A spectral resolution of the large sieve *†

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Abstract

The quadratic form $V(\varphi,Q) = \sum_{q\sim Q}\sum_{a \bmod^* q}|S(\varphi,a/q)|^2$ and its eigenvalues are well understood when $Q = o(\sqrt{N})$, while $V(\varphi,Q)$ is expected to behave like a Riemann sum when $N = o(Q)$. The behavior in the range $Q \in [\sqrt{N}, 100N]$ is still mysterious. In the present work we present a full spectral analysis when $Q \geqslant N^{7/8}$ in terms of the eigenvalues of a one-parameter family of nuclear difference operators. We show in particular that (a smoothed version of) the quadratic form $V(\varphi,Q)$ may stay *away* from $(6/\pi^2)Q\sum_n|\varphi_n|^2$ when $Q \asymp N$, though only on a vector space of positive but small dimension.

1 Introduction and results

Main consequence

We are interested in this paper in the quantity $\sum_{q\sim Q}\sum_{a \bmod^* q}|S(\varphi,a/q)|^2$ where $(\varphi_n)_{n\leqslant N}$ is any sequence of complex numbers and $S(\varphi,\alpha) = \sum_{n\leqslant N}\varphi_n e(n\alpha)$. It is this quantity that we analyze. Our main steps in this analysis are Theorem 1.2, Formula (74) and Theorem 1.6. One of the main consequence of our work is the next theorem.

Theorem 1.1. *There exists $c > 0$ such that for every N large enough and $Q \in [cN/\sqrt{\log N}, 20N]$, we have*

$$\sum_{1<q/Q\leqslant 2}\ \sum_{a \bmod^* q}|S(\varphi,a/q)|^2 \geqslant Q^2 e^{-cN/Q}\sum_m|\varphi_m|^2.$$

This is to be compared with the lower bound given by W. Duke & H. Iwaniec in [12]. Note that the summation therein extends over all classes a modulo q rather than over the reduced classes, see the remark following [35, Theorem 2.7] on this issue. In particular, the principal character is included (i.e. $q = 1$) with a definite influence. J.-C. Schlage-Puchta in [40] gives, for some random sequences, a lower bound of a large sieve quantity under the sole assumption that Q^2/N goes to infinity. Read also the papers of P. Erdös & A. Renyi [17] and of D. Wolke [47].

The proof of Theorem 1.1 will unfold in four steps:

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- By appealing to the δ-symbol technique, we relate the above sum to a sum of similar kind but where the moduli h are much smaller, namely $h \leqslant H$ for some H of size roughly N/Q.
- We then interpret, for each h, the intervening quantity as a scalar product of some function $\mathcal{R}_{N,h}(\varphi)$ together with the value of a difference operator applied at this same vector.
- After analyzing the one-parameter family of compact operators that intervene, we use their eigenvalues to derive a spectral decomposition of the large sieve quantity we are interested in.
- When W is non-negative and N/Q is small enough, we prove that these eigenvalues are < 1 by using the harmonic analysis uncertainty principle. Theorem 1.1 is a consequence of that.

Setting the horizon for a lower bound

Question. *Do we have* $\sum_{1<q/Q\leqslant 2}\sum_{a \bmod^* q}|S(\varphi,a/q)|^2 \gg N\sum_m|\varphi_m|^2$ *when* $Q \geqslant N^{1/2+\varepsilon}$ *for some positive* ε*?*

When $N = \sum_{q\leqslant Q}\phi(q)$, we gave in [35, Theorem 1.2] the (rather weak) lower bound $\|\varphi\|_2^2\exp(\frac{-1+o(1)}{2}N\log N)$ for the quantity $\sum_{q\leqslant Q}\sum_{a \bmod^* q}|S(\varphi,a/q)|^2$. Theorem 1.3 implies that the better lower bound $Q^2\|\varphi\|_2^2$ holds true as soon as φ oscillates enough along small arithmetic progressions in intervals of length about Q. The main result of [9, Theorem 2.4] by B. Conrey, H. Iwaniec and K. Soundararajan implies a similar lower bound for functions φ that are the convolution product of an oscillating factor supported on $[1, Q^{1-\varepsilon}]$ and a rather general sequence.

Some functional transforms of our weight function

The δ-symbol technique involves some functional transforms of our weight function W that we better treat before starting the analysis proper. Assumptions W being as above, we define $W^\star$ in (26), but the following expression valid for $z \in \mathbb{R}$ is better:

$$W^\star(z) = -2\sum_{n\geqslant 1}\frac{\phi(n)}{n}\int_0^\infty \cos(2\pi ny)W(z/y)dy/y.$$

By Lemma 5.9, the function $W^\star$ is even, twice differentiable outside $z = 0$ where it vanishes, and is of bounded variations over $[0, 1]$ and decreases like $1/z^{\frac{7}{2}-\varepsilon}$ at infinity. The expression for its Mellin transform, valid when $\Re s \in [0, 3/2)$ is simply $\check{W}^\star(s) = \check{W}(s)\zeta(1-s)/\zeta(1+s)$, see Lemma 5.6, where $\check{W}(s)$ is the Mellin transform of W. We finally mention the following expression for its Fourier transform, valid for $u \neq 0$ and obtained in Lemma 5.7:

$$\hat{W}^\star(u) = \frac{6}{\pi^2}\int_0^\infty W(t)dt - \frac{1}{|u|}\sum_{n\geqslant 1}\frac{\phi(n)}{n}W(n/|u|). \tag{1}$$

This Fourier transform satisfies $\hat{W}^\star(u) = \frac{6}{\pi^2}\int_0^\infty W(t)dt$ when $|u| \leqslant 1/2$ and $|u\hat{W}^\star(u)| \ll \exp -c_0\sqrt{\log|u|}$ otherwise, for some positive constant c_0, ensuring that $\hat{W}^\star(u)$ belongs to L^1. It is worth specifying that $\hat{W}^\star(u)$ varies in sign when W is non-negative[1].

A smoothed setup

Our analysis revolves around the quantity

$$\sum_{q\geqslant 1}\frac{W(q/Q)}{q}\sum_{a \bmod^* q}|S(\varphi,a/q)|^2 \tag{2}$$

for some weight function W satisfying:

(W_1) • The function W is C^3 over $]-\infty,\infty[$ and C^4 per pieces.

(W_2) • It is even and its support lies inside $[-2,-1]\cup[1,2]$.

(W_3) • We have $\int_0^\infty W(u)du \neq 0$.

We do not need W to be non-negative, though nothing is made to avoid this natural condition. We do not seek generality but on the reverse to restrict ourselves to as smooth a situation as necessary.

We define

$$I_0(W) = \sum_q \frac{\phi(q)W(q/Q)}{qQ} = \frac{6}{\pi^2}\int_0^\infty W(u)du + \mathcal{O}((\log Q)/Q). \tag{3}$$

The quantity $I_0(W)$ depends on Q, but in a very mild manner.

First step: an equality via δ-symbol

The proof of Theorem 1.1 will unfold in four steps. We start our journey with the following essential formula that is of independent interest.

Theorem 1.2. *When $1/2 \leqslant H \leqslant \sqrt{N}/(\log N)^5$ and $\log Q \ll \log N$, we have*

$$\sum_q \frac{W(q/Q)}{qQ}\sum_{a \bmod^* q}|S(\varphi,a/q)|^2 = \bigl(I_0(W)+\mathcal{O}(N(QH)^{-1})\bigr)\sum_m|\varphi_m|^2$$
$$-\sum_{h\leqslant H}\frac{1}{h}\sum_{a \bmod^* h}\int_{-\infty}^{\infty}\hat{W}^\star(u)\Bigl|S\Bigl(\varphi,\frac{a}{h}+\frac{u}{hQ}\Bigr)\Bigr|^2du.$$

The reader will find a refined version for primes in Theorem 10.2. Please note that the factor $N(QH)^{-1}$ is not polluted by any power of $\log N$ and that $\hat{W}^\star(u)$ belongs to L^1. The proof shows clearly that a polarized version is accessible of the same strength, namely:

$$\sum_q \frac{W(q/Q)}{qQ}\sum_{a \bmod^* q}S(\varphi,a/q)\overline{S(\psi,a/q)} = I_0(W)\sum_m \varphi_m\overline{\psi_m}$$
$$-\sum_{h\leqslant H}\frac{1}{hQ}\sum_{a \bmod^* h}\int_{-\infty}^{\infty}\hat{W}^\star(u)S\Bigl(\varphi,\frac{a}{h}+\frac{u}{hQ}\Bigr)\overline{S\Bigl(\psi,\frac{a}{h}+\frac{u}{hQ}\Bigr)}du$$
$$+\mathcal{O}\bigl(N(QH)^{-1}\bigr)\|\varphi\|_2\|\psi\|_2\bigr)$$

[1] Such a sign-change may be detected by using (1) for $u\in[1/2,1]$. The positivity of $\hat{W}^\star(u)$ implies that $\frac{6}{v\pi^2}\int_1^2 W(u)du \geqslant W(v)$ when $v\in[1,2]$, leading to a contradiction.

where $\|\varphi\|_2 = \sqrt{\sum_m |\varphi_m|}$ and similarly for $\|\psi\|_2$. Similar polarized versions are true for Theorems 1.3, 1.6 and Corollary 1.4. The beginning of our proof follows closely the one of B. Conrey & H. Iwaniec [8] (which has been for the most part incorporated in [9] by B. Conrey, H. Iwaniec & K. Soundararajan) and can be considered as an additive analogue of their result. Our main new ingredient at this stage, with respect to this proof, is the use of a maximal large sieve inequality. To introduce this part, we got inspired from another try at a large sieve equality due to W. Duke & H. Iwaniec and contained in [12]. The treatment of the finite parts (meaning: for $h \leqslant H$) diverges from [8], and in particular we show that what may appear like two main terms in the first coarse formula we get in fact cancels out in their leading contribution. This part of the treatment is similar to what happens for the δ-symbol of W. Duke, J. Friedlander & H. Iwaniec in [11] (see also [25, Section 20.5] by H. Iwaniec & E. Kowalski. A more precise version of this remark is documented Section 8.1).

Since $\hat{W}^\star(u)$ has its main contribution around $u = 0$, the sum over h contributes to the main term only when the sequence (φ_n) accumulates in some arithmetic progression of modulus $\leqslant H$. When it does *not*, we have the following result that implies a conditional large sieve equality.

Theorem 1.3. *When* $\frac{1}{2} \leqslant H \leqslant \sqrt{N}/(\log N)^5$ *and* $\log Q \ll \log N$*, we have*

$$\sum_q \frac{W(q/Q)}{qQ} \sum_{a \bmod^* q} |S(\varphi, a/q)|^2 = \big(I_0(W) + \mathcal{O}(N(QH)^{-1})\big) \sum_m |\varphi_m|^2 + \mathcal{O}\Bigg(\sum_{h \leqslant H} \frac{N + hQ}{hQ^2} \max_{u<v<u+2hQ} \sum_{c \bmod h} \bigg| \sum_{\substack{u<n\leqslant v,\\ n \equiv c[h]}} \varphi_n \bigg|^2\Bigg).$$

Recall that the size condition $u, v \leqslant N$ is included in the condition on the support of φ. See Theorem 9.1 for a sharper remainder term. See also the work [19] of J. Friedlander & H. Iwaniec, as well as [35, Theorem 2.6] for a large sieve equality for coefficients of a special form (convolution of a shortly supported sequence with a smooth sequence). The case $H = 1/2$ has also an interesting methodological consequence.

Corollary 1.4. *When* $\log Q \ll \log N$*, we have*

$$\sum_q \frac{W(q/Q)}{qQ} \sum_{a \bmod^* q} |S(\varphi, a/q)|^2 = (1 + \mathcal{O}(N/Q)) I_0(W) \sum_m |\varphi_m|^2.$$

Second step: Functional rephrasing

Corollary 1.4 describes the situation satisfactorily when $\tau = N/Q$ goes to zero. When τ is larger, we show that the situation is controlled by a family of embeddings $(\mathcal{R}_{N,h})_h$ of $L^2(\{1 \cdots N\})$ and a family of self-adjoined nuclear operators $\mathscr{V}_{\tau,h}$ on the subspace $L^2_*(X_h)$ of $L^2(X_h)$: we endow $X_h = \mathbb{Z}/h\mathbb{Z} \times [0,1]$ with the natural probability measure; the space $L^2_*(X_h)$ is the one of functions from $L^2(X_h)$ whose Fourier transform with respect to the first variable is supported by $(\mathbb{Z}/h\mathbb{Z})^* \times [0,1]$, see Section 11 for more details. We denote by $U_{\tilde{h}\to h}$ the orthonormal projection on this subspace.

Let us define the local embedding $\mathcal{R}_{N,h}$. We start by defining the (nearly) unitary (see Lemma 13.1) embedding $\Gamma_{N,h}$ of $L^2(\{1\cdots N\})$ in $L^2(X_h)$ by:

$$\begin{aligned}\Gamma_{N,h} : L^2(\{1\cdots N\}) &\to L^2(X_h)\\ \varphi = (\varphi_n)_{1\leqslant n\leqslant N} &\mapsto \Gamma_{N,h}(\varphi) : \mathbb{Z}/h\mathbb{Z}\times[0,1] \to \mathbb{C}\\ &\qquad (b,y)\mapsto \varphi_{\sigma_h(b)+h[N'y/h]}\end{aligned} \tag{4}$$

where $\sigma_h(z)$ is the unique integer b in $\{1\cdots h\}$ that is congruent to z modulo h; we have set $\varphi_n = 0$ when the index n is (strictly) larger than N and

$$N' = N + \sqrt{N}. \tag{5}$$

The embedding we need is given by

$$\mathcal{R}_{N,h} = U_{\tilde{h}\to h}\circ\Gamma_{N,h}. \tag{6}$$

This is to be compared with the case of integers where we send $\mathbb{Z}$ inside $\mathbb{Z}_p$ for every prime p, though we have here an "infinite place" for each modulus h (this is the factor $[0,1]$) and that we may not rely on multiplicativity. It would be interesting to show that the diagonal embedding $\varphi\mapsto(\mathcal{R}_{N,h}(\varphi))_h$ has a dense range, as in the adelic case. The situation is somewhat more intricate because of the dependence in N. We next define the one-parameter family of operators $\mathscr{V}_{\tau,h}$ by

$$\mathscr{V}_{\tau,h}(G)(b,y) = \int_0^1 G(b,y')W^\star\Big(\frac{\tau(y-y')}{h}\Big)dy'. \tag{7}$$

They are shown to be compact symmetric nuclear operators in Theorem 12.4 and to verify a Mercer like theorem (see Theorem 12.5). The fundamental formula is (74) which we repeat here:

$$\begin{aligned}\sum_q \frac{W(q/Q)}{qQ}\sum_{a \bmod^* q}|S(\varphi,a/q)|^2 &= I_0(W)\|\varphi\|_2^2(1+\mathcal{O}(\tau/H))\\ &\quad - N\sum_{h\leqslant H}\frac{\tau}{h}\big[\mathcal{R}_{N,h}(\varphi)|\mathscr{V}_{\tau,h}\,\mathcal{R}_{N,h}(\varphi)\big]_{h\times[0,1]}\\ &\qquad (H \ll N^{1/8}(\log N)^{-3/2}, \tau = N/Q \ll H, Q \ll N^2).\end{aligned} \tag{74}$$

Allowing H to be as large as a power of N requires quite some efforts and we have to rely on te moe technical formula (46) rather than on the simplified form given in Theorem 1.2. Ideally, we should be able to allow H roughly as large as $\sqrt{N}$.

Analysis of a class of difference operators

We treat in Section 12 the analysis of the intervening family of operators in an abstracted setting. For a function V satisfying the regularity assumptions (R_1), (R_2) and (R_3), we define

$$\mathscr{V}_0 :\quad G\in L^2([0,1]) \mapsto \Big(y\mapsto \int_0^1 G(y')V(y-y')dy'\Big) \tag{8}$$

Assumptions (R_1), (R_2) and (R_3) indeed hold when $V(y) = W^\star(\tau y/h)$. It is classical theory that $\mathscr{V}_0$ is a compact Hilbert-Schmidt operator, see for instance [22, Theorem 7.7]. Let $(\lambda_\ell, G_\ell)_\ell$ be a complete orthonormal system of eigenvalues / eigenfunctions, ordered with non-increasing $|\lambda_\ell|$. The Fredholm equation $\lambda G(y') = \int_0^1 K(y', y)G(y)dy$ has been intensively studied. It is not the purpose of this paper to introduce to this theory, a task for which it is better to read the complete and classical [21], or the more modern [22]. Kernel of type $V(y' - y)$ are often called *difference kernel*, and lead to operators that are distinct from convolution operators as the integration and definition interval is *not* the whole real line. The book [39] is dedicated to the operators built from such kernels. The book [7] contains also many useful informations.

Here is a summary of what we prove in Section 12.

Theorem 1.5. *The operator $\mathscr{V}_0$ is nuclear. Given a complete collection $(\lambda_\ell, G_\ell)_\ell$ of non-zero eigenvalues / eigenvectors, arranged with non-increasing $|\lambda_\ell|$ and normalized by $\int_0^1 |G_\ell(t)|^2 dt = 1$, we have the three following properties:*

- *(Explicit nuclearity)* $\sum_{\ell \geqslant 1} |\lambda_\ell| \ll \|V\|_2 e^{-c'\sqrt{1+\|V\|_\infty \|V'\|_1/\|V\|_2^2}}$ *for some positive constant c' depending only on A, B and c. The notation $\|V'\|_1$ stands for the total variation.*
- *(Mercer like property)* $V(y' - y) = \sum_{\ell \geqslant 1} \lambda_\ell G_\ell(y') G_\ell(y)$ *uniformly.*
- *(Lidskii's Theorem)* $\sum_{\ell \geqslant 1} \lambda_\ell = 0$.

This is proved in Theorem 12.4 and 12.5. These properties shows that this class of operators is indeed very regular. We recall that the Mercer Theorem concerns similar operators but having a *non-negative* reproducing kernel. On integrating the case $y = y'$ of the Mercer like property, we recover the third property.

Third step: Spectral decomposition of the large sieve

Theorem 1.6. *Assume that $\sqrt{N} \leqslant Q \leqslant N$. There exist two positive constants c_0 and c_3 such that the following holds. For each $\tau = N/Q$ and integer $h \geqslant 1$, let $(G_{\ell,\tau/h}, \lambda_\ell(\tau/h))_\ell$ be a complete family of two by two orthonormal eigenfunctions of (7) coupled with their respective non-zero eigenvalues. These eigenfunctions are all continuous and of bounded variations. The sequence $(\lambda_{h,\ell}(\tau))_{\ell \geqslant 1}$ is arranged in non-increasing absolute value, and satisfies $\lambda_\ell(\tau/h) \ll 1/\sqrt{\ell}$ uniformly in h and τ. We also have*

$$\sum_{\ell \geqslant 1} \lambda_\ell(\tau/h) = 0, \ \sum_{\ell \geqslant 1} |\lambda_\ell(\tau/h)| < \infty, \ \sum_{\ell \geqslant 1} |\lambda_\ell(\tau/h)|^2 = 2\int_0^1 W^\star\Big(\frac{\tau y}{h}\Big)^2 (1 - y) dy \tag{9}$$

and this last value is bounded uniformly in τ. Under the Riemann Hypothesis, we also have $\sum_{\ell \geqslant 1} |\lambda_\ell(\tau/h)|^p < \infty$ for any $p > 4/5$. For any sequence of complex *numbers φ, any $L \geqslant 1$, any $H \ll N^{1/8}(\log N)^{-3/2}$ and any $\xi \in [0,1]$, we have*

$$\sum_q \frac{W(q/Q)}{qQ} \sum_{a \bmod^* q} |S(\varphi, a/q)|^2 = I_0(W)\|\varphi\|_2^2 - \frac{1}{N} \sum_{\substack{h \leqslant H \\ |\lambda_\ell(\tau/h)| \geqslant \xi\eta_0(N)}} \sum_{\ell \leqslant L} \frac{\tau}{h}\lambda_\ell(\tau/h) \sum_{a \bmod^* h} \left| \sum_{n \leqslant N} \varphi_n G_{\ell,\tau/h}\Bigl(\frac{n}{N}\Bigr) e\Bigl(\frac{na}{h}\Bigr) \right|^2 + \mathcal{O}\biggl(\Bigl(\frac{\log H}{L} + \frac{1}{H} + \xi\eta_0(N)\Bigr)\tau\|\varphi\|_2^2\biggr)$$

where $\eta_0(N) = \exp -c_3\sqrt{\log N}$. We have furthermore

$$\sum_{\substack{h \leqslant H \\ |\lambda_\ell(\tau/h)| \geqslant \eta_0(N)}} \sum_{\ell \leqslant L} \sum_{a \bmod^* h} \left| \sum_{n \leqslant N} \varphi_n G_{\ell,\tau/h}\Bigl(\frac{n}{N}\Bigr) e\Bigl(\frac{na}{h}\Bigr) \right|^2 \leqslant N\|\varphi\|_2^2\bigl(1 + H^2 L\eta_0(N)\bigr).$$

When W is non-negative, the one-sided inequality $(\tau/h)\lambda_{h,\ell} \leqslant I_0(W) + o(1)$ holds true, where $o(1)$ is here a function of Q that goes to 0 with $1/Q$.

We prove that infinitely many $\lambda_\ell(\tau/h)$ are positive (resp. negative), once h is also allowed to vary; see end of Subsection 15.3. When W is further assumed to be non-negative, Theorem 12.6 shows that $(\tau/h)\lambda_\ell(\tau/h) \leqslant I_0(W) + o(1)$. The parameter ξ above has only been introduced for flexibility purpose, in case one needs a lower bound that is independent on N.

