AN EXPLICIT UPPER BOUND FOR $L(1,\chi)$ WHEN χ IS QUADRATIC

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Abstract. We consider Dirichlet L-functions $L(s,\chi)$ where χ is a non-principal quadratic character to the modulus q. We make explicit a result due to Pintz and Stephens by showing that $|L(1,\chi)| \leqslant \frac{1}{2}\log q$ for all $q \geqslant 2\cdot 10^{23}$ and $|L(1,\chi)| \leqslant \frac{9}{20}\log q$ for all $q \geqslant 5\cdot 10^{50}$.

1. Introduction and results

A central problem in number theory concerns estimates on $L(1,\chi)$, where χ is a non-principal Dirichlet character to the modulus q, and where $L(s,\chi)$ is its associated Dirichlet L-function. Bounding sums of $\chi(n)$ trivially leads to the bound $|L(1,\chi)| \leqslant \log q + O(1)$. The Pólya–Vinogradov inequality allows one to improve this to $(1/2)\log q + O(1)$. An interesting history of these developments is given by Pintz [27].

Explicit versions of the above results date back to Hua [9]. See also work by Louboutin [21] and the second author [31, 32] for finding small pairs c, q_0 such that $L(1,\chi) \leqslant (1/2)\log q + c$ for all $q \geqslant q_0$. It appears difficult to improve on these bounds for generic q.

When q is prime, the best result is due to Stephens [36], namely that $|L(1,\chi)| \leqslant \frac{1}{2}(1 - e^{-1/2} + o(1))\log q$, where $\frac{1}{2}(1 - e^{-1/2}) = 0.1967\ldots$. This result has been extended to arbitrary moduli by Pintz in [28, 27]. We aim at making the Pintz–Stephens result partially explicit in the following theorems.

Theorem 1. *Let χ be a quadratic odd primitive Dirichlet character modulo $q \geqslant 2\cdot 10^{23}$. We have $L(1,\chi) \leqslant (\log q)/2$.*

For even characters, this is proved for $q \geqslant 2$ in [31] after several papers by Louboutin, the last of which is [21]. Bounds relying on additional constraints on the characters at the small primes have been investigated by Louboutin in [22], by the second author in [32], by Saad Eddin in [35] and by Platt and Saad Eddin in [29]. On taking q to be larger, we can improve on the factor $1/2$ in Theorem 1.

Theorem 2. *Let χ be a quadratic primitive Dirichlet character modulo q. The inequality $L(1,\chi) \leqslant (9/40)\log q$ holds true when χ is even and $q \geqslant 2\cdot 10^{49}$ or χ is odd and $q \geqslant 5\cdot 10^{50}$.*

We note that, on the Generalized Riemann hypothesis much more is known. Littlewood [20] showed that $L(1,\chi) \ll \log\log q$. This has been made explicit for large q in [14] by Lamzouri, Li and Soundararajan, and then for all q in [16] by Languasco and the third author. Finally, although we do not consider lower bounds on $L(1,\chi)$, we direct the reader to a survey of explicit and inexplicit bounds of Mossinghoff, Starichkova and the third author in [24], and to the recent work [15].

The outline of this paper is follows. In §2 we collect the necessary explicit results on character sums. In §4 we prepare the technical preliminaries to Stephens'

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approach, and analyse these in §5. Our §6 is purely centred on the optimization in (an improved version of) Stephens' method, and contains no number-theoretic input. Finally, in §7 we prove Theorems 1 and 2.

We use the notation $f(x) = \mathcal{O}^*(g(x))$ to mean that $f(x) \leqslant |g(x)|$ for the range of x considered. We also make use of the following notation. We define

$$h(\chi, y) = \sum_{n \leqslant H^y} \frac{\chi(n)\Lambda(n)}{n}, \quad h(1, y) = \sum_{n \leqslant H^y} \frac{\Lambda(n)}{n} \tag{1}$$

as well as

$$f(x) = \sum_{n \leqslant H^x} \frac{\chi(n)}{H^x}, \quad F(x) = \int_0^x f(t)dt = \sum_{n \leqslant H^x} \frac{\chi(n)}{n \log H} - \frac{f(x)}{\log H}. \tag{2}$$

Our aim is to majorize $F(1)$. We further define

$$\ell(y) = H^{-y} \sum_{n \leqslant H^y} \chi(n) \log n. \tag{3}$$

It is also convenient to introduce the points

$$x_m = 1 - \frac{\log m}{\log H}. \tag{4}$$

2. Preliminary results

We now list a trivial result that follows immediately from partial summation.

Lemma 3. *When $x \geqslant 0$, we have $\sum_{n \leqslant x} 1/\sqrt{n} \leqslant 2\sqrt{x}$.*

The following result is slightly more subtle.

Lemma 4. *When $x \geqslant y \geqslant 1$, we have $\sum_{y \leqslant n \leqslant x} 1/n \leqslant 1 + \log(x/y)$.*

Proof. Using Euler–Maclaurin summation one can show that

$$\sum_{n \leqslant x} \frac{1}{n} = \log x + \gamma + \frac{\left(\frac{1}{2} - \{x\}\right)}{x} + \mathcal{O}^*\left(\frac{1}{8x^2}\right),$$

whence

$$\sum_{y \leqslant n \leqslant x} \frac{1}{n} = \log(x/y) + \mathcal{O}^*\left(\frac{1}{2x} + \frac{1}{2y} + \frac{1}{8x^2} + \frac{1}{8y^2}\right). \tag{5}$$

The lemma is clearly true when $x = 1$. Therefore, for $x \geqslant 2$ and $y \geqslant 1$ we have, by (5) that $\sum_{y \leqslant n \leqslant x} n^{-1} - \log(x/y) = \mathcal{O}^*(29/32)$, and we are done. □

We now list some bounds related to the prime number theorem. The first is (a simplification of) a classical result from Rosser and Schoenfeld, see [34, Thm 12].

Lemma 5. *When $x \geqslant 0$, we have $\psi(x) \leqslant 1.04\, x$.*

We note that the result of Rosser and Schoenfeld gives 1.03883 in Lemma 5, which is an approximation to $\psi(113)/113$. To improve the bound in Lemma 5 it would be necessary to take $x \geqslant x_0 > 113$, which, while possible, would complicate greatly the ensuing analysis for only a marginal improvement.

The second is an explicit bound of the form $\psi(x) - x = o(x)$ coming from [3, Table 15] by Broadbent, Kadiri, Lumley, Ng, and Wilk.

Lemma 6. *When $x \geqslant 10^5$, we have $|\psi(x) - x| \leqslant 0.64673\, x/(\log x)^2$.*

On the Riemann hypothesis we have $\psi(x) - x = O(x^{1/2+\epsilon})$. The following result, from [5, Thm 2] of Büthe, gives an explicit version of an even sharper bound for a finite range.

Lemma 7. *When* $11 < x \leqslant 10^{19}$*, we have* $|\psi(x) - x| \leqslant 0.94\sqrt{x}$.

We remark that slightly weaker versions of Lemma 7, but ones that hold in a longer range of x have been provided by the first author in [11]. We require the following result to be used in tandem with Lemma 7.

