EXPLICIT AVERAGES OF NON-NEGATIVE MULTIPLICATIVE FUNCTIONS: GOING BEYOND THE MAIN TERM

O. RAMARÉ AND AKHILESH P

Abstract. We produce an explicit formula to perform the evaluation of averages of type $\sum_{d\le D}(g\star \mathbb{1})(d)/d$, where $\star$ is the Dirichlet convolution and g a function that vanishes at infinity (more precise conditions are needed, a typical example of an acceptable function is $g(m)=\mu(m)/m$). This formula enables one to exploit the changes of sign of $g(m)$. We then proceed by using this formula on the classical family of sieve-related functions $G_q(D)=\sum_{\substack{d\le D,\\(d,q)=1}}\frac{\mu^2(d)}{\varphi(d)}$ for a integer parameter q, improving noticeably on earlier results. The remainder of the paper deals with the special case $q=1$ to show how to practically exploit the changes of sign of the Moebius function. It is in particular proven that $|G_1(D)-\log D-c_0|\le 4/\sqrt{D}$ and $|G_1(D)-\log D-c_0|\le 18.4/[\sqrt{D}\log D]$ when $D>1$, for a suitable constant c_0.

1. Introduction

Evaluating the average of multiplicative functions is a classical and important problem. It has been approached in several manners, see [41], [19], [11] on the "elementary" side; [15], [40], [4], [14], [17] on the tauberian side; and [10], [8], [22] following the Halász method. See also [1], [2] for better analytical estimates valid for a more narrow class of arithmetical functions.

The theory went also into special evaluations that were then systematized, like in [38] or [16]. This first batch of work essentially handles the "main term" and treats what remains as an error term. Subsequent investigations, as in [42] and [18], went to study more precisely, in some very special cases, this term and a main term is indeed extracted. The typical case corresponds to the characteristic function of the square-free integers whose Dirichlet series is $\zeta(s)/\zeta(2s)$. The first batch of investigation writes this series in the form $\zeta(s)H(s)$ where the only hypothesis on H is that it is absolutely convergent for $\Re s>1/2$. The second batch uses the fact that $1/\zeta(2s)$ can be controlled beyond $\Re s=1/2$.

The present work can be seen as an elementary and explicit counterpart of such studies. The task of getting explicit estimates has been started on particular questions by several authors as in [5], and systematically in [25, Lemma 3.2]. Concerning the analytical approach, the amount of information available is sparse; one can find a very explicit truncated Perron formula in [26, Theorem 7.1], but the bounds for the usual Dirichlet series (like $1/\zeta(s)$) are still too weak or simply missing. We develop here a strategy that achieves two things: Theorem 1.4 gets the most of the convolution method, and Theorem 1.3 goes *beyond* in some problems and incorporates cancellations from the Moebius function.

We use the following family of functions as our primary challenge:

$$G_q(D)=\sum_{\substack{d\le D,\\(d,q)=1}}\frac{\mu^2(d)}{\varphi(d)}. \tag{1.1}$$

This family has been quite extensively studied (see for instance [38], [32], [25, Lemma 3.4]) and used, see for instance [35, Lemma 4.6] and [13, (7.12), (B.5)]. It occurs in the

1991 *Mathematics Subject Classification.* Primary 11N37, Secondary 11N36, 11N56.
Key words and phrases. Multiplicative functions, explicit evaluations.

sieve, when bounding from above the number of primes in a given arithmetic progression (the so-called Brun-Titchmarsh Theorem) via the Selberg sieve [12] or via Montgomery sieve [21], or, more generally, when considering the resulting sequence as an enveloping sieve as initially done in [25], [31] (see also [9]).

The question we address here is to find as good an explicit error term as possible. We first give a fairly simple proof of the following theorem. As usual we use $g = \mathcal{O}^*(f)$ to mean that $|f| \le g$.

