula and use the functional equation for Φ.

The next step consists in elaborating on the argument $(1 \star \chi)(n^2) \geq 1$. To do so let λ be Liouville's function (whose associated Dirichlet series is $\zeta(2s)\zeta(s)^{-1}$). We check that μ^2 is the convolution inverse of λ and that $1 \star \lambda$ is the characteristic function of the squares. Defining $\nu = \mu^2 \star \chi \geq 0$, we get

$$1 \star \chi = 1 \star \lambda \star \mu^2 \star \chi = \nu \star (1 \star \lambda)$$

which yields

$$f(iz) = \frac{\sqrt{q}}{2\pi} L(1,\chi) + \sum_{m \geq 1} \nu(m)\theta_0(2\pi m z/(k\sqrt{q})) \tag{1.6}$$

where

$$\theta_0(z) = \sum_{n \geq 1} e^{-zn^2}. \tag{1.7}$$

On using the functional equation of f, we infer

$$\frac{1-x}{2}\frac{\sqrt{q}}{\pi} L(1,\chi) = \sum_{m \geq 1} \nu(m)\mathcal{H}(x, 2\pi m x/\sqrt{q}) \tag{1.8}$$

where

$$\mathcal{H}(x,y) = x\theta_0(y) - \theta_0(y/x^2). \tag{1.9}$$

The point is that $\mathcal{H}(x,y) \geq 0$ as soon as x is not too close neither to 1 nor to 0. This property is highly non-obvious since the main terms (as y is close to 0) of both summands of the RHS of (1.9) cancel each other as may be seen in the proof below. $\mathcal{H}(x,y)$ is then about $(1-x)/2$ when y is small and then decays to 0.

Lemma 1.2. *For $x = 0.1987$, we have $\mathcal{H}(x,y) \geq 0$ for $y \geq 0$. Moreover the estimate $\mathcal{H}(x,y) = \frac{1-x}{2} + \mathcal{O}(y)$ holds. Finally we have*

$$\min_{y \leq 2\pi x} \mathcal{H}(x,y) \geq \frac{1-x}{15}.$$

We believe much more to be true:

Conjecture 1.3. *We have $\mathcal{H}(x,y) \geq 0$ for every $x \in [0,1]$ and $y \geq 0$.*

Proof. We assume throughout that $0 < x < 1$. The proof is separated in two parts according as to y is small or not. We start with the case of small y's. By Euler-MacLaurin summation formula, we get

$$\theta_0(y) = \int_0^\infty e^{-yt^2}dt - \frac{1}{2} + \frac{(-1)^{h+1}}{h!}\int_0^\infty B_h(t)f^{(h)}(t)dt \qquad (h \geq 2)$$

where B_h is the h-th Bernoulli function and $f(t) = \exp(-yt^2)$. This formula is that simple because f is even: this implies that $f^{(r)}(0)B_{r+1}(0) = 0$ when $r \geq 1$. We use the above expression with $h = 3$. Recalling that $B_3(t) = t^3 - 3t^2/2 + t/2$ for $t \in [0, 1]$, we get $|B_3(t)| \leq 1/(12\sqrt{3})$ and thus

$$\theta_0(y) = y^{-1/2}\Gamma(3/2) - \tfrac{1}{2} + \mathcal{O}^*(Cy)$$

where $f(t) = \mathcal{O}^*(g(t))$ means $|f(t)| \leq g(t)$ and where

$$C = \frac{1}{72\sqrt{3}}\int_0^\infty \left|\frac{d^3}{dt^3}e^{-t^2}\right| dt = \frac{4e^{-3/2}+1}{36\sqrt{3}} = 0.030351 + \mathcal{O}^*(10^{-6}).$$

Details of this computation can be found in the appendix. We thus get

$$\mathcal{H}(x,y) \geq \frac{1-x}{2}\left(1 - 2C\,y\frac{x+x^{-2}}{1-x}\right). \tag{1.10}$$

From this expression it follows that $\mathcal{H}(x,y) \geq 0$ as soon as

$$y/x^2 \leq w(x) = \frac{1-x}{2C\,(1+x^3)}.$$

Let us assume now that $y/x^2 \geq w(x)$. Using geometrical progressions, we get

$$\begin{aligned}\theta_0(y/x^2) &\leq e^{-y/x^2} + e^{-4y/x^2} + \sum_{n\geq 3} e^{-n^2y/x^2} \\ &\leq e^{-y/x^2} + e^{-4y/x^2} + \sum_{n\geq 3} e^{-3ny/x^2} \leq \frac{e^{-y/x^2}+e^{-3y/x^2}}{1-e^{-3y/x^2}}.\end{aligned}$$