Lemma 8. *When* $e^{40} \leqslant x$*, we have* $|\psi(x) - x| \leqslant 1.994 \cdot 10^{-8}\, x$.

This is obtained directly from [3, Table 8]. The key feature here is that $e^{40} < 10^{19}$ so that Lemma 7 and Lemma 8 between them cover all values of $x > 11$. Better results are available when x is very large, say $\log x \geqslant 1000$ — see [30] by Platt and Trudgian, and [12] by the first author and Yang — but Lemmas 7 and 8 suffice for our needs.

We now turn to estimates on $\tilde{\psi}(u) := \sum_{n\leqslant u} \Lambda(n)/n$ to aid in the evaluation of $h(\chi, y)$ and $h(1, y)$ in (1). To obtain such estimates we correct a result of the second author in [33].

Lemma 9. *For* $x \geqslant 71$ *we have*

$$\sum_{n\leqslant x} \frac{\Lambda(n)}{n} = \log x - \gamma + \frac{\psi(x) - x}{x} + \frac{0.047}{\sqrt{x}} + \frac{\log(2\pi) + 10^{-4}}{x} + E(x),$$

where

$$E(x) = \begin{cases} 1.75 \cdot 10^{-12}, & 1 \leqslant x < 2R\log^2 T_0 \\ \frac{1+2\sqrt{(\log x)/R}}{2\pi} \exp(-2\sqrt{(\log x)/R}), & x \geqslant 2R\log^2 T_0, \end{cases}$$

with[1] $R = 5.69693$ *and* $T_0 = 2.44 \cdot 10^{12}$.

Proof. As discussed by Chirre, Simonič, and Valås Hagenin in [6], by fixing a couple of small typos, Lemma 2.2 in [33] can be replaced by

$$\sum_{n\leqslant x} \frac{\Lambda(n)}{n} = \log x - \gamma + \frac{\psi(x) - x}{x} - \sum_{\rho} \frac{x^{\rho-1}}{\rho(\rho - 1)} + \int_x^\infty \frac{\log 2\pi + \frac{1}{2}\log(1 - u^{-2})}{u^2} \mathrm{d}u.$$

Following [33, §5], we have

$$\left| \sum_{\rho} \frac{x^{\rho-1}}{\rho(\rho - 1)} \right| \leqslant \frac{0.047}{\sqrt{x}} + E(x).$$

Finally, since $x \geqslant 71$,

$$\left| \int_x^\infty \frac{\log 2\pi + \frac{1}{2}\log(1 - u^{-2})}{u^2} \mathrm{d}u \right| \leqslant \frac{\log(2\pi)}{x} + \frac{|\log(1 - 71^{-2})|}{2x} \leqslant \frac{\log(2\pi) + 10^{-4}}{x}.$$ □

Lemma 10. *We have*

$$\sum_{n\leqslant x} \Lambda(n)/n = \log x - \gamma + \mathcal{O}^*(1.3/\log^2 x), \qquad x > 1, \tag{6}$$

$$\sum_{n\leqslant x} \Lambda(n)/n = \log x - \gamma + \mathcal{O}^*(1/\sqrt{x}), \qquad 1 \leqslant x \leqslant 10^{19}. \tag{7}$$

[1] The value of R comes from work by Kadiri [13] on the classical zero-free region for the zeta-function. This can be lowered using more recent results [25] and [26], respectively by Mossinghoff and Trudgian and by Mossinghoff, Trudgian and Yang, but it is inconsequential for our purposes.

Proof. Using Lemma 9 with the bounds from Lemmas 6 and 7, we obtain that,

$$\sum_{n\leqslant x}\frac{\Lambda(n)}{n}=\log x-\gamma+O^*\left(\frac{0.67}{\log^2 x}\right)$$

for $x\geqslant 10^5$, and

$$\sum_{n\leqslant x}\frac{\Lambda(n)}{n}=\log x-\gamma+O^*\left(\frac{1}{\sqrt{x}}\right)$$

for $10^5\leqslant x\leqslant 10^{19}$. We then extend these estimates to smaller values of x by direct computation, giving (6) and (7). □

An immediate consequence of this result is as follows.

Lemma 11. *We have*

$$\sum_{n\leqslant x}\Lambda(n)/n\leqslant \log x-0.545,\qquad x\geqslant 10^3,$$
$$\sum_{n\leqslant x}\Lambda(n)/n\geqslant \log x-0.576,\qquad x\geqslant 10^6.$$

We now examine the weighted average of $\sum_{n\leqslant u}\Lambda(n)/n$.

Lemma 12. *We have*

$$\int_1^\infty\left|\sum_{n\leqslant u}\frac{\Lambda(n)}{n}-\log u+\gamma\right|\frac{du}{u}\leqslant 0.411.$$

This integral may be of interest in its own right. While the true value of this integral seems close to 0.41, we have no idea of the conjectured limiting value of the integral. To this end, see a similar problem discussed in [2].

Proof. We define $\Delta(u)=\sum_{n\leqslant u}\Lambda(n)/n-\log u+\gamma$. When the variable u is small, we compute directly by using the fact that $\tilde\psi(u)$ is constant on $[n,n+1)$ and that, with $\tau=\tilde\psi(n)+\gamma$, the integral $\int_n^{n+1}|\Delta(u)|du/u$ is equal to

$$\begin{cases}\frac{\log^2(n+1)-\log^2 n}{2}-\tau\log\frac{n+1}{n} & \text{when } \tau\leqslant\log n,\\ \frac{\log^2(n+1)+\log^2 n-2\tau^2}{2}+\tau(2\tau-\log(n^2+n) & \text{when } \log n<\tau<\log(n+1),\\ -\frac{\log^2(n+1)-\log^2 n}{2}+\tau\log\frac{n+1}{n} & \text{when } \tau\geqslant\log(n+1).\end{cases}$$

The second case is treated by splitting the integral at $u=e^\tau$. We compute in this manner that

$$\int_1^{10^6}|\Delta(u)|\frac{du}{u}\leqslant 0.408.$$

We use Lemma 10 to infer that

$$\int_{10^6}^{10^{19}}|\Delta(u)|\frac{du}{u}\leqslant\int_{10^6}^{10^{19}}\frac{1}{u^{3/2}}du\leqslant\frac{2}{\sqrt{10^6}}=\frac{2}{1000}=0.002.$$

We now use Lemma 8 and Lemma 9 to show that, for some $x_1\geqslant 10^{19}$,

$$\int_{10^{19}}^{x_1}|\Delta(u)|\frac{du}{u}\leqslant\int_{10^{19}}^{x_1}\left(\frac{2\cdot 10^{-8}}{u}+\frac{0.05}{u^{3/2}}\right)du$$
$$=2\cdot 10^{-8}(\log x_1-19\log 10)+\frac{0.2}{\sqrt{10^{19}}}-\frac{0.2}{\sqrt{x_1}}.$$

To handle the integration beyond x_1 we use (6) in Lemma 10, whence the total integral is

$$0.408+0.002+2\cdot 10^{-8}(\log x_1-19\log 10)+\frac{0.2}{\sqrt{10^{19}}}-\frac{0.2}{\sqrt{x_1}}+\frac{1.3}{\log x_1}.$$

Choosing $x_1 = \exp(500)$ gives the result. □

We remark that we could further divide the range to use more entries in the tables in [3], but the above result is sufficient for our purposes.