Fourth step: Uncertainty principle and eigenvalues properties

A closer study of the eigenvalues that uses F.I. Nazarov's version [31] of the uncertainty principle combined with some positivity argument leads to the following.

Theorem 1.7. *For any non-negative W satisfying the above conditions there exist $c_4, c_6, c_7 > 0$ such that we have, for any $H \leqslant \exp(c_6\sqrt{\log N})$ and any $Q \in [N\exp(-c_6\sqrt{\log N}), N^2]$,*

$$(\tau/I_0(W))|\lambda_{h,\ell}|/h \leqslant 1 - c_6 e^{-c_4\tau/h} + \mathcal{O}(\exp -c_7\sqrt{\log N})$$

for any $h \leqslant H$, any $\ell \geqslant 1$ and with $\tau = N/Q$.

P. Jaming tells me that he believes $c_4 = 120$ to be an admissible choice.

Arithmetical consequences

Corollary 1.8. *For every $\epsilon > 0$, and every $N \geqslant 1$ and $Q \geqslant 1$, there exist a constant c_4 and a subspace of dimension $\mathcal{O}(\tau^2/[\epsilon^2\log(1/\epsilon)])$ such that we have, for any (φ_n) orthogonal to this subspace and when $\log Q > c_4\log^2(N/Q)$,*

$$(1-\epsilon)\sum_m |\varphi_m|^2 \leqslant \sum_q \frac{W(q/Q)}{qQI_0(W)} \sum_{a \bmod^* q} |S(\varphi, a/q)|^2 \leqslant (1+\epsilon)\sum_m |\varphi_m|^2.$$

Moreover, when $\tau \asymp 1$ and for every integer $K \geqslant 1$, there exist $\epsilon_0 > 0$ depending only on τ and K, and $2K$ unitary sequences $(\alpha_k)_{k\leqslant K}$ and $(\beta_k)_{k\leqslant K}$, two by two almost orthogonal in the sense that

$$\forall \gamma, \gamma' \in \{\alpha_k\} \cup \{\beta_k\}, \quad [\gamma, \gamma']_N = \delta_{\gamma=\gamma'} + \mathcal{O}\big(\exp\big(-c_4\sqrt{\log N}\big)\big),$$

and such that, on one side, we have

$$\sum_q \frac{W(q/Q)}{qQI_0(W)} \sum_{a \bmod^* q} |S(\alpha_k, a/q)|^2 > (1+\epsilon_0)\sum_m |\alpha_{k,m}|^2$$

while on the other side, we have

$$\sum_q \frac{W(q/Q)}{qQI_0(W)} \sum_{a \bmod^* q} |S(\beta_k, a/q)|^2 < (1-\epsilon_0)\sum_m |\beta_{k,m}|^2.$$

The orthogonality is according to the hermitian product defined by

$$[\varphi, \psi]_N = \frac{1}{N}\sum_{1\leqslant n\leqslant N} \varphi_n \overline{\psi_n}. \tag{10}$$

The sequences (α_k) and (β_k) are pull-backs of eigenvectors. Note that the pulling-back process depends on N but that the eigenvectors do not. They are very regular and do not result from some exotic construction; in particular they are uniformly bounded and there exists $\epsilon > 0$ such that $\{n \leqslant N, |\alpha_{k,n}| \geqslant \epsilon\}$ is a set of density (in short: their "essential support" is a set of density).

Notation

We note the Mellin transform by $\check{W}(s) = \int_0^\infty W(t)t^{s-1}dt$ and the Fourier transform by $\hat{W}(u) = \int_{-\infty}^\infty W(t)e(-ut)dt$. Several other transforms of W will be used, $W^\sharp$, W, $\tilde{W}$, $W^\star$ and $W^{\star\star}$; they are described in section 5. We note here that the transform $W^\sharp$ is very close to what appears in [25, section 20.5, (20.145)] provided the changes of notation is incorporated: our $W(y)$ is their $w(y/C)$. We recall that $\|\sigma'_\infty(\psi,\cdot)\|_{1,N} = \int_1^N |\sigma'_\infty(\psi,t)|dt$. We denote by $a_{|t} = (a_n)_{n\leqslant t}$ the truncated sequence. We also define

$$\mathfrak{L}(u) = \exp\sqrt{\log(2+u)}.$$

We denote the Euler totient function by ϕ and distinguish it from the sequence by using a different script for the latter, namely φ. We use the following norms:

$$\|f\|_{1,N} = \int_1^N |f(t)|dt, \quad \|f\|_{\infty,N} = \max_{1\leqslant t\leqslant N} |f(t)|. \tag{11}$$

2 Related works

Influence of the Riemann Hypothesis

Under the Riemann Hypothesis (and not the Generalized one as one may believe), the proof we present allows to select Q as small as $N/(\log N)^{1-\varepsilon}$ for any positive ε. The coefficient $e^{-cN/Q}$ may be questioned and may well be superfluous in this range.

Eigenvalues considerations when $Q \ll \sqrt{N}$

The eigenvalues of the quadratic form $\sum_{q\leqslant Q}\sum_{a \bmod^* q}|S(\varphi,a/q)|^2$ are well understood when $Q = o(\sqrt{N})$, see the paper of I. Kobayashi [28] and this quantity is expected to behave like a Riemann sum when $N = o(Q)$ (Corollary 1.4 below gives a precise form to this statement), but the behavior in the range $Q \in [\sqrt{N}, cN]$ (for any positive constant c) is still mysterious. When $Q \sim \sqrt{N}$, F. Boca and M. Radziwiłł have shown in [3] by a very delicate analysis that the distribution of the eigenvalues of this quadratic form tend to a limiting distribution, henceforth proving a conjecture made in [34]. In fact, though this went unnoticed by the authors, the paper [6] of T.H. Chan & A.V. Kumchev can be read as also providing some informations on the eigenvalues in the case $Q \sim \sqrt{N}$. The values for the even moments of this limit distribution reveals that it is *not* a classical distribution, confirming what the (rather limited) computations from [36].

Eigenvalues considerations when $Q \geqslant N$

H. Niederreiter evaluated in [33] the discrepancy of the Farey sequence, a study refined by F. Dress in [10], and this, together with the Koksma-Hlawka's inequality, proves immediately that

$$\sum_{q\leqslant Q}\sum_{a \bmod^* q}|S(\varphi,a/q)|^2 = \sum_{q\leqslant Q}\phi(q)\sum_n|\varphi_n|^2(1+\mathcal{O}(N/Q))$$

in very much the same way P. Gallagher in [20] derived the large sieve inequality. Note that the arithmeticity of the Farey sequence is only mildly used: a discrepancy estimate is enough.

Part I
A large sieve equality

3 Large sieve ingredients

We adapt here the proof of S. Uchiyama [45] concerning the maximal large sieve to get a result which is a (weak) additive analogue of a result of P.D.T.A. Elliott [14]. This is [13, Lemma 1] or [15, Chapter 29, exercise 3, page 254].

Lemma 3.1. *Let $(x_d)_{d\leqslant D}$ be a δ-spaced sequence of points of $\mathbb{R}/\mathbb{Z}$. We have*

$$\sum_{d\leqslant D}\max_{u<v\leqslant u+L}\Big|\sum_{u<n\leqslant v}\varphi_m e(mx_d)\Big|^2 \leqslant (L+2\delta^{-1}\log(e/\delta))\sum_m|\varphi_m|^2.$$

Here is the version we shall use.

Lemma 3.2. *We have*

$$\sum_{q\leqslant Q}\sum_{a\ mod^* q}\max_{u<v\leqslant u+L}\Big|\sum_{u<n\leqslant v}\varphi_m e(ma/q)\Big|^2 \ll (L+Q^2\log Q)\sum_m|\varphi_m|^2.$$

4 A functional transform

The transform we investigate here is given by

$$W^\star(z) = \frac{-1}{2i\pi}\int_{-i\infty}^{i\infty} \check{W}(s)\frac{\zeta(1-s)}{\zeta(1+s)}z^{-s}ds. \tag{12}$$

Please note that $|\zeta(1-s)/\zeta(1+s)| = 1$ on the line $\Re s = 0$. This transform of $\check{W}$ is already the one the occurs in [34], see for instance equation numbered (48) there, and in [6], see their equation (4.19). We keep the same hypothesis as before for W. In particular, it is compactly supported and $\check{W}(s) \ll (1+|s|)^{-4}$. We follow [34, Section 9] pretty closely. We start by recalling a handy form of the complex Stirling formula.

Lemma 4.1 (Uniform complex Stirling formula)**.** *Let $\varepsilon \in]0,1]$ and a compact subset $\mathcal{A}$ of $\mathbb{C}$ be fixed. In the domain $|\arg z| \leqslant \pi - \varepsilon$ and $|z| \geqslant 1$, we have*

$$\Gamma(z+a) = \sqrt{2\pi}e^{-z}z^{z+a-1/2}\big(1+\mathcal{O}(1/|z|)\big).$$

uniformly for $a \in \mathcal{A}$.

As a (classical) conclusion and taking $z = it$ in the above, we find that

$$|\cos(\sigma+it)\Gamma(\sigma+it)| = \sqrt{\pi/2}|t|^{\sigma-1/2}\big(1+\mathcal{O}(1/|t|)\big) \tag{13}$$

uniformly in any domain $\sigma_1 \leqslant \sigma \leqslant \sigma_2$ and $|t| \geqslant 1$.

Isolating the arithmetical behavior

We proceed as in [34] and appeal to the functional equation of the Riemann ζ-function (see [44] or [25]) which may be written as

$$\zeta(1-s) = 2^{1-s}\pi^{-s}\cos(\pi s/2)\Gamma(s)\zeta(s). \tag{14}$$

To do so we first shift the line of integration in (12) to $\Re s = 9/8$. Since $|\zeta(-\sigma + it)| \ll_\varepsilon (1+|t|)^{(1+\sigma)/2+\varepsilon}$ when $\sigma \geqslant 0$ and for any $\varepsilon > 0$, it is enough to assume that $\check{W}(s) \ll (1+|s|)^{-2}$ to ensure the convergence of our integrals. Since the line shifting does not meet any pole, we get

$$\begin{aligned} W^\star(z) &= \frac{-1}{i\pi}\int_{\frac{9}{8}-i\infty}^{\frac{9}{8}+i\infty} \check{W}(s)\frac{\cos(\pi s/2)\Gamma(s)\zeta(s)}{\zeta(1+s)}(2\pi z)^{-s}ds, \\ &= 2\sum_{n\geqslant 1}\frac{\phi(n)}{n}\mathscr{F}(W)(2\pi nz) \end{aligned} \tag{15}$$

where

$$\mathscr{F}(W)(u) = \frac{-1}{2i\pi}\int_{\frac{9}{8}-i\infty}^{\frac{9}{8}+i\infty} \check{W}(s)\cos(\pi s/2)\Gamma(s)u^{-s}ds. \tag{16}$$

A bound at infinity

We infer from the estimate (13) that the line of integration in (16) can be pushed up to $\Re s = 7/2 - \varepsilon$ and thus

$$\mathscr{F}(W)(2\pi nz) \ll_\varepsilon (nz)^{-7/2+\varepsilon}. \tag{17}$$

Here is the main conclusion of this part.

Lemma 4.2. *We have $W^\star(z) \ll_\varepsilon z^{-7/2+\varepsilon}$, for any $\varepsilon > 0$.*

A real-valued formula

The next step is to proceed as in section 9 of [34], which we only sketch here. We employ equation (35) therein:

$$\cos\frac{\pi s}{2}\Gamma(s) = \int_0^\infty \cos(y)y^{s-1}dy = \int_0^\infty \cos(y)y^s dy/y \tag{18}$$

valid for $0 < \Re s < 1$ to infer that

$$\begin{aligned}\mathscr{F}(W)(u) &= -\frac{1}{2i\pi}\int_{\frac12-i\infty}^{\frac12+i\infty}\check{W}(s)\cos(\pi s/2)\Gamma(s)u^{-s}ds\\ &= -\int_0^\infty \cos(y)\frac{1}{2i\pi}\int_{\frac12-i\infty}^{\frac12+i\infty}\check{W}(s)(u/y)^{-s}\frac{dsdy}{y}\\ &= -\int_0^\infty \cos(y)W(u/y)dy/y\end{aligned}$$

by Mellin inversion formula. This yilds formula (1).

5 More auxiliary functional transforms

Several functional transforms of our bump-function W will occur. We have already seen $W^\star$ and $\hat{W}^\star$ at (1) and (1). These two functions are central in our work, but it is expedient to introduce several others. We start with the couple

$$W^\sharp(y) = \sum_{k\geqslant 1}\frac{W(y/k)}{k}, \quad W^\flat(y) = \sum_{f\geqslant 1}\frac{W(yf)}{f}. \tag{19}$$

We show in Lemma 5.1 that $W^\flat(y) = J(W) + \mathcal{O}(y)$ where

$$J(W) = \int_0^\infty \frac{W(u)du}{u}. \tag{20}$$

When y is small as in our case of application, the approximation of $W^\flat$ by $J(W)$ is efficient. The proof will then lead us to understand $W^\sharp - J(W)$, a quantity we call $-\tilde{W}$, i.e.

$$W^\sharp(y) = J(W) - \tilde{W}(y). \tag{21}$$

The situation is there more difficult than with $W^\flat$, in particular because $W^\flat(y)$ is *not* small when y is small but takes the constant value $J(W)$! See Lemma 5.2. As it turns out, we do not need to grasp $\tilde{W}$ but the average

$$W_C^\star(z) = \sum_{1\leqslant c\leqslant C}\frac{\mu(c)}{c}\tilde{W}(cz). \tag{22}$$

The value for small z, i.e. when $|z|\leqslant 1$, is now $J(W)\sum_{1\leqslant c\leqslant C}\mu(c)/c$ which tends to 0 when C is large. The rate of convergence is fast enough on the Riemann Hypothesis, but rather slow otherwise. As a consequence, we have to treat this point with care. In particular, we want to replace C by ∞ and still save a power of C. We have already defined $W^\star$ at (12) and Lemma 5.3 will show that both definitions coincide. Let us start our journey.

5.1 Approximating $W^\flat$

The transform $W^\flat$ is also studied in [25, section 20.5]: the function $V(z)$ defined there in (20.143) corresponds to $J(W) - W^\flat(z)$ where one should change $W(y)$ into $w(y/C)$ (albeit the trivial facts that w is supported on $[C, 2C]$, while our W is supported on $[1, 2]$ and extended to the negative real axis by evenness).

Lemma 5.1. *Assume that $|\hat{W}(u)| \ll 1/(1+|u|)^2$. We have, when $z > 0$,*

$$W^\flat(z) = \sum_{f \geqslant 1} \frac{W(zf)}{f} = J(W) + \mathcal{O}(z). \tag{23}$$

with $J(W)$ being defined at (20).

In practice, z is small ($\leqslant CE/Q$). The proof we present uses the Fourier transform but one could also use the Mellin transform.

Proof. We introduce Fourier transforms to write

$$\begin{aligned} W^\flat(z) &= \int_{-\infty}^{\infty} \hat{W}(u) \sum_{f \geqslant 1} \frac{e(fuz)}{f} du = -\int_{-\infty}^{\infty} \hat{W}(u) \log(1 - e(zu)) du \\ &= -\int_{-\infty}^{\infty} \hat{W}(u) \big(\log|2\sin(\pi z u)| + i\pi(\{zu\} - \tfrac{1}{2})\big) du \\ &= -\int_{-\infty}^{\infty} \hat{W}(u) \big(\log|2\sin(\pi z u)| + i\pi B_1(zu)\big) du. \end{aligned}$$

For the sake of the evaluation next to $z = 0$, it is better to adopt the expression

$$W^\flat(z) = -\int_{-\infty}^{\infty} \hat{W}(u) \big(\log \frac{|2\sin(\pi z u)|}{\pi z |u|} + \log(\pi z |u|) + i\pi(\{zu\} - \tfrac{1}{2})\big) du.$$

which we may simplify, with $\int_{-\infty}^{\infty} \hat{W}(u) du = W(0) = 0$, into

$$W^\flat(z) = -2\int_0^{\infty} \hat{W}(u) \log|u| \, du - \int_{-\infty}^{\infty} \hat{W}(u) \Big(\log \frac{|2\sin(\pi z u)|}{\pi z |u|} + i\pi\{zu\}\Big) du.$$

We split the integral according to whether $|u| \leqslant 1/z$ or not. In both cases we use $|\hat{W}(u)| \ll 1/(1+|u|)^2$ and bound $\log \frac{|\sin(\pi z u)|}{\pi z u}$ by $\mathcal{O}(zu)$ when $|u| \leqslant 1/z$ and by $\log(|zu| + 1)$ otherwise.

We proceed by getting a simpler form for $-2\int_0^\infty \hat{W}(u) \log|u| \, du$. We readily check that

$$\begin{aligned} \int_0^L \hat{W}(u) \log|u| \, du &= 2\int_0^L \int_1^2 W(t) \cos(2\pi u t) dt \log|u| \, du \\ &= 2\int_1^2 W(t) \bigg(\Big[\frac{\sin(2\pi u t)}{2\pi t} \log|u| \Big]_0^L - \frac{1}{2\pi t} \int_0^L \frac{\sin(2\pi u t)}{u} du \bigg) \\ &\to \frac{-1}{2} \int_1^2 \frac{W(t) dt}{t} \end{aligned}$$

therefore concluding the proof of our lemma. ∎

5.2 From $W^\sharp$ to $\tilde{W}$

In this part, we start from the definition of $\tilde{W}$ provided by (21) and we reach the definition (25) given below. With $v > 0$ fixed, we define

$$f(t) = W\big(v/t\big)/t. \tag{24}$$

We simply write when $v > 0$

$$\begin{aligned}
\sum_{g \geqslant 1} f(g) &= -\sum_{g\geqslant 1}\int_g^\infty f'(t)dt = -\int_0^\infty [t]f'(t)dt \\
&= \int_0^\infty f(t)dt + \int_0^\infty \{t\}f'(t)dt \\
&= \int_0^\infty W(u)\frac{du}{u} - \int_0^\infty \{t\}\bigg(\frac{vW'(v/t}{t^3} + \frac{W(v/t)}{t^2}\bigg)dt \\
&= J(W) - \frac{1}{v}\int_0^\infty \{v/u\}(W'(u)u + W(u))du.
\end{aligned}$$

This establishes Eq. (25). The condition $v > 0$ has been used on the last line: when $v < 0$, we should reverse the integration path, or divide by $|v|$ instead of by v.

5.3 Treatment of $\tilde{W}$

Define

$$\tilde{W}(z) = \frac{1}{|z|}\int_0^\infty \{z/u\}(uW'(u) + W(u))du. \tag{25}$$

The expression $\tilde{W}(z) = \int_0^\infty \{1/v\}(vzW'(vz) + W(vz))dv$ shows that $\tilde{W}$ is an even function[2].

Lemma 5.2. *The function $\tilde{W}$ is C^1 and C^2 per pieces, and both derivatives are bounded.*

When $|z| \leqslant 1$, we have $\tilde{W}(z) = J(W)$.

When $|z| \geqslant 1$, we have $\tilde{W}(z) \ll 1/z^2$.

Proof. Eq. (25) shows that the first part of the Lemma, by distinguishing whether $|z| > 1$ or not.

When $z \in [0,1)$, then $z/u \in [0,1)$ when u lies in the support of W, which implies that $\{z/u\} = z/u$ in this case. Hence the first equality. We can furthermore write, when $z \neq 0$, and with $t = z/u$, and with $B_2^*(t) = \int_0^t B_1(v)dv$:

$$\begin{aligned}
\tilde{W}(z) &= \int_0^\infty (\{t\} - \tfrac{1}{2})(zt^{-1}W'(z/t) + W(z/t))dt/t^2 \\
&= \int_0^\infty B_2^*(t)(4zt^{-2}W'(z/t) + 2W(z/t)t^{-1} + z^2t^{-3}W''(z/t))dt/t^2 \\
&= z^{-2}\int_0^\infty B_2^*(z/u)(4u^2W'(u) + 2uW(u) + u^3W''(u))du
\end{aligned}$$

from which the bound claimed in the lemma follows readily. □

[2]Still reading [25, Section 20.5] by H. Iwaniec & E. Kowalski, we find that our $\tilde{W}$ satisfies $\tilde{W}(z) = (C/|z|)\int_0^\infty \{vz/C\}(W(C/v)/v)'dv$, and is thus like their $W(C/z)$.

5.4 Study of $W_C^\star$ and $W^\star$

The function $W_C^\star(z)$ is even since so is $\tilde{W}$. Lemma 5.2 tells us that this function is constant when $|z| \leqslant 1/C$, with value $J(W)\sum_{1\leqslant c\leqslant C}\mu(c)/c$. We can even select $C=\infty$ in which case we write simply $W^\star$:

$$W_\infty^\star(z) = W^\star(z) = \sum_{c\geqslant 1}\frac{\mu(c)}{c}\tilde{W}(cz). \tag{26}$$

The next expression of $W^\star$ will in particular establish that $W^\star$ is continuous at $z=0$ where we have $W^\star(0)=0$.