Theorem 1.1. *We have*

$$G_q(D) = \frac{\varphi(q)}{q}\big(\log D + c(q)\big) + \mathcal{O}^*\big(5.9\, j(q)/\sqrt{D}\big)$$

where

$$j(q) = \prod_{\substack{p|q,\\ p\neq 2}} \frac{p^{3/2}+p-\sqrt{p}-1}{p^{3/2}-\sqrt{p}+1} \prod_{2|q} \frac{21}{25}$$

and (cf [33, (2.11)]*)*

$$c(q) = c_0 + \sum_{p|q} \frac{\log p}{p}, \quad c_0 = \gamma + \sum_{p\ge 2} \frac{\log p}{p(p-1)} = 1.332\,582\,275\cdots$$

For the moduli in the next array, the value $5.9 \cdot j(q)$ *may be replaced by the value* $j^\star(q)$ *given by:*

q	*6*	*30*	*210*	*2310*	*30 030*	*510 510*
$j^\star(q)$	*5.2*	*6.0*	*7.1*	*8.4*	*9.9*	*11.6*

As usual, we use the notation $f = \mathcal{O}^*(g)$ to mean that $|f| \le g$. The above term is for instance extremely important in [13, (7.29)]. The term "$\prod_{2|q}\frac{21}{25}$" means that the factor $\frac{21}{25}$ is present only when q is even.

The error term in Theorem 1.1 is limited to $\mathcal{O}(1/\sqrt{D})$. Theorem 1.3 below enables one to get $\mathcal{O}(1/\sqrt{D\log D})$ in a fully *explicit* manner. The details are however difficult when considering a general modulus q, so we restricted our attention to $q = 1$.

Theorem 1.2. *When* $D \ge 1$*, we have* $G_1(D) = \log D + c_0 + \mathcal{O}^*\big(3.95/\sqrt{D}\big)$.
When $D > 1$*, we have* $G_1(D) = \log D + c_0 + \mathcal{O}^*\big(18.4/\sqrt{D\log D}\big)$.

If one has access to better bounds for the summatory function of the Moebius function, better than the ones given by Lemma 7.14, or if one can rely on larger computations concerning the small values and further Lemma 8.4, then the saving will be automatically transferred on the error term under examination. For instance, if the computations in Lemma 8.4 were pushed till 10^{15}, and expecting a similar output, we could get an error term $\mathcal{O}^*\big(2.94/\sqrt{D}\big)$. It is doubtless than these parts will be improved upon. Furthermore, other sections of this proof can also be bettered.

Our main tool is the following general and fully explicit theorem.

Theorem 1.3. *Let* $(g(m))_{m\ge1}$ *be a sequence of complex numbers such that both series* $\sum_{m\ge1} g(m)/m$ *and* $\sum_{m\ge1} g(m)(\log m)/m$ *converge. We define* $G^\sharp(x) = \sum_{m>x} g(m)/m$ *and assume that* $\int_1^\infty |G^\sharp(t)|dt/t$ *converges. Let* $A_0 \ge 1$ *be a real parameter. We have*

$$\sum_{n\le D} \frac{(g\star \mathbb{1})(n)}{n} = \sum_{m\ge1} \frac{g(m)}{m}\Big(\log\frac{D}{m} + \gamma\Big) + \int_{D/A_0}^\infty G^\sharp(t)\frac{dt}{t} + \mathcal{O}^*(\mathfrak{R})$$

where $\mathfrak{R}$ *is defined by*

$$\mathfrak{R} = \left| \sum_{1\le a\le A_0} \frac{1}{a} G^\sharp\left(\frac{D}{a}\right) + G^\sharp\left(\frac{D}{A_0}\right)\left(\log\frac{A_0}{[A_0]} - R([A_0])\right)\right| + \frac{6/11}{D} \sum_{m\le D/A_0} |g(m)|$$

where $[A_0]$ *is the integer part of* A_0*, while the remainder* R *is defined in Lemma 2.1.*

By (2.1), when $[A_0]$ is large, $R([A_0])$ is equivalent to $1/(2[A_0])$, and thus, on taking $A_0 = [A_0]+\frac{1}{2}$, we conclude that $\log\frac{A_0}{[A_0]} - R([A_0])$ decreases at least like $1/A_0^2$. In our case study, we have be $G^\sharp\left(D/A_0\right)\left(\log\frac{A_0}{[A_0]} - R([A_0])\right) = \mathcal{O}(D^{-1/2}A_0^{-3/2})$ while the bound for the term $\sum_{1\le a\le A_0} G^\sharp\left(D/a\right)/a$ grows with A_0 and the last term of $\mathfrak{R}$ is be bounded by $\mathcal{O}(D^{-1/2}A_0^{-1/2})$.