3. Character sum estimates

The work of Stephens and Pintz relied on the Burgess bound from [4]. Explicit versions of this are known but are still numerically rather weak. When the modulus is prime, such bounds have been provided by Francis [7] improving on work by Treviño [37] and McGown [23]. If we restrict our attention here to quadratic characters to prime modulus congruent to 1 modulo 4, we may rely on the slightly stronger bounds of Booker in [1]. Recently, Jain-Sharma, Khale and Liu have produced in [10] an explicit version of the Burgess inequality for a composite modulus, but only for $q \leqslant \exp\exp(9.6)$.

Instead of the Burgess bound we shall rely on versions of the Pólya–Vinogradov inequality. We first require an explicit version of the Pólya–Vinogradov inequality due to Frolenkov and Soudararajan in [8, Corollary 1]. In both lemmas that follow, we let V denote the bound on the character sum. We shall, depending on the conditions, invoke these bounds for V later in the paper.

Lemma 13. *When $q \geqslant 100$ and χ is a non-principal Dirichlet character modulo q, we have*

$$\left|\sum_{A\leqslant n\leqslant B} \chi(n)\right| \leqslant \frac{1}{\pi\sqrt{2}}\sqrt{q}(\log q + 6) + \sqrt{q} = V.$$

The following is from [17, 18] by Lapkova, which makes a small improvement on the earlier result from [8, Theorem 2] by Frolenkov and Soundararajan.

Lemma 14. *When $q > 1$ and χ is a primitive Dirichlet character modulo q, we have*

$$\left|\sum_{A\leqslant n\leqslant B} \chi(n)\right| \leqslant \begin{cases} \frac{2}{\pi^2}\sqrt{q}\log q + 0.9467\sqrt{q} + 1.668 = V & \text{when } \chi(-1) = 1, \\ \frac{1}{2\pi}\sqrt{q}\log q + 0.8204\sqrt{q} + 1.0286 = V & \text{when } \chi(-1) = -1. \end{cases}$$

When $A = 0$ and χ is even, we may divide this bound by 2.

Here is a smoothed version of the Pólya–Vinogradov that we take from Levin, Pomerance and Soundararajan in [19].

Lemma 15. *Let χ be a primitive Dirichlet character modulo $q > 1$. Let M and N be real numbers with $0 < N \leqslant q$. With $H(t) = \max(0, 1 - |t-1|)$, we have*

$$\left|\sum_{M\leqslant n\leqslant M+2N} \chi(n) H\left(\frac{n-M}{N}\right)\right| \leqslant \sqrt{q} - \frac{N}{\sqrt{q}}.$$

Lemma 16. *Let χ be a primitive Dirichlet character modulo $q > 1$. Let M and N be real numbers with $0 < N \leqslant q$. When χ is odd,we have*

$$\left|\sum_{M< n\leqslant M+N} \chi(n)\right| \leqslant \sqrt{2N}q^{1/4} + \sqrt{q}.$$

When χ is even, we have

$$\left|\sum_{n\leqslant N} \chi(n)\right| \leqslant \sqrt{N}q^{1/4} + \tfrac{1}{2}\sqrt{q}.$$

Proof. We may assume that M is an integer. Notice first that the lemma is trivial when $N \leqslant \sqrt{q}$, so we may assume $N > \sqrt{q}$. Let $K \geqslant 1$ be an integer and let $A = N/K$. Keeping the notation of Lemma 15, we first notice that

$$H\Big(\frac{t-(M-A/2)}{A}\Big) + H\Big(\frac{t-(M+A/2)}{A}\Big) + \ldots + H\Big(\frac{t-(M+(K-1/2)A)}{A}\Big)$$
$$= \begin{cases} H\Big(\frac{t-(M-A/2)}{A}\Big) & \text{when } M - A/2 \leqslant t \leqslant M + A/2, \\ 1 & \text{when } M + A/2 \leqslant t \leqslant M + (K+1/2)A, \\ H\Big(\frac{t-(M+(K-1/2)A)}{A}\Big) & \text{when } M + (K-1/2)A \leqslant t \leqslant M + (K+1/2)A. \end{cases}$$

Therefore

$$\Big|\sum_{M<n\leqslant M+N}\chi(n) - \sum_{1\leqslant k\leqslant K}\sum_{n}\chi(n)H\Big(\frac{t-(M+(k-1/2)A)}{A}\Big)\Big| \leqslant \frac{4}{A}\sum_{1\leqslant a\leqslant A/2} a$$

which is readily seen to be of size at most $\frac{A}{2} + 1$. On using Lemma 15, we get

$$\Big|\sum_{M<n\leqslant M+N}\chi(n)\Big| \leqslant K\sqrt{q} - \frac{KA}{\sqrt{q}} + \frac{A}{2} + 1 \leqslant K\sqrt{q} - \frac{N}{\sqrt{q}} + \frac{N}{2K} + 1. \tag{8}$$

We let $K = 1 + [q^{-1/4}\sqrt{N/2}]$ and write $K = c + q^{-1/4}\sqrt{N/2}$ with $c \in (0, 1]$. We find that

$$K\sqrt{q} + \frac{N}{2K} = \sqrt{\frac{N\sqrt{q}}{2}} + c\sqrt{q} + \frac{N}{2c + q^{-1/4}\sqrt{2N}}.$$

By computing the derivative with respect to c, we check that this quantity is maximised at $c = 1$. The lemma follows readily. □

Lemma 17. *We have* $L(1,\chi) = F(1)\log H + \mathcal{O}^*(VH^{-1})$*, where* V *is defined in Lemma 13.*

Proof. By summation by parts, we find that

$$\sum_{n>H}\frac{\chi(n)}{n} = \int_H^\infty \sum_{H\leqslant n\leqslant t}\chi(n)\frac{dt}{t^2},$$

hence

$$\begin{aligned} L(1,\chi) &= F(1)\log H + f(1) + \int_H^\infty \sum_{H\leqslant n\leqslant t}\chi(n)\frac{dt}{t^2} \\ &= F(1)\log H + \sum_{n\leqslant H}\int_H^\infty \frac{dt}{t^2} + \int_H^\infty \sum_{H\leqslant n\leqslant t}\chi(n)\frac{dt}{t^2} \\ &= F(1)\log H + \int_H^\infty \sum_{n\leqslant t}\chi(n)\frac{dt}{t^2} = F(1)\log H + \mathcal{O}^*(V/H). \end{aligned}$$ □

4. Preliminaries to Stephens' approach

From (2) and (3) in §1 it follows that

$$\ell(x) = \sum_{m\leqslant H^x}\frac{\chi(m)\Lambda(m)}{m} f\Big(x - \frac{\log m}{\log H}\Big). \tag{9}$$

We now recast this for greater ease of use in what follows.