Lemma 5.3. *We assume that W is at least C^2. We have, when $\varepsilon>0$ and $z>0$,*

$$W_C^\star(z) = J(W)\sum_{c\leqslant C}\frac{\mu(c)}{c} - \frac{1}{2i\pi}\int_{\varepsilon-i\infty}^{\varepsilon+i\infty}\check{W}(s)\zeta(1-s)\sum_{c\leqslant C}\frac{\mu(c)}{c^{1+s}}z^{-s}ds$$

where $\check{W}(s)=\int_0^\infty W(x)x^{s-1}dx$ is the Mellin transform of W. When $C=\infty$, the expression above is correct provided we select $\varepsilon=0$ and replace $\sum_{c\geqslant 1}\mu(c)/c^{1+s}$ by $1/\zeta(1+s)$.

Proof. We first reduce the case $C=\infty$ to the case C finite. On using $\{z\} = z-[z]$, we get

$$\begin{aligned} W^\star(z) &= \sum_{c\geqslant 1}\frac{\mu(c)}{c^2 z}\int_0^\infty \{zc/u\}(uW'(u)+W(u))du \\ &= \lim_{C'\to\infty}\Bigg(\sum_{c\leqslant C'}\frac{\mu(c)}{cz}\int_0^\infty\frac{uW'(u)+W(u)}{u}du \\ &\qquad - \sum_{c\leqslant C'}\frac{\mu(c)}{c^2 z}\int_0^\infty\sum_{d\leqslant zc/u}1\,(uW)'(u)du\Bigg) \\ &= -\lim_{C'\to\infty}\sum_{c\leqslant C'}\frac{\mu(c)}{c}\sum_{d\geqslant 1}\frac{W(zc/d)}{d}. \end{aligned}$$

We introduce the Mellin transform of W and write

$$\begin{aligned} \sum_{d\geqslant 1}\frac{W(zc/d)}{d} &= \frac{1}{2i\pi}\int_{-1-i\infty}^{-1+i\infty}\check{W}(s)\zeta(1-s)(zc)^{-s}ds \\ &= \check{W}(0) + \frac{1}{2i\pi}\int_{\varepsilon-i\infty}^{\varepsilon+i\infty}\check{W}(s)\zeta(1-s)(zc)^{-s}ds \end{aligned}$$

which gives us (note that $J(W)=\check{W}(0)$)

$$W_C^\star(z) = J(W)\sum_{c\leqslant C}\frac{\mu(c)}{c} - \frac{1}{2i\pi}\int_{\varepsilon-i\infty}^{\varepsilon+i\infty}\check{W}(s)\zeta(1-s)\sum_{c\leqslant C}\frac{\mu(c)}{c^{1+s}}z^{-s}ds$$

hence the expression given, seeing that the pole of $\zeta(1-s)$ cancels out with the zero of $1/\zeta(1+s)$ at $s=0$ and that $\check{W}(s)$ is $\mathcal{O}(1/(1+|s|)^2)$. $\square$

Lemma 5.4. *For $\Re s \in (-1, 0)$, we have*

$$\int_0^1 z^s \cos(2\pi z)dz - s\int_1^\infty \frac{z^{s-1}\sin(2\pi z)dz}{2\pi} = (2\pi)^{-s-1}\Gamma(s+1)\cos\frac{\pi(s+1)}{2}.$$

Proof. We call the left-hand side $j(s)$. It is not difficult to see that (this is how is occurs below)

$$j(s) = \int_0^\infty z^s \cos(2\pi z)dz$$

and is thus the Mellin transform of $\cos(2\pi z)$. On looking at [16, (21), page 319], we readily discover that, when $\Re s \in (-1, 0)$ (note the shift or $+1$ between the s variable $j(s)$ and the one of the table we refer to), the above formula follows. Giving a full proof is not difficult by using $\cos w = (e^{iw} + e^{-iw})/2$. □

We define, when $C < \infty$,

$$\begin{aligned} W_C^{\star\star}(u) = W_C^\star(u) - W_C^\star(0) &= W_C^\star(u) - J(W)\sum_{c\leqslant C}\frac{\mu(c)}{c} \qquad (27)\\ &= -\sum_{c\leqslant C}\frac{\mu(c)}{c}W^\sharp(cu) \end{aligned}$$

on recalling (21) and (22). Note also that $W_\infty^{\star\star} = W_\infty^\star = W^\star$ by (26). We recall that $W^\sharp$ is defined at (19).

Lemma 5.5. *When $\check{W}(s) \ll 1/(1+|s|)^3$, we have, when $u > 0$,*

$$\hat{W}_C^{\star\star}(u) = \frac{-1}{2i\pi}\int_{-i\infty}^{i\infty}\frac{\check{W}(s)\zeta(s)}{u^{1-s}}\sum_{c\leqslant C}\frac{\mu(c)}{c^{1+s}}ds.$$

When $C = \infty$, we replace $\sum_{c\geqslant 1}\mu(c)/c^{1+s}$ by $1/\zeta(1+s)$. As a consequence, when $C < \infty$ and for any real number $k < 3/2$, we have $\hat{W}_C^{\star\star}(u) \ll (1 + |u|)^{-1}(1 + |u|/C)^{-k}$. Moreover, in the sense of distribution, we have $\hat{W}_C^\star(u) = J(W)\sum_{c\leqslant C}\mu(c)/c\cdot\delta_{u=0} + \hat{W}_C^{\star\star}(u)$ where $\delta_{u=0}$ is the Dirac mass at $u = 0$.

Proof. The value $\hat{W}_C^{\star\star}(u)$ is the limit, as Z goes to infinity, of

$$2\int_0^Z W_C^{\star\star}(z)\cos(2\pi uz)dz.$$

We employ Lemma 5.3 and reach the expression

$$\frac{-1}{i\pi}\int_{-\varepsilon-i\infty}^{-\varepsilon+i\infty}\check{W}(-s)\zeta(1+s)\sum_{c\leqslant C}\frac{\mu(c)}{c^{1-s}}\int_0^Z z^s\cos(2\pi zu)dz\,ds$$

which is also

$$\frac{-1}{i\pi}\int_{-\varepsilon-i\infty}^{-\varepsilon+i\infty}\frac{\check{W}(-s)\zeta(1+s)}{u^{1+s}}\sum_{c\leqslant C}\frac{\mu(c)}{c^{1-s}}\int_0^{uZ} z^s\cos(2\pi z)dz\,ds.$$

When $C = \infty$, we start with $\varepsilon = 0$ and shift the line of integration in s just to the left-hand side of $\Re s = 0$ but still within the zero-free region of $\zeta(1-s)$. Concerning the inner integral, we write

$$\int_0^{uZ} z^s \cos(2\pi z)dz = \int_0^1 z^s \cos(2\pi z)dz \\ + (uZ)^s \frac{\sin 2\pi uZ}{2\pi} - \frac{s}{2\pi}\int_1^{uZ} z^{s-1}\sin(2\pi z)dz.$$

It is then enough to use the Lebesgue dominated convergence Theorem to send Z to infinity (when $u > 0$). We next appeal to Lemma 5.4 to get that

$$\int_0^{\infty} z^s \cos(2\pi z)dz = (2\pi)^{-s-1}\Gamma(s+1)\cos(\pi(s+1)/2) \\ = -(2\pi)^{-1-s}\Gamma(1+s)\sin(\pi s/2) = \frac{1}{2}\frac{\zeta(-s)}{\zeta(1+s)}$$

by using the functional equation of the Riemann zeta-function. This gives us

$$\hat{W}_C^{\star\star}(u) = \frac{-1}{2i\pi}\int_{-i\infty}^{i\infty} \frac{\check{W}(-s)\zeta(-s)}{u^{1+s}} \sum_{c\leqslant C}\frac{\mu(c)}{c^{1-s}}ds \tag{28}$$

The bound on $\hat{W}_C^{\star\star}(u)$ comes by separating the cases $|u| \leqslant C$ and $|u| > C$ and in the latter case in shifting the line of integration to $\Re s = k$ and using $|\zeta(-s)| \ll_\varepsilon (1+|s|)^{k+1/2+\varepsilon}$ (for any positive ε) there. □

Let us mention the following consequence of Lemma 5.3 together with Mellin inversion formula.

Lemma 5.6. *The hypothesis on W being as above, we have*

$$\widetilde{W_C^{\star\star}}(s) = -\check{W}(s)\zeta(1-s)\sum_{c\leqslant C}\mu(c)/c^{1+s}.$$

for $\Re s \in (0, 3/2)$.

Lemma 5.7. *When $u > 0$ and for $C \leqslant \infty$, we have*

$$\hat{W}_C^{\star\star}(u) = \sum_{c\leqslant C}\frac{\mu^2(c)}{c^2}\hat{W}(0) - \frac{1}{u}\sum_{n\geqslant 1}\frac{\phi_C(n)}{n}W(n/u)$$

where $\phi_C(n)/n = \sum_{d|n, d\leqslant C}\mu(d)/d$. In particular, this gives

$$\hat{W}^{\star\star}(u) = \begin{cases} \frac{6}{\pi^2}\hat{W}(0) & \text{when } |u| \leqslant 1/2, \\ \frac{6}{\pi^2}\hat{W}(0) - W(1/u)/u & \text{when } 1/2 < u \leqslant 2/2. \end{cases}$$

Proof. We only treat the case $C = \infty$. Lemma 5.5 gives us

$$\hat{W}^{\star\star}(u) = \frac{-1}{2i\pi}\int_{-i\infty}^{i\infty}\frac{\check{W}(s)\zeta(s)}{u^{1-s}}\sum_{c\leqslant C}\frac{\mu(c)}{c^{1+s}}ds.$$

We shift the line of integration to $\Re s = 2$ (since we move to the right, the contribution of the pole at $s = 1$ is multiplied with a coefficient -1), use the development $\zeta(s)/\zeta(s+1) = \sum_{n\geqslant 1}\phi(n)/n^{1+s}$ and the reverse Mellin transform to get

$$\hat{W}^{\star\star}(u) = \frac{6}{\pi^2}\hat{W}(0) - \frac{1}{u}\sum_{n\geqslant 1}\frac{\phi(n)}{n}W(n/u)$$

as expected. □

Lemma 5.8. *We have* $\hat{W}^{\star\star}(u) - \hat{W}^{\star\star}_C(u) \ll \log(|u|+2)/C$.

Proof. Indeed, by Lemma 5.7, we have $\hat{W}^{\star}(u) - \hat{W}^{\star\star}_C(u) \ll 1/C$ when $|u| < C/2$. When u is larger, we use

$$\frac{\phi(n)}{n} - \frac{\phi_C(n)}{n} = \sum_{\substack{d|n,\\ d>C}}\frac{\mu(d)}{d} \ll 2^{\omega(n)}/C$$

where $\omega(n)$ is the number of prime factors of n. This implies that

$$W^{\star\star}(u) - \hat{W}^{\star\star}_C(u) \ll C^{-1} + \frac{1}{u}\sum_{u\ll n\ll u} 2^{\omega(n)}/C \ll \log(|u|+2)/C$$

as required. □

The size of $W^\star$ and $\hat{W}^\star$ is well controlled as shown in the next lemma.

Lemma 5.9. *Assume W is at least C^3. We have $W^{\star\star}_C(z) - J(W)\sum_{c\leqslant C}\mu(c)/c \ll 1/(1+z^2)$. There exists $c_0 > 0$ (depending on W only) such that, when $z \geqslant 0$ and $\delta \in (0, 1/2]$, we have $|W^\star(z+\delta) - W^\star(z)| \ll \exp -c_0\sqrt{-\log\delta}$ and, when $z \in (0,1]$, $W^{\star\prime}(z) \ll \mathfrak{L}(1/z)^{c_0}/z$. This shows in particular that W^* is of bounded variations on $[0,1]$. Under the Riemann Hypothesis, we have $|W^\star(z)| \ll_\varepsilon |z|^{\frac{1}{2}-\varepsilon}$ for any positive ε.*

When $z \leqslant 1/C$, we have $W^{\star\star}_C(z) = 0$.

When W is four times differentiable, we have $|\hat{W}^\star(u)| \ll u^{-1}\mathfrak{L}(u)^{-c_0}$. Moreover $\hat{W}^\star(0) = \frac{6}{\pi^2}\int_0^\infty W(u)du$.

Proof. We split the proof is several stages.
Bounding $W^{\star\star}_C$: When $|z| \geqslant 1$, the first bound is a direct consequence of Lemma 5.2. When $|z| \leqslant 1$, we write

$$W^{\star\star}_C(z) = \sum_{\substack{c\leqslant 1/|z|,\\ c\leqslant C}}\frac{\mu(c)}{c}J(W) + \sum_{\substack{c>1/|z|,\\ c\leqslant C}}\frac{\mu(c)}{c}W(cz) = o(1) + \mathcal{O}(1)$$

as required.
Bounding the modulus of continuity of $W^\star$: Appealing to Lemma 5.3 with the change of variable $s \mapsto -s$, we next write

$$W^\star(z+\delta) - W^\star(z) = \frac{1}{2i\pi}\int_{-i\infty}^{i\infty}\check{W}(-s)\frac{\zeta(1+s)}{\zeta(1-s)}sz^s\int_0^{\delta/z}(1+t)^{s-1}dtds.$$

Recalling that $\check{W}(-s) \ll 1/(1+|s|)^3$ and $\zeta(1+it)/\zeta(1-it) \ll (\log(2+|t|)^2$, this immediately gives us the bound $|W^\star(z+\delta) - W^\star(z)| \ll \delta/z$. This proves what we need (and more!) when $z \geqslant \sqrt{\delta}$. When z is smaller, we proceed as in the proof of the Prime Number Theorem: when $t = \Im s \in [-T, T]$, we shift the line of integration to $\Re s = \sigma = c_1/\log T$ where $c_1 > 0$ is chosen so that $\zeta(\sigma - it)^{\pm 1} \ll \log T$ when $|t| \leqslant T$. The usual prime number theory gives us such a result, see e.g. [44]. Skipping some classical steps, we reach the bound

$$\begin{aligned} W^\star(z+\delta) - W^\star(z) &\ll \frac{\delta}{z} z^{\frac{c_1}{\log T}} + \frac{(\log T)^2}{T} \\ &\ll z^{\frac{c_1}{\log T}} + \frac{(\log T)^2}{T} \ll \delta^{\frac{c_1}{2\log T}} + \frac{(\log T)^2}{T}. \end{aligned}$$

We select $T = \exp(\sqrt{\log(1/\delta)}$. The reader will easily conclude from there. This is were the hypothesis W C^3 is needed. The bound for $W^{\star\prime}$ is obtained in the same manner.

Some more upper bounds: By Lemma 5.2, we have $\tilde{W}(z) = J(W)$ when $|z| \leqslant 1$, hence $W_C^{\star\star}(z) = 0$ when $|z| \leqslant 1/C$.

The bound for the Fourier transform follows by summation by parts. Concerning the value of the Fourier transform at 0, let Z be a large parameter that goes to infinity. We write

$$\begin{aligned} 2\int_0^Z \tilde{W}(z)dz &= 2\int_0^1 J(W)dz + 2\int_1^Z\int_0^\infty \frac{B_1(z/u)}{z}(uW'(u) + W(u))dudz \\ &= 2J(W) + 2\int_0^\infty \int_{1/u}^{Z/u} \frac{B_1(z)dz}{z}(uW'(u) + W(u))du \\ &= 2J(W) - 2\int_0^\infty \left(\frac{B_1(Z/u)}{Z/u}\left(\frac{-Z}{u^2}\right) - \frac{B_1(1/u)}{1/u}\left(\frac{-1}{u^2}\right)\right)uW(u)du \\ &= 2J(W) + 2\int_0^\infty B_1(Z/u)W(u)du - 2\int_0^\infty B_1(1/u)W(u)du \\ &= 2\int_0^\infty B_1(Z/u)W(u)du + \int_0^\infty W(u)du \end{aligned}$$

and the integral depending on Z goes to 0 as Z goes to infinity by Lebesgue's Lemma. This shows that $\hat{\tilde{W}}(0) = (1/2)\hat{W}(0)$. We next employ (22) to deduce that

$$\hat{W}_C^*(u) = \sum_{c \leqslant C} \frac{\mu(c)}{c^2}\hat{\tilde{W}}(u/c)$$

hence the value at $u = 0$, whether $C < \infty$ or not. □

6 Numerical aspects related to the smoothing kernel and its transforms

It is interesting to produce some numerical datas, so as to explore our several transforms.

6.1 An explicit family of smoothing kernels

Let $1_{[-1,1]}$ be the characteristic function of the interval $[-1, 1]$. We are interested in explicit formulae for the m-th convolution-power $1_{[-1,1]}^{(*m)}$, where m is a positive integer. This function is even with support within $[-m, m]$, and of class C^{m-1}. We readily check that

$$1_{[-1,1]}^{(*2)}(t) = \begin{cases} 2-|t| & \text{when } |t| \leqslant 2, \\ 0 & \text{when } 2 \leqslant |t|. \end{cases} \tag{29}$$

Some more sweat brings the next formula:

$$1_{[-1,1]}^{(*3)}(t) = \begin{cases} 3-t^2 & \text{when } |t| \leqslant 1, \\ (3-|t|)^2/2 & \text{when } 1 \leqslant |t| \leqslant 3, \\ 0 & \text{when } 3 \leqslant |t|. \end{cases}$$

The general formula is given in [38] and reads

$$1_{[-1,1]}^{(*m)}(t) = \begin{cases} \displaystyle\sum_{j=0}^{\lfloor (m+|t|)/2 \rfloor} \frac{(-1)^j}{(m-1)!}\binom{m}{j}(m+|t|-2j)^{m-1} & \text{when } 0 \leqslant |t| \leqslant m, \\ 0 & \text{when } m < |t|. \end{cases}$$

Guessing this expression is not obvious, but verifying it by recursion is only a matter of routine. The Fourier transform of $1_{[-1,1]}$ is $\sin(2\pi u)/(\pi u)$, so the one of $1_{[-1,1]}^{(*m)}$ is $\sin(2\pi u)^m/(\pi u)^m$. Since we will use the case $m = 5$, it is worth giving its explicit expression:

$$1_{[-1,1]}^{(*5)}(t) = \begin{cases} \frac{115-30t^2+3t^4}{12} & \text{when } |t| \leqslant 1, \\ \frac{55+10|t|-30t^2+10|t|^3-t^4}{6} & \text{when } 1 \leqslant |t| \leqslant 3, \\ \frac{625-500t+150t^2-20|t|^3+t^4}{24} & \text{when } 3 \leqslant |t| \leqslant 5 \\ 0 & \text{when } 5 \leqslant |t|. \end{cases} \tag{30}$$

Formula (1) is handy for explicit computations. We introduce

$$\mathfrak{p}_m(t) = \frac{4m}{2^m} 1_{[-1,1]}^{*m}(4mt - 3m)$$

for some integer $m \geqslant 5$. Its support lies inside $[1/2, 1]$. We find that

$$\hat{\mathfrak{p}}_m(u) = e(3u/4)\left(\frac{\sin(\pi u/(2m))}{\pi u/(2m)}\right)^m.$$

Notice that $\int_0^\infty \mathfrak{p}_m(t)dt = \hat{\mathfrak{p}}_m(0) = 1$. We then select

$$W(m;t) = \mathfrak{p}_m(1/t)/t.$$

For such a choice, we readily get

$$W^\star(m;z) = 2\sum_{n \geqslant 1} \frac{\phi(n)}{n}\cos(3\pi n z/2)\left(\frac{\sin(\pi n z/(2m))}{\pi n z/(2m)}\right)^m.$$

When we truncate this series at the integer N, the error is bounded above by

$$2\left(\frac{2m}{\pi z}\right)^m \frac{1}{(m-1)N^{m-1}}. \tag{31}$$

We then use the following Sage script (see [43]):

```
def Witself(t, m = 5):
   if abs(t) > 2 or abs(t) < 1:
      return(0)
   res = 0
   z = m*(4/t-3)
   coef = 2*m/factorial(m-1)/2^m
   asign = 1
   for j in range(0, floor(float((m + abs(z))/2)) + 1):
      res += asign*binomial(m, j)*(m + abs(z) -2*j)^(m-1)
      asign = -asign
   return(res*coef/t)

plot(lambda t:Witself(t, 5), (1, 2))
```

6.2 A specific kernel

In this section, we specify $m = 5$.

On $W(5;t)$:

Here is a plot of our function.

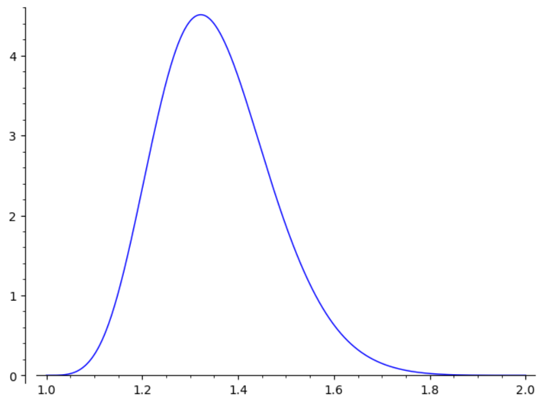


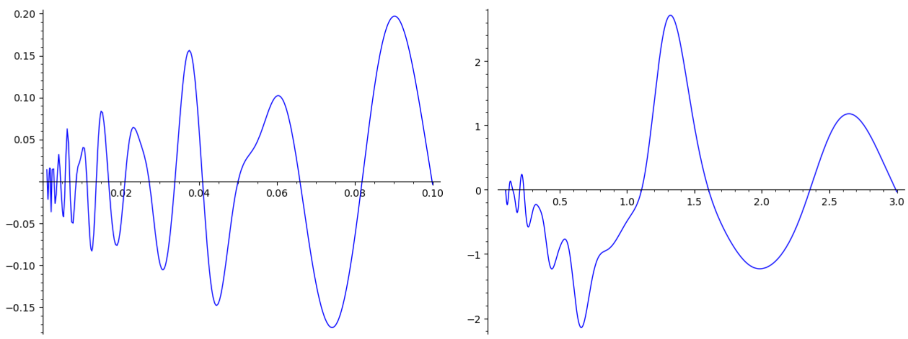
Figure 1: $W(5;t)$

The command `integral_numerical(lambda t:Witself(t,5), (1,2))` gives us

$$\frac{6}{\pi^2}\int_0^\infty W(5;t)dt = 0.816\cdots$$

On $W^\star(5;t)$:

We get the following plot on $[0.0001, 3]$:



(a) (b)

Figure 2: $W^{\star}(5;z)$ for $0.0001 \leqslant z \leqslant 0.1$ and for $0.1 \leqslant z \leqslant 3$

And here is a plot of $\hat{W}^{\star}(5;t)$. It is worth noticing that $\hat{W}^{\star}(5;1) = \hat{W}^{\star}(5;0)$.