Theorem 1.4 below corresponds to $A_0 \le 1$. Both theorems are refinements of [25, Lemma 3.2] obtained via the classical *convolution method* (see [3] for a pedagogical presentation) to which we add a finite version of Dirichlet hyperbola principle as in [27, (19)]. In particular, only on following the convolution method, I was for a very long time unable to reach an error term of size $\mathcal{O}_q(1/\sqrt{x})$ in Theorem 1.1 (see the beginning of subsection 8.2 for an explanation regarding this bound); I could only get $\mathcal{O}_q((\log x)/\sqrt{x})$, which is why I settled for the asymptotically lesser but numerically better $\mathcal{O}_q(1/x^{1/3})$.

Theorem 1.3 enables us to go even beyond $\mathcal{O}_q(1/\sqrt{x})$. The additional ingredient comes from a good upper bound for $|G^\sharp(x)|$ that is often accessible and is introduced via the Covering Remainder Lemma below (this is Lemma 3.1). In the case study of Theorem 1.1, Theorem 1.3 leads to

$$G_q(D) = \frac{\varphi(q)}{q}\bigl(\log D + c(q)\bigr) + \mathcal{O}_q\Bigl(\exp\bigl(-c_1\sqrt{\log D}\bigr)/\sqrt{D}\Bigr)$$

for some constant $c_1 > 0$.

They are several occasions when selecting $A_0 = 1$ is in fact better: when the information on $G^\sharp$ is scarce, when g is rather large (see the example $g(d) = \mu(d)/d$ below) or when g is non-negative. One can then modify the proof so as to get often a somewhat better result.

Theorem 1.4. *Let $(g(m))_{m\ge1}$ be a sequence of complex numbers such that both series $\sum_{m\ge1} g(m)/m$ and $\sum_{m\ge1} g(m)(\log m)/m$ converge. We define $G^\sharp(x) = \sum_{m>x} g(m)/m$ and assume that $\int_1^\infty |G^\sharp(t)|dt/t$ converges. We then have, for any real number $\eta\ge1$,*

$$\sum_{n\le D}\frac{(g\star\mathbb{1})(n)}{n} = \sum_{m\ge1}\frac{g(m)}{m}\Bigl(\log\frac{D}{m}+\gamma\Bigr) + \int_{\eta D}^\infty G^\sharp(t)\frac{dt}{t} - (\gamma-\log\eta)G^\sharp(\eta D) + \mathcal{O}^*\Bigl(\frac{\gamma}{D}\sum_{m\le\eta D}|g(m)|\Bigr).$$

When $\eta = 1$, the constant γ in the error term can be reduced to $6/11$.

This theorem applies for instance when $g(m) = \mu(m)/m$. The above yields an error term of size $\mathcal{O}((\log D)/D)$. On following [39] (see also [24]), one can improve slightly this bound but the paper [20] (see also the intriguing [16]) implies that one cannot even replace the $\log D$ in this error term by a constant. The introduction of the A_0 parameter in Theorem 1.1 is thus useless, and it is better to play with the η parameter of Theorem 1.4.

The major difficulty one faces when putting this program in practice, for instance for proving Theorem 1.1 or 1.2, are the coprimality conditions that come naturally into play, even for Theorem 1.2, though no coprimality condition was introduced at the beginning. This is apparent for instance in (7.7) which we recall here:

$$\frac{\mu(k)}{\varphi(k)} = \sum_{dab^2=k}\frac{\mu(d)}{d}\frac{\mu^2(abd)}{a\varphi(a)b^2\varphi(b)}.$$

The most inner and main variable is d, but the factor $\mu^2(abd)$ forces d to be coprime to ab. This question has already been addressed in [30], [28] and [29] in some special instances and these results are used here. In general though we are starting from a non-negative function, we are aiming at studying the second term of the asymptotic expansion. This implies that we may have to deal with oscillating function, as is the case here. If this second term would correspond to a pole, like for the coefficients of $\zeta(s)\zeta(2s)$, cancellations do not occur, but our series is closer to $\zeta(s)/\zeta(2s)$. Most of our effort bore on the function $\mu(k)/[k\varphi(k)]$, and for instance two of our crucial (and novel) bounds are given in Lemma 5.6 and Lemma 7.18. We state here a consequence of them for the reader to appreciate their content:

Theorem 1.5. *For any real number $K \geq 14$ and any modulus q^*, we have*

$$\Bigg|\sum_{\substack{k>K,\\(k,q^*)=1}} \frac{\mu(k)}{\varphi(k)k}\Bigg| \leq \frac{0.37}{K}.$$

When further $K > q^*$, we have

$$\left|\sum_{\substack{k>K,\\(k,q^*)=1}} \frac{\mu(k)}{k\varphi(k)}\right| \leq \frac{2.33\prod_{p|q^*}(1+p^{-1})}{K\log(K/q^*)} + \frac{5}{K}\sqrt{\frac{q^*}{K}} + \frac{6.6}{K}\sqrt{\frac{q^*}{K}}\mathbb{1}_{K\geq 1970q^*}.$$

We devised some special analysis for studying the low values of the parameters, but we face a major hurdle: the low values of K/q^* do not correspond to a finite set of parameters. Overcoming this hurdle costs us quite a lot, hence the constant 2.33 above, while for comparison, [27] shows for instance that when $D \geq 50\,000$, we have

$$\Bigg|\sum_{d<D} \frac{\mu(d)}{d}\Bigg| \leq \frac{3/100}{\log D}.$$

We also rely on precise local estimates for some particular functions. These estimates have an independent interest, but apart from some local improvements, these are essentially routine.

About the computations. We have to perform several computations by computer. We have used commonly Pari/GP [23] with the GMP [37] multiprecision library and checked the results with Sage [34] with interval arithmetic, thus using GMP, MPFR [6] and MPFI. Both systems scripts have been run on independent machines; the GP-scripts have been written and run by one author and the Sage-scripts have been written and run by the other. Whenever required, we have speeded the GP script by using GP2C, a software that converts the GP script in a C program that can then be compiled. Sage being much slower than GP and this last process being absent, the last script used for Lemma 8.4 has been run with Perl and the MPFI library for interval arithmetic. It ran for three months (while the GP counterpart ran for ten days).

Let us describe some more how to compile a GP script, say contained in the file `Check.gp`. We run the command `gp2c -g Check.gp > Check.gp.c` with the "-g" flag to enable garbage collection. At the top of the file `Check.gp.c`, the reader will find the compilation command starting by `/*-*- compile-command:` . This command has to be run to get the compiled form. From then onward, a simple way to proceed is to start GP with a large enough table of primes, say with `gp -p10000000000`. At the top of the `Check.gp.c`, the reader will find several lines starting by `GP; install`. The `install` commands have to be entered in GP, the first one being

`install("init_Check","v","init_Check","./Check.gp.so")`.

Then one has to run the command `allocatemem(7500000000)` to extend the stack to a bit more than 7 Giga, `init_Check()` to take care of the global variables and one is finally ready to use the functions in the script `Check.gp`.

Concerning the Perl scripts, the interesting part of the header reads:

```
use Math::MPFI qw(:mpfi);
use Math::Prime::Util qw(primes);
use Math::Factor::XS qw(factors prime_factors matches);
[...]
Rmpfi_set_default_prec(100); # Set default precision to 100 bits
```

2. Proof of Theorem 1.4 and beginning of proof of Theorem 1.3

Lemma 2.1. *When $X \geq 1$ is a real number, we have*

$$\sum_{n \leq X} \frac{1}{n} = \log X + \gamma + \mathcal{O}^*\Big(\frac{6}{11X}\Big).$$

When $X > 0$ is a real number, we have

$$\sum_{n \leq X} \frac{1}{n} = \log X + \gamma + \mathcal{O}^*\Big(\frac{\gamma}{X}\Big).$$

In general, we define $\sum_{n \leq X} 1/n - \log X - \gamma = R(X)$.

The first inequality with $7/12 = 0.583\cdots$ appears in [5, (3.1)]. The constant $6/11 = 0.545\cdots$ is near optimal: the optimum is $2(\log 2 + \gamma - 1) = 0.540\cdots$.

Proof.

- We inspect $x \mapsto |\log x + \gamma - 1|x$ when $x \in [1, 2)$ and find its maximum to be $2(\log 2 + \gamma - 1) = 0.540\,725\cdots$.
- We inspect $x \mapsto |\log x + \gamma - 3/2|x$ when $x \in [2, 3)$ and find its maximum to be $3(\log 3 + \gamma - 3/2) = 0.527\cdots$.
- We inspect $x \mapsto |\log x + \gamma - 11/6|x$ when $x \in [3, 4)$ and find its maximum to be $3(\log 3 + \gamma - 11/6) = 0.520\cdots$.