Lemma 18. *We have*

$$\frac{\ell(x)}{\log H} = xf(x) - \int_0^x f(u)H^u du/H^x.$$

If $H \geqslant V \geqslant 1$ we also have $\int_0^1 f(u)H^u du/H = \mathcal{O}^*(1/\log H)$ and

$$\sum_{m\leqslant H} \frac{\chi(m)\Lambda(m)}{m\log H} f(x_m) = f(1) + \mathcal{O}^*\Big(\frac{R_\chi(H,V,q)}{\log H}\Big). \tag{10}$$

where

(11)

$$R_\chi(H,V,q) = \begin{cases} (3.66+\log(V^2/q))\frac{\sqrt{q}}{H} + \log(4e^2\sqrt{q}H/V^2)\frac{V}{2H} & \text{when } \chi \text{ is even,}\\ (7.2+\log(V^2/q))\frac{\sqrt{q}}{H} + \log(2e^2\sqrt{q}H/V^2)\frac{V}{H} & \text{when } \chi \text{ is odd.}\end{cases}$$

The final proof uses only the upper bound part of (10), see (21).

Proof. We find that

$$\begin{aligned}\sum_{k\leqslant H^x}\chi(k)\log k &= \sum_{k\leqslant H^x}\chi(k)\log(H^x) - \sum_{k\leqslant H^x}\chi(k)\int_k^{H^x}\frac{dt}{t}\\ &= H^x x f(x)\log H - \int_1^{H^x} f\Big(\frac{\log t}{\log H}\Big)t\frac{dt}{t}\\ &= H^x x f(x)\log H - \int_0^x f(u)(\log H)H^u du\end{aligned}$$

and the first part of the lemma follows readily. Concerning the upper bound for $|\int_0^1 f(u)H^u du|/H$, we proceed as follows.

Case of even characters. By Lemma 14 and 16, we have three upper bounds for $|f(u)|$: either 1, $q^{1/4}H^{-u/2} + \frac12 q^{1/2}H^{-u}$ or $V/(2H^u)$. We have $q^{1/4}H^{-u/2} + \frac12 q^{1/2}H^{-u} \leqslant 1$ when $H^u/\sqrt{q} \geqslant 1+\sqrt{3}$. We momentarily set $V^* = V/2$. We define

$$u_0 = \frac{\log(1+\sqrt{3}) + \frac12\log q}{\log H}. \tag{12}$$

Define the real parameter a by $\frac12(1-a)\log H = \log(\sqrt{\sqrt{q}H}/V^*)$. We get

$$\begin{aligned}\int_0^1 |f(u)|H^u du &\leqslant \int_0^{u_0} H^u du + \int_{u_0}^a (q^{1/4}H^{u/2} + \tfrac12 q^{1/2})du + \int_a^1 V^* du\\ &\leqslant \frac{H^{u_0}-1}{\log H} + \frac{2q^{1/4}(H^{a/2}-H^{u_0/2})}{\log H} + \frac{a-u_0}{2}q^{1/2} + (1-a)V^*\\ &\leqslant \frac{\sqrt{q}}{\log H}\big(1+\sqrt{3}+2\sqrt{1+\sqrt{3}}\big) + \frac{2q^{1/4}\sqrt{H}}{\log H}\frac{V^*}{\sqrt{\sqrt{q}H}}\\ &\quad + \frac{\log\frac{V^{*2}}{q(1+\sqrt{3})}}{\log H}\sqrt{q} + \frac{\log(\sqrt{q}H/V^{*2})}{\log H}V^*\\ &\leqslant \frac{\sqrt{q}}{\log H}\left(1+\sqrt{3}+2\sqrt{1+\sqrt{3}}+\log\frac{V^{*2}}{q(1+\sqrt{3})}\right)\\ &\quad + \frac{2V^*}{\log H} + \frac{\log(\sqrt{q}H/V^{*2})}{\log H}V^*.\end{aligned}$$

Case of odd characters. Again by Lemma 14 and 16, we have three upper bounds for $|f(u)|$: either 1, $q^{1/4}\sqrt{2}H^{-u/2}+q^{1/2}H^{-u}$ or V/H^u. We have $q^{1/4}\sqrt{2}H^{-u/2}+q^{1/2}H^{-u}\leqslant 1$ when $H^u/\sqrt{q}\geqslant 2+\sqrt{3}$. We define

$$u_0=\frac{\log(2+\sqrt{3})+\frac12\log q}{\log H}. \tag{13}$$

Define the real parameter a by $\frac12(1-a)\log H=\log(\sqrt{2\sqrt{q}H}/V)$. We get

$$\begin{aligned}\int_0^1|f(u)|H^u du &\leqslant \int_0^{u_0}H^u du+\int_{u_0}^a(q^{1/4}\sqrt{2}H^{u/2}+q^{1/2})du+\int_a^1 V du\\
&\leqslant \frac{H^{u_0}-1}{\log H}+\frac{2\sqrt{2}q^{1/4}(H^{a/2}-H^{u_0/2})}{\log H}+(a-u_0)q^{1/2}+(1-a)V\\
&\leqslant \frac{\sqrt{q}}{\log H}\big(2+\sqrt{3}+2\sqrt{2}\sqrt{2+\sqrt{3}}\big)+\frac{2\sqrt{2}q^{1/4}\sqrt{H}}{\log H}\frac{V}{\sqrt{2\sqrt{q}H}}\\
&\quad+\frac{\log\frac{V^2}{2q(2+\sqrt{3})}}{\log H}\sqrt{q}+\frac{\log(2\sqrt{q}H/V^2)}{\log H}V\\
&\leqslant \frac{\sqrt{q}}{\log H}\big(2+\sqrt{3}+2\sqrt{2}\sqrt{2+\sqrt{3}}+\log\frac{V^2}{2q(2+\sqrt{3})}\big)\\
&\quad+\frac{2V}{\log H}+\frac{\log(2\sqrt{q}H/V^2)}{\log H}V.\end{aligned}$$

Resuming the proof. Inequality (10) follows: indeed, by (9), the left-hand side is $\ell(1)/\log H$ which we compute with the first formula of the present lemma. We complete the proof by using the bound above for $\int_0^1|f(u)|H^u du$. □

Lemma 19. *We have*

$$\begin{aligned}\Big(x-\frac{1}{\log H}\Big)F(x)&=\int_0^x F(x-y)dy\\
&\qquad+\sum_{m\leqslant H^x}\frac{\chi(m)\Lambda(m)}{m\log H}F\Big(x-\frac{\log m}{\log H}\Big)+\mathcal{O}^*(1/\log^2 H).\end{aligned}$$

Proof. On joining (9) and Lemma 18, we get

$$xf(x)-\int_0^x f(u)H^u du/H^x=\sum_{m\leqslant H^x}\frac{\chi(m)\Lambda(m)}{m\log H}f\Big(x-\frac{\log m}{\log H}\Big). \tag{14}$$