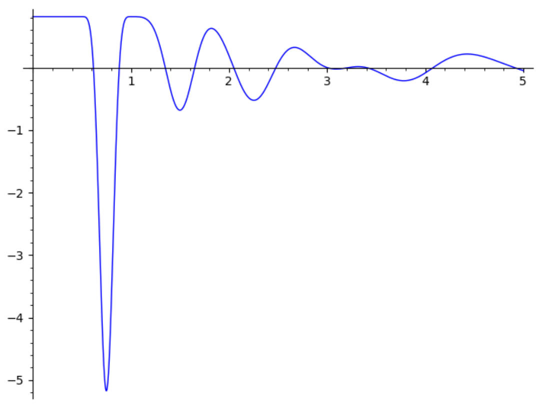


Figure 3: $\hat{W}^{\star}(5;t)$

After $u = 1$, we indeed find that $\hat{W}^{\star}(5;u) < \hat{W}^{\star}(5;0)$.

7 A general formula, first step in the proof of Theorem 1.2

In analytic number theory, when we want to detect an equality, the quantity we really study is of the shape $\sum_{m,n} \varphi_m \overline{\psi_m} \delta_{m=n}$ and that what we use in an approximation of the δ-symbol. This is not only a tautology, it also imposes a framework which decides of what are the "trivial" estimates and of what can be expected or not. It also splits the problem in two parts: a combinatorial part, where one uses the fact m and n are integers, possibly in certain subsequences, and an analytical part where the quantities arising are to be estimated. There is of course an interplay between both parts and a "good" decomposition is a decomposition that leads to quantities that we know how to estimate. It is difficult to give a precise historical date, but the contributions of M. Jutila in [26] (see also [23] and [27, Theorem 2]) and of H. Iwaniec in [11] (see also [12,] and [25, Chapter 20], in particular Proposition 20.16 therein) seem to be prominent. One can say rapidly that in some sense, Iwaniec's way is to analyze the large sieve quantity to extract a diagonal contribution, under some hypotheses, while Jutila's way is to start from the diagonal contribution and to modify the circle to keep only the rationals one knows how to handle, with a possible weight.

The present study is centered on the quantity

$$\mathscr{S}(Q,W)=\sum_q \frac{W(q/Q)}{q}\sum_{a \bmod^* q}|S(\varphi,a/q)|^2. \tag{32}$$

Moebius inversion readily yields

$$\mathscr{S}(Q,W)=\sum_d\sum_{d|q}\frac{\mu(q/d)W(q/Q)}{q}\sum_{a \bmod d}|S(\varphi,a/d)|^2.$$

We expand the square, shuffle the terms around and get

$$\mathscr{S}(Q,W)=\sum_{m,n}\varphi_m\overline{\varphi_n}\Delta(m-n) \tag{33}$$

where we have use the notation (on setting $cd=q$)

$$\Delta(v)=\sum_{\substack{c,d,\\ d|v}}\frac{\mu(c)W(cd/Q)}{c}. \tag{34}$$

Here is the decomposition of the Δ-symbol we use.

Lemma 7.1 (Iwaniec's decomposition)**.** *Let $C,E,H\geqslant 1$ be parameters that satisfy $E\leqslant \min(1Q,2Q/C)$. We have*

$$\Delta(v)=U(v)+U^\sharp(v)+L_0(v)+L(v)+L^\sharp(v)$$

where $L_0(v)$ is the diagonal contribution

$$L_0(v)=\sum_{\substack{c\leqslant C,\\ d\geqslant 1}}\frac{\mu(c)W(cd/Q)}{c}1_{v=0},$$

and $U(v)$ and $U^\sharp(v)$ are the "direct divisor" part:

$$\begin{cases} U(v)=-\displaystyle\sum_{e\leqslant E}\sum_{\substack{c\leqslant C,\\ f\geqslant 1}}\frac{\mu(c)W(cef/Q)}{cef}\sum_{a \bmod^* e}e(av/e),\\ U^\sharp(v)=\displaystyle\sum_{e>E}\sum_{\substack{c>C,\\ f\geqslant 1}}\frac{\mu(c)W(cef/Q)}{cef}\sum_{a \bmod^* e}e(av/e),\end{cases}$$

while $L(v)$ and $L^\sharp(v)$ are the "complementary divisor" part:

$$\begin{cases} L(v)=\displaystyle\sum_{h\leqslant H}\sum_{\substack{h|g,\\ c\leqslant C}}\frac{\mu(c)}{gc}\sum_{a \bmod^* h}W(cv/(gQ))e(av/h),\\ L^\sharp(v)=\displaystyle\sum_{h>H}\sum_{\substack{h|g,\\ c\leqslant C}}\frac{\mu(c)}{gc}\sum_{a \bmod^* h}W(cv/(gQ))e(av/h).\end{cases}$$

Proof. We start by splitting the range for the variable c:

$$\begin{aligned}\Delta(v) &= \sum_{\substack{c\leqslant C,\\ d|v}} \frac{\mu(c)W(cd/Q)}{c} + \sum_{\substack{c>C,\\ d|v}} \frac{\mu(c)W(cd/Q)}{c}\\ &= L(v) + U(v)\end{aligned}$$

say. When $v = 0$, the term $L(v)$ restricts to $L_0(v)$. Otherwise, we switch to the complementary divisor by setting $gd = |v|$ (and $g \geqslant 1$ since $v \neq 0$). We detect the divisibility condition by using additive characters:

$$\begin{aligned}L(v) &= \sum_{\substack{c\leqslant C,\\ g|v}} \frac{\mu(c)W(c|v|/(gQ))}{c}\\ &= \sum_{\substack{c\leqslant C,\\ g\geqslant 1}} \frac{1}{g} \sum_{b \bmod g} \frac{\mu(c)W(c|v|/(gQ))}{c} e(bv/g)\\ &= \sum_{\substack{c\leqslant C,\\ g\geqslant 1}} \frac{1}{g} \sum_{h|g} \sum_{b \bmod^* h} \frac{\mu(c)W(c|v|/(gQ))}{c} e(bv/h)\end{aligned}$$

which amounts to

$$L(v) = \sum_{h\geqslant 1} \sum_{\substack{c\leqslant C,\\ h|g}} \frac{\mu(c)}{gc} \sum_{b \bmod^* h} W(cv/(gQ))e(bv/h).$$

Note that we do not need the condition $v \neq 0$ since $W(cv/(gQ)) = 0$ when $v = 0$. We then simply split the summation over h according to whether $h \leqslant H$ or not, getting the two quantities $L(v)$ and $L^\sharp(v)$.

Concerning $U(v)$ we again detect the divisibility condition by using additive characters. This gives us

$$U(v) = \sum_{\substack{c>C,\\ d\geqslant 1}} \frac{\mu(c)W(cd/Q)}{cd} \sum_{e|d} \sum_{a \bmod^* e} e(av/e).$$

Note that $cd/Q \leqslant 2$. We set $d = ef$ and thus $e \leqslant 2Q/C$. We continue by splitting the range for e:

$$\begin{aligned}U(v) = \sum_{e\leqslant E} \sum_{\substack{c>C,\\ f\geqslant 1}} \frac{\mu(c)W(cef/Q)}{cef} \sum_{a \bmod^* e} e(av/e)\\ + \sum_{e>E} \sum_{\substack{c>C,\\ f\geqslant 1}} \frac{\mu(c)W(cef/Q)}{cef} \sum_{a \bmod^* e} e(av/e).\end{aligned}$$

We recognize $U^\sharp(v)$ in the last quantity. The first one needs a transformation.

We note that

$$\begin{aligned}\sum_{\substack{c>C,\\ f\geqslant 1}}\frac{\mu(c)W(cef/Q)}{cef} &= \sum_{\substack{c\geqslant 1,\\ f\geqslant 1}}\frac{\mu(c)W(cef/Q)}{cef}-\sum_{\substack{c\leqslant C,\\ f\geqslant 1}}\frac{\mu(c)W(cef/Q)}{cef}\\ &= \sum_{j\geqslant 1}\sum_{cf=j}\frac{\mu(c)W(je/Q)}{je}-\sum_{\substack{c\leqslant C,\\ f\geqslant 1}}\frac{\mu(c)W(cef/Q)}{cef}\\ &= \frac{W(e/Q)}{e}-\sum_{\substack{c\leqslant C,\\ f\geqslant 1}}\frac{\mu(c)W(cef/Q)}{cef}\end{aligned}$$

and the first term vanishes because of the assumption $E\leqslant Q$. □

The diagonal term is easily handled.

Lemma 7.2.

$$\sum_{m,n}\varphi_m\overline{\varphi_n}L_0(m-n)=\left(\sum_q\frac{\phi(q)W(q/Q)}{q}+\mathcal{O}(QC^{-1})\right)\|\varphi\|_2^2.$$

Proof. The contribution is

$$\sum_{c\leqslant C,d}\frac{\mu(c)W(cd/Q)}{c}\|\varphi\|_2^2=\sum_q\sum_{c|q,c\leqslant C}\frac{\mu(c)}{c}W(q/Q)\|\varphi\|_2^2.$$

Since

$$\sum_{c>C,d}\frac{\mu(c)W(cd/Q)}{c}\ll\sum_{c>C}\frac{Q}{c^2}\ll Q/C$$

we get that this diagonal term has value:

$$\left(\sum_q\frac{\phi(q)W(q/Q)}{q}+\mathcal{O}(QC^{-1})\right)\|\varphi\|_2^2$$

as announced. □

The large sieve inequality yields an efficient bound for the contribution of $U^\sharp(m-n)$.

Lemma 7.3. *We have*

$$\sum_{m,n}\varphi_m\overline{\varphi_n}U^\sharp(m-n)\ll\sum_m|\varphi_m|^2(NE^{-1}+QC^{-1})\log Q.$$

Proof. We use the bound (where c and e are fixed)

$$\sum_f\frac{W(cef/Q)}{f}\ll\sum_{Q/(ce)\leqslant f\leqslant 3Q/(ce)}\frac{1}{Q/(ce)}\ll 1$$

to get:

$$\begin{aligned}\sum_{m,n}\varphi_m\overline{\varphi_n}U^\sharp(m-n) &\ll(\log Q)\sum_{E<e\leqslant 3Q/C}e^{-1}\sum_{a\bmod^* e}|S(\varphi,a/e)|^2\\ &\ll\sum_m|\varphi_m|^2(NE^{-1}+QC^{-1})\log Q.\end{aligned}$$

□

The contribution of $L^\sharp(m-n)$ is somewhat more difficult to handle but also relies on the large sieve inequality. We shall most of the time employ the next lemma with a set I reduces to one element. It is only in the final applications that it is better to use the summation over some $i \in I$.

Lemma 7.4. *Let w be an even and C^1 function that vanishes when the variable is larger than 1. We further assume that w is piecewise C^2. Let I be a finite set. We have*

$$\sum_{i\in I}\Big|\sum_{m,n}\psi_{m,i}\overline{\psi_{n,i}}w(\alpha(m-n))\Big| \leqslant 3\|w''\|_1(N\alpha+1)\max_{u<v\leqslant u+3/\alpha}\sum_{i\in I}\Big|\sum_{u<m\leqslant v}\psi_{m,i}\Big|^2.$$

Proof. The problem is twofold: localizing the variables m and n and separating these two variables. The first problem is met by a subdivision argument: we cover the interval $[1,N]$ by at most $N\alpha+1$ disjoint intervals $(a, a+\alpha^{-1}] = I_a$ of length α^{-1} and localize n within such an interval. As a result we can assume that m lies in $[a-\alpha^{-1}, a+3\alpha^{-1}] = J_a$. We handle the separation of variables by a summation by parts and the formula

$$\begin{aligned} w(\alpha(n-m)) &= -\alpha\int_{m-\alpha^{-1}}^{n} w'(\alpha(t-m))dt \\ &= \alpha^2\int_{m-\alpha^{-1}}^{n}\int_{m}^{t+\alpha^{-1}} w''(\alpha(t-s))dsdt \end{aligned}$$

from which we infer that $\sum_{n\in I_a, m\in J_a}\psi_{m,i}\overline{\psi_{n,i}}w(\alpha(m-n))$ equals

$$\alpha^2\int_{a-2\alpha^{-1}}^{a+\alpha}\int_{a-\alpha^{-1}}^{t+\alpha^{-1}}\sum_{\substack{s\leqslant m\leqslant t-\alpha^{-1},\\ t\leqslant n\leqslant a+\alpha^{-1}}}\psi_{m,i}\overline{\psi_{n,i}}w''(\alpha(t-s))dsdt. \tag{35}$$

We find that $t-3\alpha^{-1} \leqslant a-\alpha^{-1} \leqslant s \leqslant m \leqslant t-\alpha^{-1}$ and that $t \leqslant n \leqslant a+\alpha^{-1} \leqslant t+3\alpha^{-1}$, hence the inner sum over m and n is bounded above (after introducing the summation over i, by

$$\max_{u<v\leqslant u+3/\alpha}\sum_{i\in I}\Big|\sum_{u<m\leqslant v}\psi_{m,i}\Big|^2.$$

A change of variables readily shows that

$$\alpha^2\int_{a-2\alpha^{-1}}^{a+\alpha}\int_{a-\alpha^{-1}}^{t+\alpha^{-1}}|w''(\alpha(t-s))|dsdt \leqslant 3\|w''\|_1,$$

clearing out any uniformity problem in applications. □

Lemma 7.5. *We have*

$$\sum_{m,n}\varphi_m\overline{\varphi_n}L^\sharp(m-n) \ll \big(NH^{-1}+NCQ^{-1}\big)\|\varphi\|_2^2\log^4(QN).$$

Proof. We have to control

$$\sum_{m,n}\psi_m\overline{\psi_n}W(\alpha(m-n)) \tag{36}$$

where $\psi_m = \varphi_m e(ma/h)$ and $\alpha = c/(gQ) \neq 0$. Note that the truncation in c ensures that $|\alpha|$ is small; this truncation has been introduced for this very purpose. Practically, we appeal to Lemma 7.4 and get

$$S = \sum_{c\leqslant C}\frac{\mu(c)}{c}\sum_{h>H}\sum_{h|g}g^{-1}\sum_{a \bmod^* h}\sum_{m,n}\varphi_m e(ma/h)\overline{\varphi_n e(na/h)}W(c|m-n|/(gQ))$$
$$\ll \sum_{\substack{c\leqslant C, h>H,\\ h|g\leqslant cN/Q}}\frac{1}{gc}\Big(\frac{Nc}{gQ}+1\Big)\sum_{a \bmod^* h}\max_{u<v\leqslant u+9gQ/c}\Big|\sum_{u<m\leqslant v}\varphi_m e(am/h)\Big|^2$$

The condition $v > u$ is automatically satisfied. We continue with c fixed by localizing h and using $k = g/h$. Lemma 3.2 gives us:

$$S \ll \sum_{\substack{1\leqslant k\leqslant Q/H,\\ \log H\leqslant \ell\leqslant \log\frac{cN}{Q}}}\sum_{e^{\ell-1}<h\leqslant e^{\ell}}\Big(\frac{N/Q}{k^2e^{2\ell}}+\frac{1}{ke^{\ell}c}\Big)\sum_{a \bmod^* h}\max_{v\leqslant u+\frac{9ke^{\ell}Q}{c}}\Big|\sum_{u<m\leqslant v}\varphi_m e\Big(\frac{am}{h}\Big)\Big|^2$$
$$\ll \sum_{1\leqslant k\leqslant Q/H}\sum_{\log H\leqslant \ell\leqslant \log(cN/Q)}\Big(\frac{N/Q}{k^2e^{2\ell}}+\frac{1}{ke^{\ell}c}\Big)\Big(\min(N, ke^{\ell}Qc^{-1})+\ell e^{2\ell}\Big)\|\varphi\|_2^2$$
$$\ll \sum_{1\leqslant k\leqslant Q/H}\sum_{\log H\leqslant \ell\leqslant \log(cN/Q)}\Big(\frac{N}{kce^{\ell}}+\frac{N\ell}{Qk^2}+\frac{\ell e^{\ell}}{kc}\Big)\|\varphi\|_2^2$$
$$\ll \sum_{\log H\leqslant \ell\leqslant \log(cN/Q)}\Big(\frac{N}{ce^{\ell}}+\frac{N\ell}{Q}+\frac{\ell e^{\ell}}{c}\Big)\|\varphi\|_2^2\log(Q/H)$$
$$\ll \Big(NH^{-1}c^{-1}+NQ^{-1}\Big)\|\varphi\|_2^2\log^3(QN)$$

so this contribution is at most (on summing over c), up to a multiplicative constant:

$$Q\big(N(HQ)^{-1}+NCQ^{-2}\big)\|\varphi\|_2^2\log^4(QN). \tag{37}$$

□

This approximation provided by Lemma 5.1 together with the large sieve inequality leads to the following formula (recall the definition (19) of $W^\sharp$):

$$\sum_q \frac{W(q/Q)}{qQI_0(W)}\sum_{a \bmod^* q}|S(\varphi,a/q)|^2 = \sum_m|\varphi_m|^2$$
$$-\frac{J(W)}{QI_0(W)}\sum_{\substack{c\leqslant C,\\ e\leqslant E}}\frac{\mu(c)}{ec}\sum_{a \bmod^* e}|S(\varphi,a/e)|^2$$
$$+\sum_{\substack{c\leqslant C,\\ h\leqslant H}}\frac{\mu(c)}{chQI_0(W)}\sum_{a \bmod^* h}\sum_{m,n}\varphi_m\overline{\varphi_n}W^\sharp(c|m-n|/(hQ))e((n-m)a/h)$$
$$+\mathcal{O}\bigg(\Big(\frac{N}{EQ}+\frac{N}{HQ}+\frac{1}{C}+\frac{NC}{Q^2}\Big)\|\varphi\|_2^2\log^5(QN)\bigg) \tag{38}$$

The first main term comes from L_0, the second one from U and the third one from L.

8 Proof of Theorem 1.2

8.1 From $W^\sharp$ to $\tilde{W}$: cancellation of the two main terms

We introduce $\tilde{W}$ by appealing to (21). The choice $E = H$ ensures that, in (38), the second main term is canceled out by the contribution of the factor linked with the $J(W)/h$ above, getting

$$\sum_q \frac{W(q/Q)}{qQI_0(W)} \sum_{a \bmod^* q} |S(\varphi, a/q)|^2 = \|\varphi\|_2^2 - \sum_{\substack{c\leqslant C,\\ h\leqslant H}} \frac{\mu(c)}{chQI_0(W)} \sum_{a \bmod^* h} \sum_{m,n} \varphi_m \overline{\varphi_n} \tilde{W}(c|m-n|/(hQ)) e((n-m)a/h) + \mathcal{O}\Bigl(\Bigl(\frac{N}{HQ} + \frac{1}{C} + \frac{NC}{Q^2}\Bigr)\|\varphi\|_2^2 \log^5(QN)\Bigr). \quad (39)$$

The same cancellation of the main term is what presides to the introduction of $\Delta_c(u)$ in [25, section 20.5], see the proof of Lemma 20.17 therein.