We use $x\theta_0(y) \geq xe^{-y}$, and thus $\mathcal{H}(x,y) \geq 0$ as soon as

$$\frac{1-e^{-3w(x)}}{1+e^{-2w(x)}}e^{(x^{-2}-1)y} \geq x^{-1}$$

(we have used the fact that $\rho \mapsto (1-\rho^3)/(1+\rho^2)$ is non-increasing on $[0,1]$, as a computation of its derivative readily shows). Since we want to cover the range $y \in]0,\infty[$, the condition on x reads

$$\frac{1-e^{-3w(x)}}{1+e^{-2w(x)}}e^{(1-x^2)w(x)} \geq x^{-1} \tag{1.11}$$

which holds for the chosen value of x. See the Appendix for further details. We now aim at a lower bound of the shape $\mathcal{H}(x,y) \geq \kappa(1-x)$ valid for every $y \leq 2\pi x$. We first check that (1.10) implies it when $y \leq (1-2\kappa)x^2 w(x)$. We extend this range by using

$$\mathcal{H}(x,y) \geq xe^{-y} - \frac{e^{-y/x^2}+e^{-3y/x^2}}{1-e^{-3\xi/x}} \quad \text{for} \quad y \geq \xi x. \tag{1.12}$$

This lower bound, say $\mathcal{H}_0(x,y,\xi)$, is first increasing and then decreasing as a function of y, as an examination of its derivative readily shows. This implies that

$$\min_{\xi_1 x \leq y \leq \xi_2 x} \mathcal{H}(x,y) \geq \min(\mathcal{H}_0(x,\xi_1 x,\xi_1),\mathcal{H}_0(x,\xi_2 x,\xi_1)). \tag{1.13}$$

We then select $x = 0.1987$ and $\kappa = 1/15$. The lower bound stemming from (1.13) gives our lower bound when y lies in $[1.7, 2\pi]$ and we extend this bound by using (1.10) since $(1-2\kappa)xw(x) > 1.7$. Let us mention that we could have split the range $[1.7, 2\pi]$ of ξ in several subintervals. The Lemma follows readily. □

We now continue the proof of Theorem 1.1. Using (1.8) and the Lemma, we get

$$L(1,\chi)\sqrt{q} \geq \pi(1+\mathcal{O}(1/\sqrt{q}))$$

by discarding all the terms except the one corresponding to $m = 1$. By discarding only the terms corresponding to $m > \sqrt{q}$, we get

Theorem 1.4. *We have*

$$L(1,\chi)\sqrt{q} \geq \frac{2\pi}{15}\sum_{m\leq\sqrt{q}}\nu(m)$$

To end the proof of Theorem 1.1, note that $\nu(p) = 1$ if $p|q$. Since half the divisors of q are less than $\sqrt{q}$, we conclude easily.

Theorem 1.4 is to be compared with what one can get by using the theory of quadratic forms. For instance (cf [2, (1.4)]), Oesterlé proves

$$L(1,\chi)\sqrt{q}/\pi \geq \sum_{m\leq\sqrt{q}/2}\nu(m),$$

and Goldfeld in [1] already proved similar inequalities. Modifying this technique we can extend the range of summation to $m \leq \sqrt{q}$ if we divide the lower bound by 3.

2. Numerical Appendix

Computing C

With $g(t) = e^{-t^2}$, we get

$$g''(t) = (-2 + 4t^2)g(t), \quad g'''(t) = 4t(3 - 2t^2)g(t).$$

The function $g'''(t)$ is negative except for $t \in [0, \sqrt{3/2}]$ where it is non-negative. The L^1-norm of g''' is $-g''(0) + 2g''(\sqrt{3/2}) = 2(4e^{-3/2} + 1)$, which gives the values of C.

Checking the positivity

Note that the RHS of (1.11) is increasing as a function of w and that $w(x)$ is a decreasing function of x. By splitting the interval $[0, 1]$ in 1 000 subintervals $[x_-, x_+]$, and using the fact that the RHS of (1.12) is non-negative as soon as

$$\frac{1 - e^{-3w(x_-)}}{1 + e^{2w(x_-)}} e^{(1-x_+^2)w(x_+)} - x_-^{-1} \geq 0 \qquad (x_- \leq x \leq x_+) \tag{2.1}$$

we check that $\mathcal{H}(x, y) \geq 0$ for all $y \geq 0$ provided x verifies:

$$0.001 \leq x \leq 0.882.$$

Plotting $\mathcal{H}(x, y)$

We have presented a simple approach, but we could have relied more heavily on computational material. The function θ_0 satisfies a functional equation inherited from the one of the theta function (and the latter is equivalent via Hecke's correspondence to the one of the Riemann zeta function). It reads:

$$\theta_0(y) = \sqrt{\frac{\pi}{y}}\theta_0(\pi^2/y) + \frac{\sqrt{\pi}}{2\sqrt{y}} - \frac{1}{2}.$$

This leads to the following GP/PARI-script:

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{thetabase0(y)= sumpos(n = 1, exp(-n^2*y))}

{theta0(y)=
   if(y < Pi,
      return(thetabase0(Pi^2/y)/sqrt(y/Pi)-1/2+1/2/sqrt(y/Pi)),
      return(thetabase0(y)))}

{IH(x,y)=(x*theta0(y)-theta0(y/x^2))/(1-x)}

ploth(y = 0.1, 2*Pi*0.1987, IH(0.1987, y));
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This produces a high definition plot of $\mathcal{H}(0.1987, y)/(1 - 0.1987)$. We see on this that we can replace the 15 in our bound by a 14.

Bibliography

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