This is the equivalent of [36, (55)] by Stephens. The next step is to integrate the above relation:

$$\begin{aligned}\int_0^x yf(y)dy-\int_0^x\int_0^y f(u)H^{u-y}dudy&=\int_0^x\sum_{m\leqslant H^y}\frac{\chi(m)\Lambda(m)}{m\log H}f\Big(y-\frac{\log m}{\log H}\Big)dy\\
&=\sum_{m\leqslant H^x}\frac{\chi(m)\Lambda(m)}{m\log H}\int_{\frac{\log m}{\log H}}^x f\Big(y-\frac{\log m}{\log H}\Big)dy\\
&=\sum_{m\leqslant H^x}\frac{\chi(m)\Lambda(m)}{m\log H}F\Big(x-\frac{\log m}{\log H}\Big).\end{aligned}$$

As for the left-hand side, we first check that

$$\int_0^x yf(y)dy=xF(x)-\int_0^x F(x-y)dy. \tag{15}$$

And finally

$$\begin{aligned}\int_0^x\int_0^y f(u)H^{u-y}dudy &= \int_0^x f(u)H^u\int_u^x H^{-y}dydu\\ &= \int_0^x f(u)H^u\frac{H^{-u}-H^{-x}}{\log H}du = \frac{F(x)}{\log H}+\mathcal{O}^*(1/\log^2 H)\end{aligned}$$

by bounding $|f(u)|$ by 1. □

Lemma 20. *We have, when $x\geqslant 0$*

$$\int_0^x F(x-y)dy = \sum_{m\leqslant H^x}\frac{\Lambda(m)}{m\log H}F\biggl(x-\frac{\log m}{\log H}\biggr) - F(x)\frac{\gamma}{\log H}+\mathcal{O}^*\biggl(\frac{0.411}{\log^2 H}\biggr).$$

Proof. We start from the right-hand side:

$$\begin{aligned}\sum_{m\leqslant H^x}\frac{\Lambda(m)}{m\log H}F\biggl(x-\frac{\log m}{\log H}\biggr) &= \sum_{m\leqslant H^x}\frac{\Lambda(m)}{m\log H}\int_0^{x-\frac{\log m}{\log H}} f(t)dt\\ &= \int_0^x f(t)\sum_{m\leqslant H^{x-t}}\frac{\Lambda(m)}{m\log H}dt.\end{aligned}$$

We approximate $\tilde{\psi}(H^{x-t})$ by $(x-t)\log H-\gamma$, getting the main term and this is $\int_0^x F(t)dt - F(x)\frac{\gamma}{\log H}$ and treat the error term by bounding $|f(t)|$ by 1:

$$\begin{aligned}\int_0^x\left|f(t)\biggl(\frac{\tilde{\psi}(H^{x-t})}{\log H}-x-t-\frac{\gamma}{\log H}\biggr)\right|dt &\leqslant \int_0^x\left|\frac{\tilde{\psi}(H^{x-t})}{\log H}-x-t-\frac{\gamma}{\log H}\right|dt\\ &\leqslant \int_0^x\left|\frac{\tilde{\psi}(H^{t})}{\log H}-t-\frac{\gamma}{\log H}\right|dt\\ &\leqslant \int_1^{H^x}|\tilde{\psi}(u)-\log u-\gamma|\frac{du}{u\log^2 H}.\end{aligned}$$

We then majorize this last term by Lemma 12: it is not more than $0.411/\log^2 H$. □

Lemma 21. *We have, when $x\geqslant 0$,*

$$xF(x) = \sum_{m\leqslant H^x}\frac{\Lambda(m)(1+\chi(m))}{m\log H}F\biggl(x-\frac{\log m}{\log H}\biggr)+F(x)\frac{1-\gamma}{\log H}+\mathcal{O}^*\biggl(\frac{1.411}{\log^2 H}\biggr).$$

Proof. Join the first equality of Lemma 19 together with Lemma 20. □

5. A comparison and the main inequality

This section is devoted to the comparison between

$$\sum_{m\leqslant H}\frac{\Lambda(m)}{m\log H}f(x_m)$$

and $F(1)$. The important observation, essentially due to Stephens, is that since f has tame variations, both should be about equal. One look at the final proof discloses that it is enough to bound the initial sum *from below* by $F(1)$ plus some error term.

We first connect $F(1)$ with the bounds on character sums, that is, with the V from Lemmas 13 and 14.

Lemma 22. *We have, for any $D\geqslant 1$,*

$$F(1) = \sum_{n\leqslant H/D}\frac{\chi(n)}{n\log H}-\frac{f(1)}{\log H}+\mathcal{O}^*\biggl(\frac{(D-1)V/H}{\log H}\biggr).$$

Proof. We have

$$\sum_{H/D<n\leqslant H}\frac{\chi(n)}{n} = \sum_{H/D<n\leqslant H}\chi(n)\Big(\int_n^H\frac{dt}{t^2}+\frac{1}{H}\Big) = \int_{H/D}^H\sum_{H/D<n\leqslant t}\chi(n)\frac{dt}{t^2}+\frac{1}{H}\sum_{H/D<n\leqslant H}\chi(n).$$

Using that $|\sum_{A\leqslant n\leqslant B}\chi(n)|\leqslant V$ yields the desired result. □

Lemma 23. *Let $D_0\geqslant 1$, and A_2 be such that*

$$|\psi(x)-x|\leqslant A_2\sqrt{x},\quad D_0\leqslant x\leqslant 10^{19}. \tag{16}$$

We then have, for any $D\geqslant D_0$,

$$\sum_{mn\leqslant H}\chi(n)\Lambda(m) = H\big(F(x_D)\log H+f(x_D)\big)+\mathcal{O}^*\big(1.6\cdot 10^{-5}H+2A_2HD^{-1/2}+1.04DV\big).$$

Please notice that we would need only the lower estimate in the last bound.

Proof. We write

$$\begin{aligned}\sum_{mn\leqslant H}\chi(n)\Lambda(m) &= \sum_{n\leqslant H/2}\chi(n)\sum_{m\leqslant H/n}\Lambda(m)\\ &= \sum_{n\leqslant H/D}\chi(n)\sum_{m\leqslant H/n}\chi(m)+\sum_{m\leqslant D}\Lambda(m)\sum_{H/D<n\leqslant H/m}\chi(n). \end{aligned}\tag{17}$$

By Lemma 3, the last sum over n is bounded in absolute value by V. It then follows by Lemma 5 that the second summand of (17) satisfies

$$\sum_{m\leqslant D}\Lambda(m)\sum_{H/D<n\leqslant H/m}\chi(n)\leqslant 1.04DV.$$

Concerning the first summand of (17), we use three steps. For the first step, we restrict to the range $H/10^{19}<n\leqslant H/D$ and use (16). Note that Lemma 7 tells us that we can take $A_2=0.94$ provided $D_0>11$. A quick calculation also shows that $A_2=\sqrt{2}$ works for $D_0\geqslant 1$, or $A_2=0.956$ works for $D_0\geqslant 7$. Now,

$$\begin{aligned}&\sum_{\frac{H}{10^{19}}<n\leqslant\frac{H}{D}}\chi(n)\sum_{m\leqslant H/n}\Lambda(m)\\ &= \sum_{\frac{H}{10^{19}}<n\leqslant\frac{H}{D}}\chi(n)\sum_{m\leqslant H/n}\Lambda(m)\\ &= H\sum_{\frac{H}{10^{19}}<n\leqslant\frac{H}{D}}\frac{\chi(n)}{n}+\mathcal{O}^*\left(\sum_{\frac{H}{10^{19}}<n\leqslant\frac{H}{D}}A_2\sqrt{\frac{H}{n}}\right)\\ &= H\sum_{\frac{H}{10^{19}}<n\leqslant\frac{H}{D}}\frac{\chi(n)}{n}+\mathcal{O}^*\left(2A_2HD^{-1/2}\right),\end{aligned}$$

where for the second equality we used Lemma 3.