8.2 Sharpening the error term in its H-dependence

One of the error term in Eq. (39) is $\mathcal{O}(\frac{N}{HQ}\|\varphi\|_2^2 \log^5(QN))$ and we want to (and need to!) remove the $\log^5(QN)$. We have to consider

$$\Sigma(H_1, H_2) = \sum_{H_1 < h \leqslant H_2} \frac{1}{h} \sum_{a \bmod^* h} \sum_{m,n} \varphi_m \overline{\varphi_n} W_C^\star(|m-n|/(hQ)) e((n-m)a/h). \quad (40)$$

We somehow go backwards and use $W_C^{\star\star}$ from (27) to write

$$\Sigma(H_1, H_2) = W_C^\star(0) \sum_{H_1 < h \leqslant H_2} \frac{1}{h} \sum_{a \bmod^* h} \Bigl|S\Bigl(\varphi, \frac{a}{h}\Bigr)\Bigr|^2 + \Sigma'(H_1, H_2)$$

with

$$\Sigma'(H_1, H_2) = \int_{-\infty}^{\infty} \sum_{H_1 < h \leqslant H_2} \frac{1}{h} \sum_{a \bmod^* h} \Bigl|S\Bigl(\varphi, \frac{a}{h} + \frac{u}{Qh}\Bigr)\Bigr|^2 \hat{W}_C^{\star\star}(u) du.$$

The large sieve inequality readily yields (since $W_C^\star(0) \ll 1$)

$$\Sigma(H_1, H_2) - \Sigma'(H_1, H_2) \ll \Bigl(\frac{N}{H_1} + H_2\Bigr)\|\varphi\|_2^2.$$

The treatment of $\Sigma'(H_1, H_2)$ is somewhat more difficult. When $|u/Q| \leqslant 1/2$, by combining a summation by parts together with the large sieve inequality, we find that

$$\sum_{H_1 < h \leqslant H_2} \frac{1}{h} \sum_{a \bmod^* h} \Bigl|S\Bigl(\varphi, \frac{an}{h} + \frac{un}{hQ}\Bigr)\Bigr|^2 \leqslant \|\varphi\|_2^2 \Bigl(\frac{N}{H_1} + 8H_2\Bigr)$$

since the points $(\frac{a}{h}+\frac{u}{hQ})_{a,h}$ are $\frac12 H_2^{-2}$-well spaced. When $|u/Q|\geqslant 1/2$, we use the large sieve inequality for every h. In this case the shift by $u/(hQ)$ is constant and the points are h^{-1}-well-spaced, giving

$$\sum_{H_1<h\leqslant H_2}\frac{1}{h}\sum_{a \bmod^* h}\Big|S\Big(\varphi,\frac{an}{h}+\frac{un}{hQ}\Big)\Big|^2 \ll \|\varphi\|_2^2\Big(N\log\frac{2H_2}{H_1}+H_2\Big).$$

As a consequence

$$\Sigma'(H_1,H_2)/\|\varphi\|_2^2 \ll \frac{N}{H_1}+8H_2+\frac{C}{Q}\Big(N\log\frac{2H_2}{H_1}+H_2\Big)$$

on using the bound $|\hat{W}_C^{\star\star}(u)|\ll C/(1+|u|^2)$ from Lemma 5.5 when $|u|\geqslant Q/2$. This implies that

$$\Sigma'(H_0,\sqrt{N})\ll\frac{N}{H_0}+\sqrt{N}+\frac{CN}{Q}\log N.$$

We can use formula (39) with $H=\sqrt{N}$ and shorten the summation by the process above. On renaming $H_0=H$, we have reached:

$$\begin{aligned}\sum_q \frac{W(q/Q)}{qQI_0(W)}\sum_{a \bmod^* q}|S(\varphi,a/q)|^2 &= \|\varphi\|_2^2\\ &-\sum_{h\leqslant H}\frac{1}{hQI_0(W)}\sum_{a \bmod^* h}\sum_{m,n}\varphi_m\overline{\varphi_n}W_C^\star(|m-n|/(hQ))e((n-m)a/h)\\ &+\mathcal{O}\Big(\Big(\frac{N}{HQ}+\frac{\log^5(QN)}{C}+\frac{NC\log^5(QN)}{Q^2}\Big)\|\varphi\|_2^2\Big).\end{aligned}\quad (41)$$

The effect of the previous treatment is neat: the log-factor attached to $N/(HQ)$ has disappeared while the rest of the remainder term is still of the same order of magnitude.

8.3 Direct extension of the c-variable

We handle the sum over c essentially trivially. The contribution from the diagonal term $m=n$ is bounded above by $\sum_{c>C}\frac{\mu(c)}{c}H\|\varphi\|_2^2/Q$. When $|m-n|\leqslant hQ/c$, we bound $\tilde{W}(c|m-n|/(hQ))$ by $\mathcal{O}(1)$, getting a contribution bounded above, up to a multiplicative constant, by

$$\sum_{\substack{h\leqslant H,\\ c>C}}\frac{1}{chQI_0(W)}\sum_{a \bmod^* h}\sum_{|m-n|\leqslant hQ/c}|\varphi_m\varphi_n| \ll \sum_{\substack{h\leqslant H,\\ c>C}}\frac{1}{cQ}\sum_m|\varphi_m|^2\frac{hQ}{c}\ll\|\varphi\|_2^2H^2/C.$$

We use $\tilde{W}(z) \ll 1/(1+z^2)$ when $|m-n| > hQ/c$, getting a contribution bounded above, up to a multiplicative constant, by

$$\sum_{\substack{h\leqslant H,\\ c>C}} \frac{1}{chQI_0(W)} \sum_{a \bmod^* h} \sum_{|m-n|>hQ/c} \frac{|\varphi_m\varphi_n|}{1+c^2(m-n)^2/(h^2Q^2)} \ll \sum_{\substack{h\leqslant H,\\ c>C}} \frac{Q^2h^3}{c^3hQ} \sum_m |\varphi_m|^2 \frac{c}{hQ} \ll \|\varphi\|_2^2 H^2/C.$$

We thus get, for any $C' \geqslant C$:

$$\sum_q \frac{W(q/Q)}{qQI_0(W)} \sum_{a \bmod^* q} |S(\varphi, a/q)|^2 = \|\varphi\|_2^2 - \sum_{h\leqslant H} \frac{1}{hQI_0(W)} \sum_{a \bmod^* h} \sum_{m,n} \varphi_m \overline{\varphi_n} W^\star_{C'}(|m-n|/(hQ)) e((n-m)a/h) + \mathcal{O}\Big(\Big(\frac{N}{HQ} + \frac{H^2}{C} + \frac{\log^5(QN)}{C} + \frac{NC\log^5(QN)}{Q^2}\Big)\|\varphi\|_2^2\Big).$$

The optimal choice $C = QH/N^{1/2}$ (provided that $H \leqslant N^{1/4}$; Indeed we recall that Lemma 7.1 asks for $E \leqslant \min(Q, 2Q/C)$ and that we have chosen $E = H$) may be too large. Instead we select

$$C = \min\Big(\frac{QH}{\sqrt{N}}, \frac{2Q}{H}, C'\Big) = \min\Big(\frac{QH}{\sqrt{N}}, C'\Big) \tag{42}$$

and get

$$\sum_q \frac{W(q/Q)}{qQI_0(W)} \sum_{a \bmod^* q} |S(\varphi, a/q)|^2 = \|\varphi\|_2^2 - \sum_{h\leqslant H} \frac{1}{hQI_0(W)} \sum_{a \bmod^* h} \sum_{m,n} \varphi_m \overline{\varphi_n} W^\star_{C'}(|m-n|/(hQ)) e((n-m)a/h) + \mathcal{O}\Big(\Big(\frac{N}{HQ} + \frac{H^2+\log^5(QN)}{C}\Big)\|\varphi\|_2^2\Big). \tag{43}$$

We may reformulate this equality by using the Fourier transform of $W^\star$:

$$\sum_q \frac{W(q/Q)}{qQI_0(W)} \sum_{a \bmod^* q} |S(\varphi, a/q)|^2 = \|\varphi\|_2^2 - \sum_{h\leqslant H} \frac{1}{hQI_0(W)} \sum_{a \bmod^* h} \int_{-\infty}^{\infty} \hat{W}^\star_{C'}(u) \Big|S\Big(\varphi, \frac{an}{h} + \frac{un}{hQ}\Big)\Big|^2 du + \mathcal{O}\Big(\Big(\frac{N}{HQ} + \frac{H^2+\log^5(QN)}{C}\Big)\|\varphi\|_2^2\Big).$$

Later, to prove (71), it will be better to restrict the range of integration (note that the Fourier transform has two parts: a Dirac mass and a regular part; only the regular part is concerned, as the Dirac mass is concentrated at $u = 0$). We use the large sieve inequality with u and h fixed to infer that

$$\sum_q \frac{W(q/Q)}{qQI_0(W)} \sum_{a \bmod^* q} |S(\varphi, a/q)|^2 = \|\varphi\|_2^2 - \sum_{h \leqslant H} \frac{1}{hQI_0(W)} \sum_{a \bmod^* h} \int_{-U}^{U} \hat{W}^\star_{C'}(u) \Big|S\Big(\varphi, \frac{a}{h} + \frac{u}{hQ}\Big)\Big|^2 du + \mathcal{O}\bigg(\Big(\frac{N}{HQ} + \frac{NC' \log H}{UQ} + \frac{H^2 + \log^5(QN)}{C}\Big)\|\varphi\|_2^2\bigg). \quad (44)$$

We can however proceed in a different fashion: majorize $|\hat{W}^*(u)|$ when $|u| \geqslant U$ by $\mathcal{O}(1/U)$, uniformly in C, and use $\int_{-\infty}^{\infty} |S(\alpha + u/(hQ)|^2 du = hQ\|\varphi\|_2^2$ by Parseval. This leads to

$$\sum_q \frac{W(q/Q)}{qQI_0(W)} \sum_{a \bmod^* q} |S(\varphi, a/q)|^2 = \|\varphi\|_2^2 - \sum_{h \leqslant H} \frac{1}{hQI_0(W)} \sum_{a \bmod^* h} \int_{-U}^{U} \hat{W}^\star_{C'}(u) \Big|S\Big(\varphi, \frac{a}{h} + \frac{u}{hQ}\Big)\Big|^2 du + \mathcal{O}\bigg(\Big(\frac{N}{HQ} + \frac{H^2}{U} + \frac{H^2 + \log^5(QN)}{C}\Big)\|\varphi\|_2^2\bigg). \quad (45)$$

The difference from $\hat{W}^\star_{C'}$ to $\hat{W}^{\star\star}_{C'}$ is $J(W)\sum_{c \leqslant C'} \mu(c)/c \cdot \delta_{u=0}$ by Lemma 5.5. On using that $J(W) \ll 1$, that $\sum_{c \leqslant C'} \mu(c)/c \ll 1$ and the large sieve inequality, we get a contribution which is $\ll NH^{-1}\|\varphi\|_2^2$, thus incorporable in the already existing error term. We have obtained:

$$\sum_q \frac{W(q/Q)}{qQI_0(W)} \sum_{a \bmod^* q} |S(\varphi, a/q)|^2 = \|\varphi\|_2^2 - \sum_{h \leqslant H} \frac{1}{hQI_0(W)} \sum_{a \bmod^* h} \int_{-U}^{U} \hat{W}^{\star\star}_{C'}(u) \Big|S\Big(\varphi, \frac{a}{h} + \frac{u}{hQ}\Big)\Big|^2 du + \mathcal{O}\bigg(\Big(\frac{N}{HQ} + \min\Big(\frac{NC' \log H}{UQ}, \frac{H^2}{U}\Big) + \frac{H^2 + \log^5(QN)}{C}\Big)\|\varphi\|_2^2\bigg). \quad (46)$$

We can send U to infinity and Theorem 1.2 follows by keeping $U = \infty$ and sending also C' to infinity.

9 A case of large sieve equality. Proof of Theorem 1.3

We prove a first result that is suited for some applications.

Theorem 9.1. *When* $\frac{1}{2} \leqslant H \leqslant \sqrt{N}/(\log N)^5$ *and* $\log Q \ll \log N$, *we have*

$$\sum_q \frac{W(q/Q)}{qQ} \sum_{a \bmod^* q} |S(\varphi, a/q)|^2 = \big(I_0(W) + \mathcal{O}(N(HQ)^{-1})\big)\|\varphi\|_2^2 + \mathcal{O}\Big(\sum_{h \leqslant H} \frac{N + hQ}{h^2Q^2} \max_{u<v<u+2hQ} \sum_{a \bmod^* h} \Big|\sum_{u<n\leqslant v} \varphi_n e(na/h)\Big|^2\Big).$$

Proof. Ideally, we would simply combine Theorem 1.2 (but we convert back $\hat{W}^\star$ in $W^\star$ as in (43)) together with Lemma 7.4 applied to $W^\star$, the set I being $\{a \bmod^* h\}$. The function $W^\star$ is however not regular enough, and we have to revert to $W_C^\star$ and more precisely to Eq. (43). We select $C = QH/\sqrt{N}$. When $z \leqslant 1/C$, we have $(W_C^\star)'' = 0$ while Lemma 5.3 with $\varepsilon = 0$ implies that $(W_C^\star)''(z) \ll 1$ in general. The theorem follows readily. □

Proof of Theorem 1.3. We employ Theorem 9.1 and simplify the remainder term by appealing to

$$\sum_{a \bmod^* h} \Big|\sum_{u<n\leqslant v} \varphi_n e(na/h)\Big|^2 \leqslant \sum_{a \bmod h} \Big|\sum_{u<n\leqslant v} \varphi_n e(na/h)\Big|^2 \leqslant h \sum_{c \bmod h} \Big|\sum_{\substack{u<n\leqslant v,\\ n\equiv c[h]}} \varphi_n\Big|^2.$$

Such an extension of the variable a may look a weak step, but since this theorem is aimed at sequences oscillating highly in small arithmetic progressions, the loss is not noticeable (at least in the examples I could think of). □

10 A refinement for primes

When the sequence φ is supported on integers prime to every integer $h \leqslant H$, we may refine Theorem 1.2 further, thanks to the next improved large sieve inequality. This is [35, Theorem 5.3]. See also [37, Corollary 1.5].

Lemma 10.1. *If* $(\varphi_n)_{n\leqslant N}$ *is such that* φ_n *vanishes as soon as* n *has a prime factor less than* $\sqrt{N}$, *then*

$$\sum_{q \leqslant Q_0} \sum_{a \bmod^* q} \big|S(\varphi, a/q)\big|^2 \leqslant 7\frac{N \log Q_0}{\log N}\|\varphi\|_2^2$$

for any $Q_0 \leqslant \sqrt{N}$ *and provided* $N \geqslant 100$.

This lemma enables us to improve Theorem 1.2 into the next result.

Theorem 10.2. *When* $1/2 \leqslant H \leqslant \sqrt{N}/(\log N)^5$, $Q \leqslant 10N$ *and* φ_n *vanishes when* n *has a prime factor below* $\sqrt{N}$, *we have*

$$\sum_q \frac{W(q/Q)}{qQ} \sum_{a \bmod^* q} |S(\varphi, a/q)|^2 = \Big(I_0(W) + \mathcal{O}\Big(\frac{N \log 3H}{QH \log N}\Big)\Big)\sum_m |\varphi_m|^2 - \sum_{h \leqslant H} \frac{1}{h} \sum_{a \bmod^* h} \int_{-\infty}^{\infty} \hat{W}^\star(u)\Big|S\Big(\varphi, \frac{a}{h} + \frac{u}{hQ}\Big)\Big|^2 du.$$

Proof. We start from Theorem 1.2, but with say H' rather than H and now shorten the sum over h. To do so, we write

$$\sum_{h\sim H_1}\frac{1}{hQ}\sum_{a \bmod^* h}\int_{-\infty}^{\infty}\hat{W}^\star(u)\Big|S\Big(\varphi,\frac{a}{h}+\frac{u}{hQ}\Big)\Big|^2du$$
$$=\frac{1}{H_1Q}\int_{-\infty}^{\infty}\max_{h\sim H_1}\Big|\hat{W}^\star\Big(\frac{hv}{H_1}\Big)\Big|\sum_{h\sim H_1}\sum_{a \bmod^* h}\Big|S\Big(\varphi,\frac{a}{h}+\frac{v}{H_1Q}\Big)\Big|^2dv.$$

Lemma 10.1 tells us that this quantity is $\ll \frac{N\log 3H_1}{QH_1\log N}\|\varphi\|_2^2$ from which, after noticing the bound for $\hat{W}^\star$ from Lemma 5.9, the theorem follows readily. □

Part II
Operator Decomposition of the Large Sieve

11 A local geometrical space

We consider $X_h=\mathbb{Z}/h\mathbb{Z}\times[0,1]$, equipped with the product of the probability measures. We denote by $L^2_*(X_h)$ the space of functions from $L^2(X_h)$ whose Fourier transform with respect to the first variable is supported by $(\mathbb{Z}/h\mathbb{Z})^*\times[0,1]$, i.e. functions f such that

$$\forall y\in[0,1],\forall d\in\mathbb{Z}/h\mathbb{Z}\ /\gcd(d,h)>1,\qquad \sum_{b \bmod h}f(b,y)e(-db/h)=0.$$

It is maybe simpler to say that this is the space generated by the functions $(c,y)\mapsto e(ac/q)f(y)$ for all $f\in L^2([0,1])$ and (this is where a restriction occurs) a prime to q. We reproduce rapidly the theory developed in [35, Chapter 4]. Let $k|h$ be two moduli. We consider

$$\begin{aligned} L_h^k: L^2(X_k)&\to L^2(X_h)\\ F&\mapsto L_h^k(F):\mathbb{Z}/h\mathbb{Z}\times[0,1]\to\mathbb{C}\\ &\qquad (b,y)\mapsto F(\sigma_k(b),y)\end{aligned}\tag{47}$$

and correspondingly

$$\begin{aligned} J_k^h: L^2(X_h)&\to L^2(X_k)\\ F&\mapsto J_k^h(F):\mathbb{Z}/k\mathbb{Z}\times[0,1]\to\mathbb{C}\\ &\qquad (b,y)\mapsto \frac{1}{h/k}\sum_{\substack{c \bmod h,\\ c\equiv b[k]}}F(\sigma_h(c),y).\end{aligned}\tag{48}$$

We finally define

$$U_{\tilde h\to\tilde k}=L_h^kJ_k^h,\quad U_{\tilde h\to k}=\sum_{d|k}\mu(k/d)U_{\tilde h\to\tilde d}.\tag{49}$$

Here is the structure theorem we need.[3]

Theorem 11.1. *The maps L_h^k and J_k^h are adjoined one to the other. The collection $(U_{\tilde{h}\to\tilde{k}})_{k|h}$ is a family of commuting orthogonal projectors. Furthermore*

$$U_{\tilde{h}\to\tilde{k}} = \sum_{d|k} U_{\tilde{h}\to d}$$

while, for any two divisors k_1 and k_2 of h, we have $U_{\tilde{h}\to k_1} U_{\tilde{h}\to k_2} = \delta_{k_1=k_2} U_{\tilde{h}\to k_2}$. We have $L^2_*(X_h) = U_{\tilde{h}\to h} L^2(X_h)$.

An explicit expression

At the heart of this matter are the Gauss sums

$$\tau_h(\chi,\cdot) = \sum_{b \bmod h} \chi(b) e(b\cdot/h). \tag{50}$$

Theorem 11.2. *For any $h \geqslant 1$, any class b modulo c, any real number y and any function $F \in L^2(X_h)$, the orthonormal projection $U_{\tilde{h}\to h}$ on $L^2_*(X_h)$ has the following explicit form:*

$$U_{\tilde{h}\to h} F(b,y) = \frac{1}{h} \sum_{c \bmod h} c_h(b-c) F(c,y).$$

Given a hilbertian orthonormal basis $(f_k)_k$ of $L^2([0,1])$, the family $(\mathscr{E}_{h,\chi}\otimes f_k)_{\chi,k}$ where $\mathscr{E}_{h,\chi} = \tau_h(\chi,\cdot)/\sqrt{\phi(h)}$ and χ ranges the Dirichlet characters modulo h is a hilbertian orthonormal basis of $L^2_*(X_h)$.

Proof. We first check that

$$\begin{aligned}\sum_{b \bmod h} c_h(b-c) e(bd/h) &= \sum_{a \bmod^* h} e(-ac/h) \sum_{b \bmod h} e(b(a+d)/h) \\ &= \begin{cases} he(dc/h) & \text{when } (d,h)=1, \\ 0 & \text{else.}\end{cases}\end{aligned}$$

and since $(b \mapsto e(bd/h))_{d \bmod h}$ generates the whole space of functions over X_h, this proves our first assertion. The introduction of the Dirichlet character may be arbitrary, but in fact $(\tau_h(\chi,\cdot))_\chi$ is the full set of eigenfunctions of $f \mapsto \sum_{c \bmod h} c_h(b-c)f(c)/h$ that are associated to a non-zero eigenvalue. We simply have

$$\forall b \in \mathbb{Z}/h\mathbb{Z}, \quad \tau_h(\chi,b) = \frac{1}{h} \sum_{c \bmod h} c_h(b-c)\tau(\chi,c). \tag{51}$$

Note finally that

$$\begin{aligned}\frac{1}{h} \sum_{c \bmod h} \tau_h(\chi_1,c)\overline{\tau_h(\chi_2,c)} &= \sum_{a,b \bmod h} \chi_1(a)\overline{\chi_2(b)} \frac{1}{h} \sum_{c \bmod h} e\Big(\frac{c(a-b)}{h}\Big) \\ &= 1_{\chi_1=\chi_2}\varphi(h)\end{aligned}$$

as required. □

[3] These results are easily proved. Details may be found in [35, Chapter 4], though with no y-component. This component is inert here, so the proofs carry through mutatis mutandis.

12 Analysis of a class of difference operators

We treat here the analysis of the intervening family of operators in an abstracted setting. Let V be a function satisfying the following assumptions:

(R_1) • V is a continuous real-valued even function of bounded variations and integrable over $\mathbb{R}$.

(R_2) • $V(0) = 0$.

(R_3) • There exist $B \geqslant \|V\|_\infty$, $c \in (0, 1]$ and $A > 0$ such that, for every $\delta \in (0, 1)$ and $x \in [0, 1-\delta]$, we have $|V(x+\delta)-V(x)| \leqslant B\exp(-c\sqrt{-\log\min(1, A\delta)})$.

Recall that we defined

$$\mathscr{V}_0: \quad G \in L^2([0,1]) \mapsto \left(y \mapsto \int_0^1 G(y')V(y-y')dy'\right) \tag{8}$$

It is classical theory that $\mathscr{V}_0$ is a compact Hilbert-Schmidt operator, see for instance [22, Theorem 7.7]. Let $(\lambda_\ell, G_\ell)_\ell$ be a complete orthonormal system of eigenvalues / eigenfunctions, ordered with non-increasing $|\lambda_\ell|$. The Fredholm equation $\lambda G(y') = \int_0^1 K(y', y)G(y)dy$ has been intensively studied. It is not the purpose of this paper to introduce to this theory, a task for which it is better to read the complete and classical book [21] by I. Gohberg, I. C. & M.G. Kreĭn, or the more modern [22] by I. Gohberg, S. Goldberg & N. Krupnik. Kernel of type $V(y'-y)$ are often called *difference kernel*, and lead to operators that are distinct from convolution operators as the integration and definition interval is *not* the whole real line. The book [39] by L. Sakhnovich is dedicated to the operators built from such kernels. The book [7] by J. Cochran contains also many useful informations.