For the second step, we use Lemma 8 and consider the range $H/A<n\leqslant H/10^{19}$, where $A\geqslant\exp(40)$ is to be chosen later. That is,

$$\begin{aligned}\sum_{\frac{H}{A}<n\leqslant\frac{H}{10^{19}}}\chi(n)\sum_{m\leqslant H/n}\Lambda(m) &= H\sum_{\frac{H}{A}<n\leqslant\frac{H}{10^{19}}}\frac{\chi(n)}{n}+\mathcal{O}^*\left(1.93378\cdot 10^{-8}\sum_{\frac{H}{A}<n\leqslant\frac{H}{10^{19}}}\frac{H}{n}\right)\\ &= H\sum_{\frac{H}{A}<n\leqslant\frac{H}{10^{19}}}\frac{\chi(n)}{n}+\mathcal{O}^*\left(1.93378\cdot 10^{-8}(1+\log(A/10^{19}))H\right),\end{aligned}\tag{18}$$

where for the second equality we used Lemma 4.

For the third step we consider the sum over $n \leqslant H/A$. First, if $H < A$ then there is nothing to add. On the other hand, if $H \geqslant A$ we use Lemma 6 to get

$$\sum_{n\leqslant \frac{H}{A}} \chi(n) \sum_{m\leqslant H/n} \Lambda(m) = H \sum_{n\leqslant \frac{H}{A}} \frac{\chi(n)}{n} + \mathcal{O}^*\left(\sum_{n\leqslant \frac{H}{A}} \frac{1.83H}{n\log^2(H/n)}\right). \tag{19}$$

Since $n\log^2(H/n)$ is increasing when $n \leqslant H/e^2$, we have that

$$\begin{aligned}\sum_{n\leqslant H/A} \frac{1.83H}{n\log^2(H/n)} &\leqslant \frac{1.83H}{\log^2 H} + H\int_1^{H/A} \frac{1.83dt}{t\log^2(H/t)}\\ &\leqslant \frac{1.83H}{\log^2 H} + 1.83H\left(\frac{1}{\log A} - \frac{1}{\log H}\right)\\ &= 1.83H\left(\frac{1}{\log^2 H} + \frac{1}{\log A} - \frac{1}{\log H}\right).\end{aligned}$$

Since the above is decreasing in H, and $H \geqslant A$, we can set $A = \exp(574)$ to bound the $\mathcal{O}^*$ terms in (18) and (19) by $1.6\cdot 10^{-5}H$. □

Lemma 24. *For any $D \geqslant D_0 \geqslant 1$*

$$\left|\sum_{m\leqslant H} \frac{\Lambda(m)}{m\log H} f(x_m) - F(1) - \frac{f(1)}{\log H}\right| \leqslant \frac{1.6\cdot 10^{-5} + 2A_2D^{-1/2} + (2.04D-1)VH^{-1}}{\log H},$$

where A_2 is as in Lemma 23.

The main proof only requires a lower bound for $\sum_{m\leqslant H} \Lambda(m)f(x_m)/m$, see (23).

Proof. By using the definition of f, we get

$$\sum_{m\leqslant H} \frac{\Lambda(m)}{m\log H} f(x_m) = \sum_{mn\leqslant H} \frac{\chi(n)\Lambda(m)}{H\log H}$$

and we appeal to Lemma 23. This leads to

$$\sum_{m\leqslant H} \frac{\Lambda(m)}{m\log H} f(x_m) = \sum_{n\leqslant H/D} \frac{\chi(n)}{n\log H} + \mathcal{O}^*\left(\frac{1.6\cdot 10^{-5} + 2A_2D^{-1/2} + 1.04DVH^{-1}}{\log H}\right).$$

Note further that, by Lemma 22 (we need only the upper estimate), we have

$$F(1) = \sum_{n\leqslant H/D} \frac{\chi(n)}{n\log H} - \frac{f(1)}{\log H} + \mathcal{O}^*\left(\frac{(D-1)V/H}{\log H}\right). \tag{20}$$

□

We are now in a position to prove the following crucial lemma.

Lemma 25. *Let $H \geqslant 10^6$ and $x \geqslant 1/2$. Then we have*

$$0 \leqslant \frac{h(1,x) + h(\chi,x)}{\log H} \leqslant 2x$$

as well as, if H also satisfies $H \geqslant V$,

$$\begin{aligned}\frac{h(1,x) + h(\chi,x)}{\log H} \leqslant& 2 - F(1) + f(1) - \frac{f(1)}{\log H}\\ &+ \frac{-1.15 + 3.81A_2^{2/3}(V/H)^{1/3} - VH^{-1} + R_\chi(H,V,q)}{\log H},\end{aligned}$$

where A_2 is as in Lemma 23 when D is taken to be $\left(\frac{A_2}{2.04}\frac{H}{V}\right)^{2/3}$ (see (24)).

This is the equivalent of [36, Lemma 2] by Stephens.

Proof. The first inequality follows by Lemma 11. Concerning the second one, we proceed as follows. Define

$$S = \sum_{m\leqslant H}\frac{\Lambda(m)}{m\log H}\bigl(1-f(x_m)\bigr) - \sum_{m\leqslant H}\frac{\chi(m)\Lambda(m)}{m\log H}\bigl(1-f(x_m)\bigr). \tag{21}$$

Since $|\chi(m)|$, $|f(x_m)|\leqslant 1$, we have $S\geqslant 0$. Furthermore, on expanding and using the second part of Lemma 18, we find that

$$S = \sum_{m\leqslant H}\frac{\Lambda(m)}{m\log H} - \sum_{m\leqslant H}\frac{\Lambda(m)}{m\log H}f(x_m) - \frac{h(\chi,1)}{\log H} + f(1) + \mathcal{O}^*\Bigl(\frac{R_\chi(H,V,q)}{\log H}\Bigr). \tag{22}$$

Now we use Lemma 24. Since $S\geqslant 0$, this leads to the inequality

$$\begin{aligned}\frac{h(\chi,1)}{\log H} \leqslant& \frac{h(1,1)}{\log H} - F(1) - \frac{f(1)}{\log H} + f(1)\\ &+ \frac{1.6\cdot 10^{-5} + 2A_2D^{-1/2} + (2.04D-1)VH^{-1} + R_\chi(H,V,q)}{\log H}.\end{aligned} \tag{23}$$

We select

$$D = \Bigl(\frac{A_2}{2.04}\frac{H}{V}\Bigr)^{2/3} \tag{24}$$

so that the expression in ((23)) involving D is minimised. This then gives

$$\begin{aligned}\frac{h(\chi,1)}{\log H} \leqslant& \frac{h(1,1)}{\log H} - F(1) - \frac{f(1)}{\log H} + f(1)\\ &+ \frac{1.6\cdot 10^{-5} + 3.81A_2^{2/3}(V/H)^{1/3} - VH^{-1} + R_\chi(H,V,q)}{\log H}.\end{aligned}$$

Let us extend this inequality to $h(\chi,x)$. We simply write

$$\begin{aligned}h(\chi,x) &= h(\chi,1) - \sum_{H^x<m\leqslant H}\frac{\chi(m)\Lambda(m)}{m}\\ &\leqslant h(\chi,1) + h(1,1) - h(1,x),\end{aligned}$$

hence the result, since $2h(1,1)/\log H \leqslant 2 - 2\times 0.576/\log H$ by Lemma 11 and $1.6\cdot 10^{-5} - 2\times 0.576 \leqslant -1.15$. □

6. A result in optimization

This section contains a refined version of a theorem of Stephens. No further arithmetical material is being introduced. We start with a technical lemma.

Lemma 26. *We have*

$$-4\int_\theta^x (x-u)\log u\,du + 2\int_{x-\theta}^{\theta} udu + \int_\theta^x 2\theta du = 2x(x - x\log x - \theta) + (2x-\theta)\theta(1+2\log\theta).$$

Proof. Notice that $2\int u\log udu = u^2\log u - (u^2/2)$ and thus

$$\begin{aligned}4\int_\theta^x (x-u)\log u\,du &= 4x(x\log x - x - \theta\log\theta + \theta) - 2x^2\log x + x^2 + 2\theta^2\log\theta - \theta^2\\ &= 4x(-\theta\log\theta + \theta) + 2x^2\log x - 3x^2 + 2\theta^2\log\theta - \theta^2.\end{aligned}$$

Next,

$$2\int_{x-\theta}^{\theta} udu + \int_{\theta}^{x} 2\theta du = \theta^2 - (x-\theta)^2 + 2\theta(x-\theta) = -x^2 + 4x\theta - 2\theta^2,$$

and thus

$$\begin{aligned} -4\int_{\theta}^{x}(x-u)\log u\,du + 2\int_{x-\theta}^{\theta} udu + \int_{\theta}^{x} 2\theta du \\ = 4x(\theta\log\theta - \theta) - 2x^2\log x + 3x^2 - 2\theta^2\log\theta + \theta^2 - x^2 + 4x\theta - 2\theta^2, \end{aligned}$$

whence the lemma follows after some simple algebraic rearrangement. □

Lemma 27. *Let $H > 1$ be a real parameter. Suppose we are given a sequence of non-negative real numbers $(u_m)_{1\leqslant m\leqslant H}$ and a continuous function G over $[0,1]$. Assume we have, for every $x \in [0,1]$, that*

$$(H_0) \qquad G(x) \leqslant x,$$

that for some parameters a and ε_2, we have, when $x \geqslant 1/2$,

$$(H_1) \qquad (x+a)G(x) \leqslant \sum_{m\leqslant H^x} \frac{u_m}{\log H} G\left(x - \frac{\log m}{\log H}\right) + \varepsilon_2,$$

that

$$(H_2) \qquad 0 \leqslant \sum_{m\leqslant H^x} \frac{u_m}{\log H} \leqslant 2x$$

and that, for some parameter ε_1 we have, when $x \geqslant 1/2$,

$$(H_3) \qquad \sum_{m\leqslant H^x} \frac{u_m}{\log H} \leqslant 2 - G(1) + \varepsilon_1.$$

Then either $G(1) \leqslant 2(1 - 1/\sqrt{e})$ or

$$(25) \qquad 2a\theta\log\theta - 2\theta(1/\sqrt{e} - \theta)(2 + \log\theta) + \varepsilon_1 + \varepsilon_2 \geqslant 0$$

where $\theta = 1 - G(1)/2$ belongs to $[1/2, 1/\sqrt{e}]$.

Proof. Set

$$(26) \qquad \theta = 1 - G(1)/2, \quad \varphi(y) = 2(y - y\log y - \theta).$$

The function φ is increasing (its derivative is $-2\log y$) on $(0,1]$ and takes the positive value $-2\theta\log\theta$ at $y = \theta$. Note that $\theta \geqslant 1/2$ since $G(1) \leqslant 1$, and that when $\theta \geqslant 1/\sqrt{e}$, our result is immediate. Let us assume that $\theta < 1/\sqrt{e}$ so that $\theta + 2\theta\log\theta < 0$. Assume that, when $\theta \leqslant y \leqslant Z$, we have $G(y) \leqslant \varphi(y)$. This latter inequality translates into

$$G(1) - G(y) \geqslant 2(1 - y + y\log y).$$

Our initial remark is that θ is such a number.

Proof. Indeed, if it where not, we would have

$$G(1) = G(\theta) + G(1) - G(\theta) \leqslant \theta + 2(1 - \theta + \theta\log\theta)$$

since $G(x) \leqslant x$. We notice next that $G(1) = 2 - 2\theta$, so that the above inequality can be rewritten as $G(1) \leqslant G(1) + \theta + 2\theta\log\theta < G(1)$ by the inequality assumed for θ, leading to a contradiction. □

We define for this proof

$$g(y) = \sum_{m \leqslant H^y} u_m / \log H. \tag{27}$$

We find that, for $Z \geqslant x \geqslant \theta$,

$$\sum_{m \leqslant H^x} \frac{u_m}{\log H} G\Big(x - \frac{\log m}{\log H}\Big) \leqslant \sum_{m \leqslant H^{x-\theta}} \frac{u_m}{\log H} \varphi\Big(x - \frac{\log m}{\log H}\Big) + \sum_{H^{x-\theta} < m \leqslant H^x} \frac{u_m}{\log H}\Big(x - \frac{\log m}{\log H}\Big)$$

by bounding above $G(y)$ by y (H_0) when $y \leqslant \theta$. We study separately the two right-hand side sums, say S_1 and S_2. First we note that, on recalling the definition (27) of g:

$$\begin{aligned} S_1 &= \sum_{m \leqslant H^{x-\theta}} \frac{u_m}{\log H} \int_\theta^{x - \frac{\log m}{\log H}} \varphi'(u) du + g(x-\theta)\varphi(\theta) \\ &= g(x-\theta)\varphi(\theta) - 2\int_\theta^x g(x-u) \log u \, du \end{aligned}$$

while

$$S_2 = \sum_{H^{x-\theta} < m \leqslant H^x} \frac{u_m}{\log H} \int_{\frac{\log m}{\log H}}^x du = \int_{x-\theta}^x (g(u) - g(x-\theta)) du$$

and this amounts to

$$S_1 + S_2 = g(x-\theta)(\varphi(\theta) - \theta) - 2\int_\theta^x g(x-u)\log u \, du + \int_{x-\theta}^x g(u) du.$$