12.1 $\mathbf{L}^2$-norm

We readily find that

$$\int_0^1\int_0^1 |V(y'-y)|^2dy = \int_{-1}^1 |V(z)|^2(1-|z|)dz. \tag{52}$$

Hence

$$\sum_{\ell\geqslant 1}|\lambda_\ell|^2 = \int_{-1}^1 |V(z)|^2(1-|z|)dz \leqslant 2\int_0^1 |V(z)|^2dz.$$

As a consequence, and enumerating the eigenvalues in such a way that $|\lambda_\ell|$ is non-increasing, we find that

$$|\lambda_\ell| \leqslant \sqrt{2\int_0^1 |V(z)|^2dz}/\sqrt{\ell}. \tag{53}$$

Theorem 12.4 will enable us to replace $\sqrt{\ell}$ by ℓ, but it uses the above bound.

12.2 Properties of the eigenvectors

The eigenvectors of $\mathscr{V}_0$ attached to non-zero eigenvalues are classically shown to be continuous. Since the L^1-norm is not more than the L^2-norm squared here, we have $\|G\|_1 \leqslant 1$. Each of them thus satisfies

$$|\lambda|\|G\|_\infty \leqslant 2\int_0^1 |V(z)|(1-z)dz. \tag{54}$$

Furthermore, we find that

$$|\lambda||G(y+\delta)-G(y)| \leqslant \|G\|_1\omega(V,\delta) = \|G\|_1 \max_{-1\leqslant z\leqslant 1}|V(z+\delta)-V(z)|. \tag{55}$$

These functions are also of bounded variation. Indeed, with obvious notation, we find that

$$\begin{aligned}|\lambda|\sum_{1\leqslant i\leqslant n}|G(y_{i+1})-G(y_i)| &\leqslant \int_0^1 |G(y)|\sum_{1\leqslant i\leqslant n}\bigl|V(y_{i+1}-y)-V(y_i-y)\bigr|dy\\ &\leqslant \int_{-1}^1|V'(y)|dy\int_0^1|G(y)|dy \leqslant \int_{-1}^1|V'(y)|dy\end{aligned}$$

since $\|G\|_1 \leqslant 1$.

12.3 Nuclearity

A consequence of a theorem of Fredholm from [18] is that, when $y\mapsto V(y)$ is Hölder of exponent α, then the eigenvalues verify $\sum_{\ell\geqslant 1}|\lambda_\ell|^p<\infty$ for every $p>2/(1+2\alpha)$. This proof is reproduced in the book [22, Chapter IV, Theorem 8.2] by I. Gohberg, S. Gohberg & N. Krupnik. This is too strong a condition for us if we are to avoid the Riemann Hypothesis (in which case $\alpha=1/2+\varepsilon$ would be accessible). D. Swann in [42] considered the effect of bounded variation on a general kernel, but his theorem asks again for too strong hypotheses since the function $(y',y)\mapsto V(y'-y)$ is a priori not of bounded variation. However, each function $y\mapsto V(y'-y)$ is uniformly of bounded variation (i.e. its total variation is, as function of y' integrable; in our case, it is even bounded), a case that is mentioned (with more generality) in the paragraph preceding [42, Theorem 3] and more formally in [7, Theorem 16.2] in the monograph of J. Cochran. We follow this approach.

In this subsection, we use

$$\log^- t = \log\min(1,t); \tag{56}$$

We consider the coefficients of the Carleman determinant, see [7, Chapter 4, (3)], for $\nu\geqslant 2$:

$$d_\nu = \frac{(-1)^\nu}{\nu!}\int_0^1\!\!\cdots\!\int_0^1\begin{vmatrix}0 & V(y_1-y_2) & \cdots & V(y_1-y_\nu)\\ V(y_2-y_1) & 0 & \cdots & V(y_2-y_\nu)\\ \vdots & \vdots & & \vdots\\ V(y_\nu-y_1) & V(y_\nu-y_2) & \cdots & 0\end{vmatrix}dy_1dy_2\cdots dy_\nu. \tag{57}$$

As $V(y-y)=0$, this is also the Fredholm determinant, see [22, Chapter VI, (1.5)]. The above determinant, say $K(y_1,\cdots,y_\nu)$, can be rewritten as

$$\begin{vmatrix} 0 & V(y_1-y_2)-V(y_1-y_1) & \cdots & V(y_1-y_\nu)-V(y_1-y_{\nu-1}) \\ V(y_2-y_1) & V(y_2-y_2)-V(y_2-y_1) & \cdots & V(y_2-y_\nu)-V(y_2-y_{\nu-1}) \\ \vdots & \vdots & & \vdots \\ V(y_\nu-y_1) & V(y_2-y_\nu)-V(y_2-y_{\nu-1}) & \cdots & V(y_\nu-y_\nu)-V(y_\nu-y_{\nu-1}) \end{vmatrix}.$$

We use the symmetry of the integral and now assume that $0\leqslant y_1<y_2<\cdots<y_\nu\leqslant 1$ (when an equality occurs between these variables, the determinant vanishes). We define $\delta_i=y_{i+1}-y_i$ so that $\sum_{1\leqslant i\leqslant \nu-1}\delta_i\leqslant 1$. We divide the second column by $\sqrt{B}\exp(-(c/2)\sqrt{-\log^-(A\delta_1)})$, the third one by $\sqrt{B}\exp(-(c/2)\sqrt{-\log^-(A\delta_2)})$ and so on, getting a factor

$$B^{(\nu-1)/2}\prod_{1\leqslant i\leqslant \nu-1}\exp(-(c/2)\sqrt{-\log^-(A\delta_i)})$$

in front of our determinant. We first note the following lemma.

Lemma 12.1. *We have $\sum_{1\leqslant i\leqslant n}\sqrt{-\log^-(A\delta_i)}\geqslant n\sqrt{\log n}$ when the δ_i's are positive real numbers such that $\sum_{1\leqslant i\leqslant n}\delta_i\leqslant 1$.*

Proof. Given an n-tuple $(\delta_1,\cdots,\delta_n)$, we note that the n-tuple obtained by replacing each δ_i by $\min(A^{-1},\delta_i)$ satisfies the same constraint with an equal sum of $\sqrt{-\log^-(A\cdot)}$. In order to find the minimum required, we may thus restrict our attention to variables that verify $\delta_i\leqslant 1/A$. Set $x_i=(-\log^-(A\delta_i))^{1/4}$. This variable ranges possibly $(0,\infty)$. The condition on δ_i now reads $\sum_{1\leqslant i\leqslant n}e^{-x_i^4/A}=\delta$ for some $\delta\in(0,1]$, while we seek to minimize $\sum_{1\leqslant i\leqslant n}x_i^2$ and we forget the condition $e^{-x_i^4/A}\leqslant 1/A$. We use the Lagrange method and consider

$$Y(x_1,\ldots,x_n,\lambda)=\sum_{1\leqslant i\leqslant n}x_i^2-\lambda\Big(\sum_{1\leqslant i\leqslant n}e^{-x_i^4/A}-\delta\Big).$$

Its critical points, obtained by equating all the partial derivatives to 0, satisfy:

$$\begin{cases}\forall i\leqslant n, \quad 2x_i+4A^{-1}x_i^3\lambda e^{-x_i^4/A}=0,\\ \displaystyle\sum_{1\leqslant i\leqslant n}e^{-x_i^4/A}=\delta.\end{cases}$$

This implies that[4] $\lambda/A=-2e^{-x_i^4/A}/x_i^2$. The function $y\mapsto 2e^{-y^4/A}/y^2$ is decreasing, from which we conclude that all x_i's are equal, which in turn implies that all δ_i's are equal, and equal to δ/n. The choice $\delta=1$ is also optimal. □

Next we use Hadamard's inequality (as in all such proofs!) together with

[4] Any choice $x_i=0$ means that $\delta_i=1$, which implies that any other δ_j vanishes, leading to the maximum being ∞ when $n\geqslant 2$.

the previous lemma (and $(\nu-1)\geqslant \nu/2$) and get

$$\frac{|K(y_1,\cdots,y_\nu)|}{B^{\nu/2}e^{-(c/4)\nu\sqrt{\log(\nu-1)}}}\leqslant \prod_i\Big(\sum_j |a_{i,j}|^2\Big)^{1/2}$$
$$\leqslant \prod_i\Big(\|V\|_\infty\big|V(y_i-y_1)-V(y_i-y_i)\big|+B\sum_{2\leqslant j\leqslant\nu}\big|V(y_i-y_{j-1})-V(y_i-y_j)\big|\Big)^{1/2}$$
$$\leqslant \Big(2c\int_{-1}^1\big|V'(y)\big|dy\Big)^{\nu/2}$$

since $B\geqslant \|V\|_\infty$. As a consequence, we find that the Carleman determinant

$$\mathcal{D}(V,z)=1+\sum_{\nu\geqslant 2}d_\nu z^\nu=\prod_{\ell\geqslant 1}(1-\lambda_\ell z)e^{\lambda_\ell z} \tag{58}$$

satisfies, with $M=\sqrt{2B\int_0^1|V'(y)|dy}$,

$$|\mathcal{D}(V,z)|\leqslant 1+\sum_{\nu\geqslant 2}\frac{M^\nu|z|^\nu e^{-\frac{c}{4}\nu\sqrt{\log(\nu-1)}}}{\nu!}$$
$$\leqslant (M|z|)^{N+1}e+e^{M|z|e^{-\frac{c}{4}\sqrt{\log N}}}\leqslant H^{N+1}e+e^{He^{-\frac{c}{4}\sqrt{\log N}}}$$

with $H=M|z|\geqslant 1$ and for any real valued parameter $N\geqslant 1$ that we may choose. When $H\leqslant e^2$, we use the upper bound $|\mathcal{D}(V,z)|\leqslant e^{e^2}$. When $\log H\geqslant 2$, we select

$$N=He^{-\frac{c}{4}\sqrt{\log(H+1)}}/\log H. \tag{59}$$

When $\log H\geqslant 2$, we check that (recall that we have assumed that $c\leqslant 1$)

$$\log N=\log H-\frac{c}{4}\sqrt{\log H}-\log\log H$$
$$=\log(H+1)\Big(\frac{\log H}{\log(H+1)}-\frac{1}{4\sqrt{\log(H+1)}}-\frac{\log\log H}{\log(H+1)}\Big)$$
$$\geqslant \log(H+1)\Big(\frac{\log 2}{\log 3}-\frac{1}{4\sqrt{\log 3}}-\frac{1}{e}\Big)\geqslant\frac{\log(H+1)}{49}.$$

We thus find that, in this case, we have

$$|\mathcal{D}(V,z)|\leqslant He^{He^{-\frac{c}{4}\sqrt{\log(H+1)}}}e+e^{He^{-\frac{c}{28}\sqrt{\log(H+1)}}}$$
$$\leqslant 6He^{He^{-\frac{c}{28}\sqrt{\log(H+1)}}}.$$

Next, $He^{-\frac{c}{28}\sqrt{\log(H+1)}}+\log H$ is certainly not more than $He^{-\frac{c}{30}\sqrt{\log(H+1)}}$ provided H be larger than some constant depending on c. So, in general, we find that $He^{-\frac{c}{28}\sqrt{\log(H+1)}}+\log H\leqslant He^{-\frac{c}{30}\sqrt{\log(H+1)}}+c''$, where c'' is a constant depending solely on c. We have proved that

$$|\mathcal{D}(V,z)|\leqslant 6e^{c''}e^{He^{-\frac{c}{30}\sqrt{\log(H+1)}}}$$

when $H\geqslant e^2$. The minimum of $6e^{c''}e^{He^{-\frac{c}{30}\sqrt{\log(H+1)}}}$ when H ranges $[0,e^2]$ is some positive constant, say c''', depending only on c (we have introduced

$\log(H+1)$ rather tha $\log H$ earlier for this very purpose). As a consequence, we have, for any $H \geqslant 0$,

$$|\mathcal{D}(V,z)| \leqslant e^{e^2} \frac{6e^{c''} e^{He^{-\frac{c}{30}\sqrt{\log(H+1)}}}}{\min(1, c''')}.$$

Here is the lemma we have proved.

Lemma 12.2. *There exists a positive constant $c' = c'(c)$ such that we have*

$$|\mathcal{D}(V,z)| \leqslant c' e^{M|z|e^{-\frac{c}{30}\sqrt{\log(M|z|+1)}}}$$

with $M = \sqrt{2B\int_0^1 |V'(y)|dy}$.

We continue with the following general lemma.

Lemma 12.3. *Let f be an entire function of finite order and such that $f(0) = 1$ and let (ρ_ℓ) be an enumeration of its zeroes with non-decreasing $|\rho_\ell|$. Let g be a C^2-function over $(0,\infty)$. Assume that, as t goes to infinity,*

$$\frac{1}{2\pi}\int_0^{2\pi} \log|f(te^{i\theta})| d\theta \, tg'(t) \to 0.$$

Then, provided the RHS converges absolutely, we have

$$\sum_{\ell \geqslant 1} g(|\rho_\ell|) = \frac{1}{2\pi}\int_a^\infty \int_0^{2\pi} \log|f(te^{i\theta})| d\theta (tg''(t) + g'(t)) dt$$

for any $a \in [0, |\rho_1|]$.

The reader may want to read [2], for instance Theorem 8.4.1, for general results on entire functions having only real zeroes.

Proof. We denote by $n(t)$ the number of zeroes of f (counted with multiplicities) that are of modulus not more than t. We use an integration by parts to write

$$\begin{aligned}\sum_{\ell \geqslant 1} g(|\rho_\ell|) &= -\sum_{\ell \geqslant 1} \int_{|\rho_\ell|}^\infty g'(t)dt = -\alpha \int_a^\infty \frac{n(t)}{t} tg'(t) dt \\ &= -\left[\int_0^t \frac{n(u)du}{u} tg'(t)\right]_a^\infty + \int_a^\infty \int_0^t \frac{n(u)du}{u}(tg''(t) + g'(t))dt\end{aligned}$$

We only have to introduce Jensen's formula in the RHS and use our hypothesis to get our lemma. □

When used with $g(t) = 1/t$ and appealing to Lemma 12.2, we get the following important result.

Theorem 12.4. *The hypothesis on V being as above, the operator $\mathscr{V}_0$ is nuclear. Furthermore, it satisfies $\sum_{\ell \geqslant 1} \lambda_\ell = 0$ and*

$$\sum_{\ell \geqslant 1} |\lambda_\ell| \ll \sqrt{\int_0^1 |V(z)|^2 dz} \; e^{-c_3\sqrt{\log(1+\|V\|_\infty \int_0^1 |V'(t)|dt / \int_0^1 |V(t)|^2 dt)}}$$

for some positive constant c_3 that depends only on B and c. In particular, we have

$$|\lambda_\ell| \ll \sqrt{\int_0^1 |V(z)|^2 dz/\ell}. \tag{60}$$

In our case of application, the L^2-norm of V is controlled by Lemma 15.1.

Proof. On combining Lemma 12.3 together with Lemma 12.2, we readily find that

$$\begin{aligned}\sum_{\ell\geqslant 1}|\lambda_\ell| &\ll M\int_{1/|\lambda_1|} t\, e^{-\frac{c}{30}\sqrt{\log(Mt+1)}}\frac{dt}{t^2} \\ &\ll M\int_{M/|\lambda_1|} e^{-\frac{c}{30}\sqrt{\log(t+1)}}\frac{dt}{t} \ll |\lambda_1|e^{-\frac{c}{30}\sqrt{\log(M|\lambda_1|^{-1}+1)}}.\end{aligned}$$

By (53) with $\ell = 1$, we find that $|\lambda_1| \leqslant \sqrt{2\int_0^1 |V(z)|^2 dz}$, hence the bound for $\sum_{\ell\geqslant 1}|\lambda_\ell|$. Lidskii's Theorem then applies giving us that $\sum_{\ell\geqslant 1}\lambda_\ell = \int_0^1 V(y-y)dy = 0$. ☐

12.4 Oscillation of the eigenvalues

Let us consider the eigenvalues of $\mathscr{V}_0$. At least one of them is positive and at least one of them is negative because

$$\sum_{\ell\geqslant 1}\lambda_\ell = 0$$

and V is not identically 0. Proving that infinitely many of them are positive or negative seems to be more difficult, if true.

12.5 A Mercer Theorem

Let us select a complete system of non-zero eigenvectors $(G_\ell)_{\ell\geqslant 1}$ associated with the eigenvalues $(\lambda_\ell)_\ell$ that are repeated according to multiplicity and arranged in non-increasing order of their absolute values.

Theorem 12.5. *For every positive integer N, we have*

$$\max_{y,y'\in[0,1]}\left|V(y'-y) - \sum_{\ell\leqslant N}\lambda_\ell\overline{G_\ell}(y)G_\ell(y')\right| \leqslant |\lambda_{N+1}|.$$

This theorem contains the value of the trace. Indeed, on selecting $y' = y$, we get $\sum_{\ell\geqslant 1}\lambda_\ell|G_\ell(y)|^2 = 0$; we then integrate this equality over y and recover the trace $\sum_{\ell\geqslant 1}\lambda_\ell = 0$.

Proof. We have, for any y' in $[0,1]$ and any L^2-function f:

$$\begin{aligned}\int_0^1 V(y'-y)f(y)dy &= \sum_{\ell\geqslant 1}\lambda_\ell(f|G_\ell)G_\ell(y') \\ &= \sum_{\ell\leqslant N}\lambda_\ell\int_0^1 f(y)\overline{G_\ell}(y)dyG_\ell(y') + \sum_{\ell\geqslant N+1}\lambda_\ell(G_\ell|f)G_\ell(y').\end{aligned}$$

This implies that, for any test function h, we have

$$\int_0^1 h(y')\Big|\int_0^1 \Big(V(y'-y) - \sum_{\ell\leqslant N}\lambda_\ell\overline{G_\ell}(y)G_\ell(y')\Big)f(y)dy\Big|dy' \ll |\lambda_{N+1}|\|f\|\|h\| \quad (61)$$

by using Cauchy's inequality and

$$\begin{aligned}\int_0^1\Big|\sum_{\ell\geqslant N+1}\lambda_\ell(G_\ell|f)G_\ell(y')\Big|^2 dy &\leqslant \int_0^1 \sum_{\ell\geqslant N+1}|\lambda_\ell(G_\ell|f)|^2|G_\ell(y')|^2 dy \\ &\leqslant |\lambda_{N+1}|^2\sum_{\ell\geqslant N+1}|(G_\ell|f)|^2 \leqslant |\lambda_{N+1}|^2\|f\|^2.\end{aligned}$$

Select a point y_0 from $(0,1)$ and a positive ε such that $[y_0-\varepsilon, y_0+\varepsilon]\subset[0,1]$. We take $f = 1_{[y_0-\varepsilon,y_0+\varepsilon]}$ and get

$$\int_0^1 h(y)\Big|\frac{1}{2\varepsilon}\int_{y_0-\varepsilon}^{y_0+\varepsilon}\Big(V(y'-y) - \sum_{\ell\leqslant N}\lambda_\ell\overline{G_\ell}(y)G_\ell(y')\Big)dy\Big|dy' \ll |\lambda_{N+1}|\|h\|.$$

However we have

$$\begin{aligned}V(y'-y_0) - \frac{1}{2\varepsilon}\int_{y_0-\varepsilon}^{y_0+\varepsilon}V(y'-y)dy &\ll \frac{1}{2\varepsilon}\int_{y_0-\varepsilon}^{y_0+\varepsilon}\exp\big(-c\sqrt{\min(1,A\varepsilon)}\big)dy \\ &\ll \exp\big(-c\sqrt{\min(1,A\varepsilon)}\big)\end{aligned}$$

which tends to zero with ε. The same applies to $y\mapsto\sum_{\ell\leqslant N}\lambda_\ell\overline{G_\ell}(y)G_\ell(y')$. In case of the two endpoints $y_0=0$ and $y_0=1$, we simply select $f=1_{[0,\varepsilon]}$ in the first case and $f=1_{[1-\varepsilon,1]}$ in the second one. We then employ the same trick regarding the variable y'. We leave the details to the reader. □

12.6 Influence of the Riemann Hypothesis

As we already mentioned, under the Riemann Hypothesis, the function $y'\mapsto V(y'-y)$ is uniformly Hölder with exponent $1/2-\varepsilon$ for any $\varepsilon>0$. In which case, [7, Theorem 16.3-1] gives us that

$$\sum_{\ell\geqslant 1}|\lambda_\ell|^p \ll_p 1$$

for every $p>4/5$. This implies that the number of eigenvalues below t, say $n(t)$, satisfies $n(t)\ll_\varepsilon t^{4/5+\varepsilon}$ under the Riemann Hypothesis.

12.7 Bounds from Fourier analysis and non-negativity

Since the function V is even over $\mathbb{R}$ its Fourier transform is (a cosine transform and hence) real valued. In practice, we will use $V(u)=W^\star(\tau u/h)$ where $\hat{W}^\star$ is also given by (1); hence we can bound above the values of the eigenvalues when W is assumed to be non-negative.