In the first integral, we bound above $g(x-u)$ by $2(x-u)$ by (H_2). We split the second integral at $u = \theta$; between $x - \theta$ and θ, we bound above $g(u)$ again by $2u$ while in the later range, we bound above $g(u)$ by $2\theta + \varepsilon_1$ by $(H-3)$ (valid since $u \geqslant \theta \geqslant 1/2$). We infer in this manner that

$$\begin{aligned} S_1 + S_2 \leqslant{}& g(x-\theta)(\varphi(\theta) - \theta) \\ &- 4\int_\theta^x (x-u)\log u \, du + 2\int_{x-\theta}^\theta u du + \int_\theta^x (2\theta + \varepsilon_1) du. \end{aligned}$$

By Lemma 26 and noticing that $\varphi(\theta) - \theta = -\theta(1 + 2\log\theta)$, we get (again bounding above $g(x-u)$ by $2(x-u)$ by (H_2))

$$\begin{aligned} S_1 + S_2 &\leqslant x\varphi(x) + \big(g(x-\theta) + \theta - 2x\big)(\varphi(\theta) - \theta) + \varepsilon_1 \\ &\leqslant x\varphi(x) + \theta^2(1 + 2\log\theta) + \varepsilon_1. \end{aligned}$$

By (H_1) and the above, we infer that

$$(x+a)G(x) \leqslant x\varphi(x) + \theta^2(1 + 2\log\theta) + \varepsilon_1 + \varepsilon_2.$$

We also find that, when $\theta \leqslant 1/\sqrt{e}$, we have

$$0 - \theta^2(1 + 2\log\theta) = \int_\theta^{1/\sqrt{e}} 2u(2 + \log u) du \geqslant 2\theta(1/\sqrt{e} - \theta)(2 + \log\theta).$$

Hence, we get

$$(x+a)G(x) \leqslant (x+a)\varphi(x) - a\varphi(x) - 2\theta(1/\sqrt{e} - \theta)(2 + \log\theta) + \varepsilon_1 + \varepsilon_2.$$

We can now use $\varphi(x) \geqslant \varphi(\theta) = -\theta\log\theta$, getting

$$(x+a)G(x) \leqslant (x+a)\varphi(x) + 2a\theta\log\theta - 2\theta(1/\sqrt{e} - \theta)(2 + \log\theta) + \varepsilon_1 + \varepsilon_2.$$

When $2a\theta\log\theta - 2\theta(1/\sqrt{e}-\theta)(2+\log\theta)+\varepsilon_1+\varepsilon_2<0$, we would have $G(x)<\varphi(x)$. However the function G is continuous and $G(1)=\varphi(1)$, there exists an x_0 between θ and 1 for which $G(x_0)=\varphi(x_0)$ and $G(x)\leqslant\varphi(x)$ for x between θ and x_0. The above inequality then leads to a contradiction. Hence we have

$$2a\theta\log\theta - 2\theta(1/\sqrt{e}-\theta)(2+\log\theta)+\varepsilon_1+\varepsilon_2\geqslant 0. \qquad \square$$

7. Proof of Theorems 1 and 2

We use Lemma 27 with $G(x)=F(x)$ and $u_m=(1+\chi(m))\Lambda(m)/m$.

Initial upper bound. Lemma 17 gives us

$$L(1,\chi)\leqslant F(1)\log H+\frac{V}{H}. \tag{28}$$

Hypotheses (H_0), (H_1), (H_2) *and* (H_3). Hypothesis (H_0) is granted by the bound $|f(u)|\leqslant 1$. By Lemma 21 we can then set

$$a\leqslant\frac{\gamma-1}{\log H},\quad \varepsilon_2\leqslant\frac{1.411}{\log^2 H}, \tag{29}$$

and this gives us Hypothesis (H_1).

Lemma 10 is enough to grant Hypothesis (H_2). Finally, by Lemma 25 and provided that $H\geqslant\max(V,10^6)$, Hypothesis (H_3) is satisfied with

$$\varepsilon_1\leqslant f(1)-\frac{f(1)}{\log H}+\frac{-1.15+3.81A_2^{2/3}(V/H)^{1/3}-VH^{-1}+R_\chi(H,V,q)}{\log H}. \tag{30}$$

We will further majorize $f(1)$ by $V/2$ when χ is even and by V/H when χ is odd.

Using Lemma 27. So we infer that

$$L(1,\chi)\leqslant 2(1-\theta)\log H+\frac{V}{H} \tag{31}$$

where $\theta\in[1/2,1/\sqrt{e}]$ satisfies

$$2a\theta\log\theta-2\theta(1/\sqrt{e}-\theta)(2+\log\theta)+\varepsilon_1+\varepsilon_2\geqslant 0. \tag{32}$$

Since $a<0$, if this inequality is satisfied for H_0 then it remains true for $H\geqslant H_0$. We select $H=BV$, for some parameter B, and bound $|f(1)|$ by V/H. Therefore, for $q\geqslant q_0$ we have that

$$D\geqslant\left(\frac{A_2B}{2.04}\right)^{2/3}=D_0. \tag{33}$$

So, here are possible choices:

$$B\geqslant\sqrt{2}/2.04\to A_2=\sqrt{2}, \tag{34}$$

$$B\geqslant 39.6\to A_2=0.956, \tag{35}$$

$$B\geqslant 79.5\to A_2=0.94. \tag{36}$$

Setting the numerics. We can now prove Theorems 1 and 2. We use the expression for V given in Lemma 13. We take $H=BV$ which we assume to be $\geqslant 10^6$, we also assume that $q\geqslant q_0$ so that $V\geqslant V_0$. Given a choice of B, we select

$$a=\frac{\gamma-1}{\log(BV_0)}, \tag{37}$$

$$\varepsilon_2=\frac{1.411}{\log(BV_0)^2}, \tag{38}$$

$$\varepsilon_1=\frac{\delta(\chi)}{B}-\frac{\delta(\chi)}{B\log(BV_0)}+\frac{-1.15+3.81A_2^{2/3}B^{-1/3}-B^{-1}+R(BV_0,V_0,q_0)}{\log(BV_0)}. \tag{39}$$

where $\delta(\chi) = (3-\chi(-1))/4$. We then compute the smallest solution θ^* to (31) and infer that

$$\frac{L(1,\chi)}{\log q} \leqslant 2(1-\theta^*)\frac{\log B + \log V_0}{\log q_0} + \frac{1}{B \log q_0}. \tag{40}$$

Result for χ even and primitive: We select $B = 51$, and infer that $L(1,\chi) < \frac{1}{2}\log q$ when $q \geqslant 7\cdot 10^{22}$. But this is already known for all q's by [31]. Even more is true if we combine the theorem of Saad Eddin in [35] together with [32, Corollary 1].

We select $B = 80$, and infer that $L(1,\chi) < \frac{9}{20}\log q$ when $q \geqslant 2\cdot 10^{49}$.

Result for χ odd and primitive: We select $B = 90$, and infer that $L(1,\chi) < \frac{1}{2}\log q$ when $q \geqslant 2\cdot 10^{23}$.

We select $B = 145$, and infer that $L(1,\chi) < \frac{9}{20}\log q$ when $q \geqslant 5\cdot 10^{50}$.

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