Theorem 12.6. *Assume that $\hat{V}(u)\leqslant M_1$ when $u\in\mathbb{R}$. Then the eigenvalues of $\mathscr{V}_0$ are not more than M_1. There exists a positive constant c_4 such that, if we further assume that $\hat{V}(u)\leqslant M_2$ when $|u|\geqslant U_2$ for some positive parameters $M_1>M_2$ and U_2, then the eigenvalues of $\mathscr{V}_0$ are not more than $M_1-ce^{-c_4U_2}$ for some positive constant c depending on M_1 and M_2 (but not on V nor on U_2).*

The proof uses F.I. Nazarov's form [31], [32] of the Amrein-Berthier Theorem [1] (see also [24, Section 4.11] in the monograph of V. Havin & B. Jöricke) that we now recall.

Theorem 12.7 (Nazarov)**.** *There exist two positive constants c_4, c_8 such that, for any measurable subsets E and Σ of $\mathbb{R}$ of finite measure, and for any $f \in L^2(\mathbb{R})$, we have*

$$\|f\|_2^2 \leqslant c_8\, e^{c_4|E||\Sigma|}\left(\int_{x\in\mathbb{R}\setminus E}|f(x)|^2dx + \int_{u\in\mathbb{R}\setminus\Sigma}|\hat{f}(u)|^2du\right).$$

We thank P. Jaming for giving some advice on this result, for pointing out that a theorem of V.N. Logvinenko Ju.F. Sereda [30] would be enough here (since we consider only the case when E and Σ are intervals), and for giving us the reference to the paper [29] of O. Kovrijkine that gives a simpler proof. P. Jaming also told us that he believes $c_8 = 300$ and $c_4 = 120$ to be an admissible choice.

Proof of Theorem 12.6. We write

$$V(y-y') = \int_{-\infty}^{\infty}\hat{V}(u)e(-u(y-y'))du$$

and thus, for any $G \in L^2([0,1])$, we have

$$[G, \mathscr{V}_0(G)] = \int_{-\infty}^{\infty}\hat{V}(u)|\hat{G}(u)|^2du. \tag{62}$$

Some comments are called for here. We have

$$\hat{G}(u) = \int_{-\infty}^{\infty}G(v)e(-uv)dv$$

i.e. we have extended G from $[0,1]$ to $\mathbb{R}$ by 0 outside. By the result of Nazarov cited above, its Fourier transform is not accumulated on an interval. More precisely, on selecting $E = [0,1]$ and $\Sigma = [-U_2, U_2]$ in Theorem 12.7, we find that

$$\int_{|u|\geqslant U_2}|\hat{G}(u)|^2du \geqslant e^{-c_4U_2}\|G\|_2^2/c_8$$

and thus

$$[G, \mathscr{V}_0(G)] \leqslant (M_1 - e^{-c_4U_2}c_8^{-1}(M_1-M_2))\|G\|_2^2.$$

The theorem follows readily. □

In between, (62) implies the following.

Lemma 12.8. *The eigenvalues of $\mathscr{V}_0$ lie inside $[-\min\hat{V}(u), \max\hat{V}(u)]$.*

12.8 Spectral decomposition of $\mathscr{V}$ from the one of $\mathscr{V}_0$

Now that we have the spectral decomposition of $\mathscr{V}_0$ with couples (λ_ℓ, G_ℓ), we recover a spectral decomposition of $\mathscr{V}|_{L^2_*(\mathbb{Z}/h\mathbb{Z})}$ (the restriction of $\mathscr{V}$ to $L^2_*(\mathbb{Z}/h\mathbb{Z})$), by considering the eigenvectors $\mathscr{E}_{h,\chi}\otimes G_\ell$, where $\mathscr{E}_{h,\chi}$ comes from Theorem 11.2. These eigenvectors are of norm 1 and are associated with the eigenvalues λ_ℓ. When we want to refer to the eigenvalues of $\mathscr{V}|_{L^2_*(\mathbb{Z}/h\mathbb{Z})}$, we use the notation λ_ℓ and we add the superscript for $\mathscr{V}_0$. We go from the latter to the former by repeating $\phi(h)$ times each eigenvalue.

13 From global to local: two embeddings

The hermitian product on $\{1 \cdots N\}$ is given by (10).

From the sequence φ to a local function

We explore the embedding defined in (4).

Concerning (5), we specify here that we could select a uniform value for N', typically $N + H$ where H is a bound to be chosen (like $\exp c_1\sqrt{\log N}$). Since N is supposed to be much larger than H, the introduction of this parameter in the next definition is only to correct some effects on the border of our domain, see the proof of Lemma 13.1 below. There are several ways to handle this situation, we could have considered $[0, 2]$ rather than $[0, 1]$ in the definition of X or we could also have kept N and $[0, 1]$ and simply replaced the equality of Lemma 13.1 by an equality with an error term and carried this error term throughout the proofs. The choice above has the advantage of being independent of an external upper bound (but is not henceforth canonical).

As a consequence, we note directly[5] here that

$$\Gamma_{N,h}(\varphi)(b, y) = 0 \quad \text{when } y \geqslant [N'/h]h/N'. \tag{63}$$

The fundamental property of $\Gamma_{N,h}$ is that it preserves the hermitian product up to a multiplicative constant (but is *not* isometric as it is not onto).

Lemma 13.1. *For any positive integer $h \leqslant N' - N$, we have*

$$\tfrac{N}{N'}[\varphi, \psi]_N = \langle \Gamma_{N,h}(\varphi), \Gamma_{N,h}(\psi)\rangle_h.$$

The reader should notice a notational difficulty here: the norm $\|\varphi\|_2$ that we have used up to now corresponds to the scalar product only *up to the scalar $1/N$. We will thus refrain from using $\|\varphi\|_N^2$ as a shortcut to $[\varphi, \varphi]_N$.*

Proof. Indeed, we have

$$\begin{aligned}
\langle \Gamma_{N,h}(\varphi),&\Gamma_{N,h}(\psi)\rangle_h \\
&= \frac{1}{h}\sum_{1\leqslant b\leqslant h}\int_0^1 \Gamma_{N,h}(\varphi)(b, y)\overline{\Gamma_{N,h}(\psi)(b, y)}dydz \\
&= \frac{1}{h}\sum_{1\leqslant b\leqslant h}\sum_{0\leqslant k\leqslant \frac{N'}{h}-1}\int_{kh/N'}^{(k+1)h/N'} \Gamma_{N,h}(\varphi)(b, y)\overline{\Gamma_{N,h}(\psi)(b, y)}dydz \\
&\qquad + \frac{1}{h}\sum_{1\leqslant b\leqslant h}\int_{[\frac{N'}{h}]h/N'}^{1} \Gamma_{N,h}(\varphi)(b, y)\overline{\Gamma_{N,h}(\psi)(b, y)}dy \\
&= \frac{1}{N'}\sum_{1\leqslant b\leqslant h}\sum_{n\equiv b[h]}\varphi_n\overline{\psi_n}
\end{aligned}$$

on employing (63). Hence the result. □

[5]Indeed, under the stated condition on y, we have $[N'y/h] \geqslant [N'/h] \geqslant N/h$ and thus the index $b + h[N'y/h]$ is strictly larger than N.

Local adjoint

For every φ, the linear functional $f \mapsto \langle \Gamma_{N,h}(\varphi), f\rangle$ can be uniquely represented in the form $[\varphi, \Gamma^*_{N,h}(f)]_N$, i.e. we have

$$\langle \Gamma_{N,h}(\varphi), f\rangle = [\varphi, \Gamma^*_{N,h}(f)]_N. \tag{64}$$

The functional $f \mapsto \Gamma^*_{N,h}(f)$ is of course linear. We find that

$$\begin{aligned}[\varphi, \Gamma^*_{N,h}(f)]_N &= \frac{1}{h}\sum_{c \bmod h}\int_0^1 \varphi_{c+h[Nx/h]}\overline{f(c,y)}dy \\ &= \frac{1}{h}\sum_{c \bmod h}\sum_{\substack{n\leqslant N,\\ n \bmod h=c}} \varphi_n \int_{[\frac{n-c}{N'},\frac{n-c+h}{N'}]} \overline{f(c,y)}dy\end{aligned}$$

and thus, for any integer $n \leqslant N$, we deduce the following explicit expression:

$$\Gamma^*_{N,h}(f)(n) = \frac{N}{h}\int_{[\frac{n-\sigma_h(n)}{N'},\frac{n-\sigma_h(n)+h}{N'}]} f(\sigma_h(n), y)dy. \tag{65}$$

We conclude from $\frac{N}{N'}[\varphi,\psi]_N = \langle \Gamma_{N,h}(\varphi), \Gamma_{N,h}(\psi)\rangle = [\Gamma^*_{N,h}\Gamma_{N,h}\varphi, \psi]_N$ that

$$\Gamma^*_{N,h}\Gamma_{N,h} = \tfrac{N}{N'}\,\mathrm{Id}\,. \tag{66}$$

And some easy manipulations tell us that $\Gamma_{N,h}\Gamma^*_{N,h} = \frac{N}{N'}P_h$ where P_h is the orthogonal projector on $\mathrm{Im}\,\Gamma_{N,h} = \Gamma_{N,h}\big(L^2(\{1\cdots N\})\big)$.

Proof. Indeed, we find that, for any φ and ψ, we have

$$\begin{aligned}\langle \Gamma_{N,h}\Gamma^*_{N,h}\Gamma_{N,h}\Gamma^*_{N,h}\varphi, \psi\rangle &= [\Gamma^*_{N,h}\Gamma_{N,h}\Gamma^*_{N,h}\varphi, \Gamma^*_{N,h}\psi]_N \\ &= \tfrac{N}{N'}[\Gamma^*_{N,h}\varphi, \Gamma^*_{N,h}\psi]_N = \tfrac{N}{N'}[\Gamma_{N,h}\Gamma^*_{N,h}\varphi, \psi]_N.\end{aligned}$$

We conclude from these equalities that $(\Gamma_{N,h}\Gamma^*_{N,h})^2 = \frac{N}{N'}\Gamma_{N,h}\Gamma^*_{N,h}$. The conclusion is easy. □

Pure embeddings

It will be clear in a moment that, if $\Gamma_{N,h}(\varphi)$ is easier to grasp from a geometrical viewpoint, our object is in fact $\mathcal{R}_{N,h} = U_{\tilde{h}\to h}\circ\Gamma_{N,h}$ as already defined in (6), i.e. the orthonormal projection of $\Gamma_{N,h}$ on the space $L^2_*(X_h)$ (see section 11)[6]. We call the function $\mathcal{R}_{N,h}$ the *pure embedding*. From Theorem 11.2, we get

$$\mathcal{R}_{N,h}(\varphi)(b,y) = \frac{1}{h}\sum_{c \bmod h} c_h(b-c)\Gamma_{N,h}(\varphi)(c,y) \tag{67}$$

from which we readily compute that

$$\mathcal{R}^*_{N,h}(\varphi)(n) = \frac{1}{h^2}\sum_{b \bmod h} c_h(b-n)\int_{\frac{n-\sigma_h(n)}{N'}}^{\frac{n-\sigma_h(n)+h}{N'}} \varphi(b,z)dz. \tag{68}$$

[6] The choice of notation $\Gamma^*_{N,h}$ would lead to confusion since adjoints are present in the latter theory.

Note that

$$\|\mathcal{R}_{N,h}(\varphi)\|^2 = \frac{1}{h^2}\sum_{b \bmod h}\int_0^1\Big|\sum_{c \bmod h} c_h(b-c)\Gamma_{N,h}(\varphi)(c,y)\Big|^2 dy$$
$$= \frac{1}{h^2}\sum_{b \bmod^* h}\int_0^1\Big|\sum_{0\leqslant a<h}\Gamma_{N,h}(\varphi)(a,y)e(-ab/h)\Big|^2 dy. \quad (69)$$

Eq. (69) shows immediately (by extending the summation in b to all of $\mathbb{Z}/h\mathbb{Z}$) that $\|\mathcal{R}_{N,h}(\varphi)\| \leqslant \|\Gamma_{N,h}(\varphi)\|$, a fact that could have been more easily obtained by noticing that the norm of an orthogonal projection is surely not more than the initial norm. We can also get an explicit expression of $\|\mathcal{R}_{N,h}(\varphi)\|^2$ in terms of φ:

$$\|\mathcal{R}_{N,h}(\varphi)\|^2 = \frac{1}{h^2}\sum_{b \bmod^* h}\sum_{k\geqslant 0}\int_{kh/N'}^{(k+1)h/N'}\Big|\sum_{0\leqslant a<h}\varphi_{a+kh}e(-ba/h)\Big|^2 dy$$
$$= \frac{1}{hN'}\sum_{b \bmod^* h}\sum_{k\geqslant 0}\Big|\sum_{n/[n/h]=k}\varphi_n e(-bn/h)\Big|^2. \quad (70)$$

14 Theorem 1.2 in functional form

We start with an easy lemma.

Lemma 14.1. *We have*

$$S\Big(\varphi; \frac{a+\vartheta}{h}\Big) = \frac{N'}{h}\sum_{1\leqslant b\leqslant h}\int_0^1 \Gamma_{N,h}(\varphi)(b,y)e\Big(\frac{ab}{h}\Big)e\Big(\Big(\frac{b}{h}+[N'y/h]\Big)\vartheta\Big)dy.$$

Proof. When $m \equiv b[h]$ with $1 \leqslant b$, we have

$$\varphi_m = \frac{N'}{h}\int_{\frac{m-b}{N'}}^{\frac{m-b}{N'}+\frac{h}{N'}} \Gamma_{N,h}(\varphi)(b,y)dy.$$

It is straightforward to get the lemma from this expression. □

When $H \leqslant N^{1/8}(\log N)^{-3/2}$, $N \ll QH$ and $Q \leqslant N^2$ (this condition is only to control $\log Q$ in the error term. In practice, Q is not more than N, but we may want to select $Q = \text{constant} \times N$), we have the following.

$$\sum_q \frac{W(q/Q)}{qQI_0(W)}\sum_{a \bmod^* q}|S(\varphi, a/q)|^2 = \|\varphi\|_2^2\Big(1+\mathcal{O}\Big(\frac{N}{QH}\Big)\Big)$$
$$-\sum_{h\leqslant H}\sum_{1\leqslant b\leqslant h}\int_0^1\int_0^1 \frac{\mathcal{R}_{N,h}(\varphi)(b,y)\overline{\mathcal{R}_{N,h}(\varphi)(b,y')}}{h^2}\frac{W^\star\big(\tau\frac{y-y'}{h}\big)dydy'}{QI_0(W)/N^2}. \quad (71)$$

Remark 14.2. Most of the work below is to allow H to be a power of N. If one can control the continuity of $W^\star$, like under the Riemann Hypothesis, then the proof is much simpler. We instead rely heavily on the bilinear structure.

Proof. We start from (46) and Lemma 14.1 to get that:

$$\sum_{a \bmod{}^* h} \int_{-U}^{U} \hat{W}^{\star\star}_{C'}(u) \Big| S\Big(\varphi, \frac{a}{h} + \frac{u}{hQ}\Big)\Big|^2 du =$$
$$\frac{N'^2}{h^2} \sum_{1 \leqslant b_1, b_2 \leqslant h} \int_0^1 \int_0^1 \Gamma_{N,h}(\varphi)(b_1, y)\Gamma_{N,h}(\varphi)(b_2, y') \sum_{a \bmod{}^* h} e\Big(\frac{(b_1 - b_2)a}{h}\Big)$$
$$\int_{-U}^{U} \hat{W}^{\star\star}_{C'}(u) e\Big(\frac{b_1 - b_2}{h}\frac{u}{Q} + \Big([N'y/h] - [N'y'/h]\Big)\frac{u}{Q}\Big) du dy dy'. \quad (72)$$

In the inner integration, we replace

$$e\Big(\frac{b_1 - b_2}{h}\frac{u}{Q} + \Big([N'y/h] - [N'y'/h]\Big)\frac{u}{Q}\Big)$$

by $e((y - y')Nu/(hQ))$. We call $\Delta_h(b_1, b_2, y, y')$ the difference of the two, integrated against $\hat{W}^{\star\star}_{C'}(u)$. We have

$$\Delta_h(b_1, b_2, y, y') \ll \int_{-U}^{U} \Big|\hat{W}^{\star\star}_{C'}(u)\Big| \min(1, |u|/Q) du.$$

This gives rise to the error term

$$\frac{N'^2}{h^2} \sum_{1 \leqslant b_1, b_2 \leqslant h} c_h(b_1 - b_2) \int_0^1 \int_0^1 \Gamma_{N,h}(\varphi)(b_1, y)\Gamma_{N,h}(\varphi)(b_2, y')\Delta_h(b_1, b_2, y, y') dy dy'.$$

We get $\max|\Delta_h(b_1, b_2, y, y')|$ out, separate $\Gamma_{N,h}(\varphi)(b_1, y)$ from $\Gamma_{N,h}(\varphi)(b_2, y')$ by using $2|z_1 z_2| \leqslant |z_1|^2 + |z_2|^2$ and have to bound

$$\Sigma = \frac{N'^2}{h^2} \sum_{1 \leqslant b_1, b_2 \leqslant h} |c_h(b_1 - b_2)| \int_0^1 |\Gamma_{N,h}(\varphi)(b_1, y)|^2 dy \max_{y, y', b_1, b_2} |\Delta_h(b_1, b_2, y, y')|.$$

We use

$$\sum_{1 \leqslant b_2 \leqslant h} |c_h(b_1 - b_2)| = \sum_{1 \leqslant b \leqslant h} |c_h(b)| = \sum_{d|h} \sum_{\substack{b \bmod h, \\ \gcd(b,h) = h/d}} \mu^2(d)\frac{\phi(h)}{\phi(d)}$$
$$= \sum_{d|h} \mu^2(d)\phi(d)\frac{\phi(h)}{\phi(d)} = 2^{\omega(h)}\phi(h).$$

This and the isometrical property of Γ leads to

$$\Sigma = \frac{2^{\omega(h)}\phi(h)N'}{h} \|\varphi\|_2^2 \max_{b_1, b_2, y, y'} |\Delta_h(b_1, b_2, y, y')|.$$

Next by using Lemma 5.5, we check that $|\Delta_h(b_1, b_2, y, y')| \ll C'/Q$. The total error term is $\ll \sum_{h \leqslant H} (Qh)^{-1} C' 2^{\omega(h)} N^{3/2} \|\varphi\|_2^2 / Q \ll C' N^{3/2} \|\varphi\|_2^2 (\log H)^2 / Q^2$ which we call E_1. Thus we have reduced the right-hand side de (72) to

$$\frac{N'^2}{h^2} \sum_{1 \leqslant b_1, b_2 \leqslant h} \int_0^1 \int_0^1 \Gamma_{N,h}(\varphi)(b_1, y)\Gamma_{N,h}(\varphi)(b_2, y') c_h(b_1 - b_2)$$
$$\int_{-U}^{U} \hat{W}^{\star\star}_{C'}(u) e\Big((y - y')\frac{Nu}{hQ}\Big) du dy dy'. \quad (73)$$

By (67), this is also

$$\frac{N'^2}{h}\sum_{1\leqslant b_2\leqslant h}\int_0^1\int_0^1 \mathcal{R}_{N,h}(\varphi)(b_2,y)\overline{\Gamma_{N,h}(\varphi)(b_2,y')}$$
$$\int_{-U}^{U}\hat{W}^{\star\star}_{C'}(u)e\Big((y-y')\frac{Nu}{hQ}\Big)dudydy',$$

which, by orthogonality, is also

$$\frac{N'^2}{h}\sum_{1\leqslant b\leqslant h}\int_0^1\int_0^1 \mathcal{R}_{N,h}(\varphi)(b,y)\overline{\mathcal{R}_{N,h}(\varphi)(b,y')}$$
$$\int_{-U}^{U}\hat{W}^{\star\star}_{C'}(u)e\Big((y-y')\frac{Nu}{hQ}\Big)dudydy'.$$

We want to replace $\hat{W}^{\star\star}_{C'}(u)$ by $\hat{W}^{\star}$. We assume $U\leqslant C'/2$, hence $\hat{W}^{\star}(u)=\hat{W}^{**}_{C'}(u)+\text{Constant}$ when $|u|\leqslant U$ and this constant is $\mathcal{O}(1/C')$. Again using $2|z_1z_2|\leqslant|z_1|^2+|z_2|^2$ on $\mathcal{R}_{N,h}(\varphi)$, and noting that (with $s=NU/hQ$)

$$\int_0^1\frac{\sin((y'-y)s)}{(y'-y)s}dy=\int_{y's-y}^{y's}\frac{\sin x}{x}dx\ll 1$$

uniformly in s and y', we get an error term of size $\mathcal{O}((\log H)N'\|\varphi\|_2^2/(QC'))$. We finally want to extend the path of integration in u to infinity. Again using $2|z_1z_2|\leqslant|z_1|^2+|z_2|^2$, this means bounding

$$A=\int_0^1\Big|\int_U^\infty\hat{W}^{\star}(u)e\Big((y-y')\frac{Nu}{hQ}\Big)du\Big|dy$$

and similarly with y'. We employ Cauchy's inequality and open the square, getting:

$$A^2\ll\int_U^\infty\int_U^\infty\hat{W}^{\star}(u_1)\overline{\hat{W}^{\star}(u_2)}\int_0^1 e\Big((y-y')\frac{N(u_1-u_2)}{hQ}\Big)du.$$

We employ Lemma 5.8 on u_1 and u_2. When $|u_1-u_2|\leqslant 1$, we get the contribution $\mathcal{O}(1/U)$; When $|u_1-u_2|\geqslant 1$, we integrate in y and get the contribution

$$\int_U^\infty\int_U^\infty\frac{du_1du_2}{u_1u_2(1+|u_1-u_2|)}.$$

On splitting the path of integration on u_2 in $[U,\max(U,u_1/2)]$, followed by $[\max(U,u_1/2),2u_1]$ and finally by $[2u_1,\infty)$, we readily see that this integral is $\mathcal{O}((\log U)/U)$. Summing over h gives the contribution

$$\frac{N}{QI_0(W)}\sum_{h\leqslant H}\frac{1}{h}\|\varphi\|_2^2\sqrt{\frac{hQ(\log U)}{NU}}\ll\frac{\sqrt{NH\log U}}{\sqrt{UQ}}\|\varphi\|_2^2=E_2.$$

In total, we get the error term bounded above by a constant multiple of

$$\Big(\frac{N}{QH}+\frac{N\log H}{QC'}+\sqrt{\frac{NH\log U}{UQ}}+\frac{C'N^{3/2}}{Q^2}(\log H)^2+\frac{H^2}{U}+\frac{H^2+(\log N)^5}{C}\Big)\|\varphi\|_2^2.$$

It is best to take U as large as possible, so we select $U = C'/2$. In turn, we select $C' = QH^{1/3}/N^{2/3}$ and we check that $C' = C$ (see (42)). The error term becomes not more than a constant multiple times

$$\left(\frac{N}{QH} + \frac{N^{5/3}\log H}{Q^2H^{1/3}} + \frac{N^{5/6}H^{1/3}}{Q}(\log N)^2 + \frac{H^2+(\log N)^5}{QH^{1/3}}N^{2/3}\right)\|\varphi\|_2^2.$$

We then check that this reduces to

$$\left(\frac{N}{QH} + \frac{N}{QH}\frac{NH^{5/3}}{QHN^{1/3}} + \frac{N^{5/6}H^{1/3}}{Q}(\log N)^2\right)\|\varphi\|_2^2$$

when $H \leqslant N^{1/8}$. And we check further that $N^{5/6}H^{1/3}Q^{-1}(\log N)^2 \ll N/(QH)$ when $H \leqslant N^{1/8}(\log N)^{-3/2}$. The second term equally disappears, as $N \ll QH$. □

Hervé Queffélec has kindly pointed out to me that when $q = 1$, this process bears similarities with the one devised independently by [41] and [46], and which is nicely presented in [5, Section 3].

On recalling the definition of the operator $\mathscr{V}_{\tau,h}$ in (7), here is another manner of writing (71):

$$\begin{aligned}\sum_q \frac{W(q/Q)}{qQ}\sum_{a \bmod^* q}|S(\varphi,a/q)|^2 &= I_0(W)\|\varphi\|_2^2(1+\mathcal{O}(\tau/H))\\ &\quad - N\sum_{h\leqslant H}\frac{\tau}{h}\big[\mathcal{R}_{N,h}(\varphi)|\mathscr{V}_{\tau,h}\,\mathcal{R}_{N,h}(\varphi)\big]_{h\times[0,1]}\\ &\qquad (H \ll N^{1/8}(\log N)^{-3/2}, N \ll QH, Q \ll N^2). \quad (74)\end{aligned}$$

15 Using spectral analysis

Formula (74) involves the operators $\mathscr{V}_{\tau,h} \circ U_{\tilde{h}\to h}$. In this section, we first diagonolize them as local operators (i.e. on a space that depends on h), and control the dependance in h and τ. We then lift this diagonalization to the global space (where the sequence φ lives) and show that the resulting family of eigenvectors, h varying, is near-orthonormal (see Lemma 15.3). We encounter a problem (that may be only technical): the control we have of the modulus of continuity of these eigenfunctions is weak when they are associated with very small eigenvalues. But then, their total contribution is small, and we then introduce a trade-off point with the condition $|\lambda_{h,\ell}| \geqslant \xi\eta_0(N)^{1/8}$. We conclude this part with another consequence of the near-orthonormality which enables us to control the quadratic form resulting from taking some upper bound for the eigenvalues. This is required because, when using (80) to simplify our statement, the near-orthogonality is not apparent anymore.

15.1 Decomposing the implied operators

The operator $\mathscr{V}_{\tau,h}$ does not touch the b-variable, from which we infer that (recall the definition of the rothonormal projector $U_{\tilde{h}\to h}$ in (49))

$$U_{\tilde{h}\to h} \circ \mathscr{V}_{\tau,h} = \mathscr{V}_{\tau,h} \circ U_{\tilde{h}\to h}.$$

This has two consequences: first the image of $\mathscr{V}_{\tau,h}$ lies inside $L^2_*(X_h)$ and second, its couples eigenvalues / eigenvectors are simply (tensor) products of the respective couples coming from the two operators:

$$F \in L^2(\mathbb{Z}/h\mathbb{Z}) \mapsto \left(b \mapsto \frac{1}{h} \sum_{c \bmod h} c_h(b-c)F(c)\right)$$

where the only difference with the operators $U_{\tilde{h}\to h}$ and $\mathscr{V}_{\tau,h}$ are the spaces. The first operator is covered by Theorem 11.2. We are left with the second one which belongs to the class described in Section 12 (if we ignore the first variable, as we may). The regularity assumptions (R_1), (R_2) and (R_3) are met by Lemma 5.9.

15.2 Diagonalisation in the local spaces

We use the eigenvectors / eigenvalues $(G_{h,\ell,\chi}, \lambda_{h,\ell})_{\chi,\ell}$ of $\mathscr{V}_{\tau,h}$ as well as the ones of $\mathcal{R}_{\tau,h}$ (see Theorem 11.2) to write

$$\left[\mathcal{R}_{N,h}(\varphi)|\mathscr{V}_{\tau,h}\,\mathcal{R}_{N,h}(\varphi)\right]_{h\times[0,1]} = \sum_{\ell\geqslant 1} \lambda_{h,\ell} \sum_{\chi \bmod h} \left[\mathcal{R}_{N,h}(\varphi)|\,\mathscr{E}_{h,\chi}\otimes G_{h,\ell}\right]^2_{h\times[0,1]}.$$

We then divide this quantity by h and sum that over h. Before proceeding, let us note the following lemma.

Lemma 15.1.

$$\int_0^1 \left|W^\star\left(\frac{\tau z}{h}\right)\right|^2 dz \ll \begin{cases} \exp -2c_0\sqrt{\log(2+h/\tau)}, \\ h/\tau. \end{cases}$$

We will use the latter when $h \leqslant 2\tau$ and the former otherwise. It is however better for questions of uniformly to state them in general

Proof. When $h \geqslant 2\tau$, we use Lemma 5.9 and bound the value $|W^\star(\tau z/h)|$ by $\mathcal{O}(\exp -c_0\sqrt{\log(h/\tau)})$. When $h \leqslant 2\tau$, we use

$$\int_0^1 W^\star\left(\frac{\tau z}{h}\right)^2 dz \leqslant 2\int_0^1 W^\star\left(\frac{\tau w}{h}\right)^2 dw \leqslant \frac{2h}{\tau}\int_0^{\tau/h} W^\star(w)^2 dw \ll h/\tau.$$

The lemma is proved. □

Since $|\lambda_{h,\ell}| \ll 1/\ell$ by (60) and Lemma 15.1, we can explicitly shorten the spectral decomposition in (recall also Lemma 13.1)

$$\left[\mathcal{R}_{N,h}(\varphi)|\mathscr{V}_{\tau,h}\,\mathcal{R}_{N,h}(\varphi)\right]_{h\times[0,1]} = \\ \sum_{\ell\leqslant L} \lambda_{h,\ell} \sum_{\chi \bmod h} \left[\mathcal{R}_{N,h}(\varphi)|\,\mathscr{E}_{h,\chi}\otimes G_{h,\ell}\right]^2_{h\times[0,1]} + \mathcal{O}(N^{-1}\|\varphi\|_2^2/L).$$

We can similarly restrict the summation to $|\lambda_{h,\ell}| \geqslant \eta_0(N)^{1/4}$ (with $\eta_0(x) = \exp -\frac{c_0}{2}\sqrt{\log x}$) and get, for any $\xi \in [0,1]$:

$$\left[\mathcal{R}_{N,h}(\varphi)|\mathscr{V}_{\tau,h}\,\mathcal{R}_{N,h}(\varphi)\right]_{h\times[0,1]} = \sum_{\substack{\ell\leqslant L,\\ |\lambda_{h,\ell}|\geqslant\xi\eta_0(N)^{1/4}}} \lambda_{h,\ell} \sum_{\chi \bmod h} \left[\mathcal{R}_{N,h}(\varphi)|\,\mathscr{E}_{h,\chi}\otimes G_{h,\ell}\right]^2_{h\times[0,1]} + \mathcal{O}\big(N^{-1}\|\varphi\|_2^2(\eta_0(N)^{1/4}+1/L)\big).$$

The parameter ξ is here for flexibility, in case we want the sum not to depend on the parameter N. We may rewrite formula (74) by introducing the adjoint $\mathcal{R}^*_{N,h}$ of $\mathcal{R}_{N,h}$, as follows.

$$\sum_q \frac{W(q/Q)}{qQ} \sum_{a \bmod^* q} |S(\varphi,a/q)|^2 = I_0(W)\|\varphi\|_2^2 - N\sum_{h\leqslant H}\frac{\tau}{h} \sum_{\substack{\ell\leqslant L,\\ |\lambda_{h,\ell}|\geqslant\xi\eta_0(N)^{1/4}}} \lambda_{h,\ell} \sum_{\chi \bmod h} \left[\varphi|\,\mathcal{R}^*_{N,h}\,\mathscr{E}_{h,\chi}\otimes G_{h,\ell}\right]^2_N + \mathcal{O}\left(\left(\frac{\xi\log H}{\exp\frac{c_0}{8}\sqrt{\log N}} + \frac{\log H}{L} + \frac{1}{H}\right)\tau\|\varphi\|_2^2\right). \tag{75}$$

Our task is now to replace $\mathcal{R}^*_{N,h}\,\mathscr{E}_{h,\chi}\otimes G_{h,\ell}$ by a simpler expression.

15.3 Approximate diagonalization in the global space

We define

$$g_{h,\ell,\chi,N,\tau} = \mathcal{R}^*_{N,h}\,\mathscr{E}_{h,\chi}\otimes G_{h,\ell} = \Gamma^*_{N,h}\,\mathscr{E}_{h,\chi}\otimes G_{h,\ell}, \tag{76}$$

as well as

$$g^\flat_{h,\ell,\chi,N,\tau}(n) = \frac{\tau_h(\chi,n)}{\sqrt{\phi(h)}}G_{h,\ell}\Big(\frac{n}{N}\Big). \tag{77}$$

The function $g_{h,\ell,\chi,N,\tau}$ inherits from $\mathscr{E}_{h,\chi}\otimes G_{h,\ell}$ a similar separation of behaviour between arithmetic and size characters.

Lemma 15.2. *When $|t-n| \leqslant N^{1/2}$ and $Q \leqslant N$, we have*

$$g_{h,\ell,\chi,N,\tau}(n) = \frac{\tau_h(\chi,n)}{\sqrt{\phi(h)}}G_{h,\ell}\Big(\frac{t}{N}\Big) + \mathcal{O}\bigg(\frac{\sqrt{\phi(h)}}{|\lambda_{h,\ell}(\tau)|}\exp\Big(-\frac{c_0}{2}\sqrt{\log N}\Big)\bigg)$$

where c_0 is defined in Lemma 5.9. Moreover, we have

$$|\lambda_{h,\ell}(\tau)|\|g_{h,\ell,\chi,N,\tau}\|_\infty \ll \sqrt{\phi(h)}.$$

In particular, $g^\flat_{h,\ell,\chi,N,\tau}$ approximates $g_{h,\ell,\chi,N,\tau}$.

Proof. We have by (65):

$$g_{h,\ell,\chi,N,\tau}(n) = \frac{N}{h}\frac{1}{\sqrt{\phi(h)}}\tau_h(\chi,n)\int_{\frac{n-\sigma_h(n)}{N'}}^{\frac{n-\sigma_h(n)+h}{N'}} G_{h,\ell}(y)dy.$$

We next use (55) together with Lemma 5.9 to infer that, when $\delta \in [0,1]$, we have, for any $y \in [0, 1-\delta]$,

$$|\lambda_{h,\ell}(\tau)||G_{h,\ell}(y+\delta) - G_{h,\ell}(y)| \ll \exp -c_0\sqrt{-\log\min\Big(1, \frac{\tau\delta}{h}\Big)}. \tag{78}$$

We note that $\tau \leqslant 1$ and that $h \geqslant 1$. Hence, for any t such that $|t-n| \leqslant \sqrt{N}$, we have

$$g_{h,\ell,\chi,N,\tau}(n) = \tau_h(\chi,n)\frac{N}{\sqrt{\phi(h)}N'}G_{h,\ell}\Big(\frac{t}{N'}\Big) + \mathcal{O}\Big(\frac{\tau_h(\chi,n)}{|\lambda_{h,\ell}(\tau)|\sqrt{\phi(h)}}\exp\Big(-\frac{c_0}{4}\sqrt{\log N}\Big)\Big)$$

from which the stated estimate readily follows, up to two blemishes: the factor $N/N' = 1 + \mathcal{O}(N^{-1/2})$ and the $G(t/N')$ instead of $G(t/N)$. This last modification follows from (78), the former one being trivial. For the L^∞-norm, note that (see (54))

$$\|g_{h,\ell,\chi,N,\tau}\|_\infty \leqslant 2\|W^\star(\tau\cdot/h)\|_1\sqrt{\phi(h)}/|\lambda_{h,\ell}(\tau)|.$$

□

Lemma 15.3. *When $h, h' \leqslant H \leqslant N^{1/5}$ and $N' \leqslant N + \sqrt{N}$, we have*

$$[g_{h,\ell,\chi,N,\tau}, g_{h',\ell',\chi',N,\tau}]_N = \delta_{h=h'}\delta_{\ell=\ell'}\delta_{\chi=\chi'} + \mathcal{O}\Big(\frac{1}{\sqrt{N}} + \frac{H\exp\big(-\frac{c_0}{4}\sqrt{\log N}\big)}{|\lambda_{h,\ell}(\tau)||\lambda_{h'\ell'}(\tau)|}\Big)$$

where c_0 is defined in Lemma 5.9. The same applies when replacing $g_{h,\ell,\chi,N,\tau}$ and $g_{h',\ell',\chi',N,\tau}$ respectively by $g^\flat_{h,\ell,\chi,N,\tau}$ and $g^\flat_{h',\ell',\chi',N,\tau}$.

Proof. In order to compute $[g_{h,\ell,\chi,N,\tau}, g_{h',\ell',\chi',N,\tau}]_N$, we split the interval $[1, N]$ in $\mathcal{O}(N/(hh'))$ sub-intervals containing hh' consecutive integers and a remaining one. We employ Lemma 15.2 on each sub-interval, selecting a t that is independent on the point n, for instance choosing it at the origin of such a segment, but we shall use the freedom on choice in t to shorten the argument below. We bound the L^∞-norm of the other factor by Lemma 15.2. The error term for each interval is

$$\ll hh'\frac{\max(\|W^\star(\tau\cdot/h)\|_1, \|W^\star(\tau\cdot/h')\|_1)\sqrt{\phi(h')\phi(h)}}{|\lambda_{h',\ell'}(\tau)||\lambda_{h,\ell}(\tau)|}\exp\Big(-\frac{c_0}{4}\sqrt{\log N}\Big)$$

which we have to sum over all intervals and divide by N (since the scalar product $[,]_N$ is scaled in this manner). The total error term incurred is thus

$$\frac{H\max(\|W^\star(\tau\cdot/h)\|_1, \|W^\star(\tau\cdot/h')\|_1)}{|\lambda_{h',\ell'}(\tau)||\lambda_{h,\ell}(\tau)|}\Big(\frac{H^2}{N} + \exp\Big(-\frac{c_0}{4}\sqrt{\log N}\Big)\Big).$$

The summand H^2/N comes from the end interval. Concerning this end interval, we should have had $\|W^\star(\tau\cdot/h')\|_1 \cdot \|W^\star(\tau\cdot/h)\|_1$ rather than the maximum, but each norm is bounded (uniformly in τ), which legitimates the bound above.

Whenever $h \neq h'$ or $\chi \neq \chi'$, the summation over the remaining intervals vanishes by orthogonality. We are left with the case when $h = h'$ and $\chi = \chi'$, in which case we have to evaluate

$$\frac{1}{N}\sum_{n \bmod h^2}\frac{|\tau_h(\chi,n)|^2}{\phi(h)}\sum_t G_{h,\ell}\Big(\frac{t}{N}\Big)\overline{G_{h,\ell'}\Big(\frac{t}{N'}\Big)}.$$

The sum upon n is $h^2\|\mathscr{E}_{h,\chi}\|_2^2 = h^2$. Concerning the sum upon t, we employ the following trick: given any interval we can use any t from within, hence we can integrate over t and divide by the length h^2 of the interval. Concerning the final interval, the reader will check that the contribution to include it is not more than what we already paid for discarding it. As a result, we get as a main term

$$\int_0^1 G_{h,\ell}(u)\overline{G_{h,\ell'}(u)}du$$

which is $\delta_{\ell=\ell'}$. □

15.4 External control of the eigenvectors

Let us recall an inequality due to Selberg (given in [4, Proposition 1] or in extended form in [35, Lemma 1.1-1.2]).

Lemma 15.4. *Let $(g_i)_{i\in I}$ be a finite family of vectors in the Hilbert space $\mathcal{H}$, and f be some fixed vector in this same space. We have*

$$\sum_{i\in I}|[f|g_i]|^2/\sum_{j\in I}|[g_i|g_j]| \leqslant \|f\|^2.$$

We apply Lemma 15.4 to the family

$$\{g^\flat_{h,\ell,\chi,N,\tau} : h \leqslant H, \chi \bmod h, \ell \leqslant L, |\lambda_{h,\ell}(\tau)| \geqslant \eta_0(N)^{1/4}\}.$$

By Lemma 15.3, we infer that

$$N\sum_{\substack{h\leqslant H\\ |\lambda_{h,\ell}|\geqslant \exp -\frac{c_0}{8}\sqrt{\log N}}}\sum_{\ell\leqslant L}\sum_{\chi \bmod h}[\varphi|g^\flat_{h,\ell,\chi,N,\tau}]_N^2 \leqslant \|\varphi\|_2^2\Big(1 + H^2L\exp -\frac{c_0}{8}\sqrt{\log N}\Big). \quad (79)$$

Finally we use the identity:

$$\sum_{\chi \bmod h}\left|\sum_{n\leqslant N}\varphi(n)\frac{\tau_h(\chi,n)}{\sqrt{\phi(h)}}G\Big(\frac{n}{N}\Big)\right|^2 = \sum_{a \bmod {*h}}\left|\sum_{n\leqslant n}\varphi(n)e(na/h)G\Big(\frac{n}{N}\Big)\right|^2. \quad (80)$$

16 Deducing Theorem 1.6 and 1.1

16.1 Proof of Theorem 1.6

The spectral decomposition is treated in Subsection 15.2. The family $g_{h,\ell,\chi,N,\tau}$ is defined in the next subsection at (76) and its near orthonormal property in proved in Lemma 15.3. The global decomposition is given in (75) once $\mathcal{R}^*_{N,h}\mathscr{E}_{h,\chi}\otimes G_{h,\ell}$ is replaced by $g_{h,\ell,\chi,N,\tau}$ and the relative sizes are taken into account. The final property is in (79).

Note that, for each h, we have at a positive and a negative eigenvalue. Recalling (9), we see that $\max_\ell |\lambda_{h,\ell}(\tau)|$ goes to zero. Hence these positive or negative values of $\lambda_{h,\ell}(\tau)$ cannot be the same one save for finitely many h's. This is how we prove that infinitely many of them are positive (resp. negative).

16.2 Proof of Theorem 1.1

To prove Theorem 1.1, we first introduce a smooth non-negative function W verifying (W_1), (W_2) and (W_3) stated in the introduction and write

$$\sum_{1<q/Q\leqslant 2}\sum_{a \bmod^* q}|S(\varphi,a/q)|^2 \geqslant \sum_{q\geqslant 1}\frac{W(q/Q)}{q}\sum_{a \bmod^* q}|S(\varphi,a/q)|^2.$$

We then use Theorem 1.2. Theorem 1.6 is our next step, with $\xi = 1$. We select $H = L = \exp c\sqrt{\log N}\tau$ for some small but positive c. Given $h \leqslant H$, we may first employ the first statement of Theorem 12.6 together with (1) and (3) to get that $\frac{\tau}{h}\lambda_{h,\ell} \leqslant I_0(W) + \mathcal{O}(1/\sqrt{Q})$. This already ensures us that

$$N\sum_{h\leqslant H}\frac{\tau}{h}\sum_{\substack{\ell\leqslant L,\\ |\lambda_{h,\ell}|\geqslant \xi \exp -c_3\sqrt{\log N}}}\lambda_{h,\ell}\sum_{a \bmod^* q}|S(\varphi,a/q)|^2$$
$$\leqslant I_0(W)N\|\varphi\|_2^2\Big(1 + H^2L\exp -c_3\sqrt{\log N} + Q^{-1/2}\Big).$$

This is not quite enough. The full strength of Theorem 1.6 uses the non-negativity of W. We employ this theorem with $U_2 = \tau/h$, and this gives us that
$$\frac{\tau}{h}\lambda_{h,\ell} \leqslant I_0(W)(1 - ce^{-c_4\tau/h}) + \mathcal{O}(1/\sqrt{Q}).$$
Theorem 1.1 readily follows.

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