We proceed in a similar fashion for all intervals up to 11 and obtained the following maxima, all of them reached (almost!) at the endpoint of the studied interval:

[4,5)	[5,6)	[6,7)	[7,8)	[8,9)	[9,10)	[10,11)
0.516...	0.513...	0.511...	0.510...	0.508...	0.508...	0.507...

For larger values, the Euler-MacLaurin formula gives us (see for instance [36, Théorème 5])

$$\sum_{n \leq N} \frac{1}{n} = \log N + \gamma + \frac{1}{2N} - \frac{1}{12N^2} + \mathcal{O}^*\Big(\frac{1}{60N^4}\Big) \tag{2.1}$$

for any positive integer N. We have $\frac{N+1}{N}(\frac{1}{2} - \frac{1}{12N} + \frac{1}{60N^3}) \leq 0.538$ when $N \geq 11$. The lemma follows readily. □

Proof of Theorem 1.4. We write

$$\sum_{n \leq D} \frac{(g \star \mathbb{1})(n)}{n} = \sum_{m \geq 1} \frac{g(m)}{m} \sum_{n \leq D/m} \frac{1}{n}.$$

The summation in m can be restricted to $m \leq D$, and more generally to $m \leq \eta D$ for some $\eta \geq 1$. This together with the definition of $R(x)$ in Lemma 2.1 gives us

$$\sum_{n \leq D} \frac{(g \star \mathbb{1})(n)}{n} = \sum_{m \leq \eta D} \frac{g(m)}{m}\Big(\log \frac{D}{m} + \gamma\Big) + \sum_{m \leq \eta D} \frac{g(m)}{m} R\left(\frac{D}{m}\right).$$

The next task is to complete the summation over m, for which we add and substract the quantity $\sum_{m > \eta D} \frac{g(m)}{m}(\log \frac{D}{m} + \gamma)$ and we rewrite the substracted part by using $G^\sharp$:

$$\begin{aligned}
-\sum_{m > \eta D} \frac{g(m)}{m}\Big(\log \frac{D}{m} + \gamma\Big) &= \sum_{m > \eta D} \frac{g(m)}{m}\Big(\log \frac{m}{\eta D} - \gamma + \log \eta\Big) \\
&= \sum_{m > \eta D} \frac{g(m)}{m} \int_{\eta D}^{m} \frac{dt}{t} - (\gamma - \log \eta) G^\sharp(\eta D) \\
&= \int_{\eta D}^{\infty} \sum_{m > t} \frac{g(m)}{m} \frac{dt}{t} - (\gamma - \log \eta) G^\sharp(\eta D) \\
&= \int_{\eta D}^{\infty} G^\sharp(t) \frac{dt}{t} - (\gamma - \log \eta) G^\sharp(\eta D)
\end{aligned}$$

This leads to the following key expression:

$$\sum_{n\le D}\frac{(g\star\mathbb{1})(n)}{n}=\sum_{m\ge1}\frac{g(m)}{m}\Big(\log\frac{D}{m}+\gamma\Big)+\int_{\eta D}^{\infty}G^\sharp(t)\frac{dt}{t}\\ -(\gamma-\log\eta)G^\sharp(\eta D)+\sum_{m\le\eta D}\frac{g(m)}{m}R\left(\frac{D}{m}\right). \tag{2.2}$$

On appealing to Lemma 2.1, one gets Theorem 1.4. □

The main point above has been to write (when $\eta=1$)

$$\sum_{d>D}\frac{g(d)}{d}\log\frac{d}{D}=\int_D^\infty\sum_{d>t}\frac{g(d)}{d}\frac{dt}{t}$$

while the usual treatment is comparable to using $\log\frac{d}{D}=\log d-\log D$ and treating the two resulting sums independently.

Second proof of (2.2). Equation (2.2) being a linear identity in g, we can use algrebraic means to prove it. The following decomposition holds:

$$g=\sum_{k\ge1}g(k)\delta_{\cdot=k}.$$

It is thus enough to prove our identity for $\delta_{\cdot=k}$. In this case, we readily check that

$$G^\sharp(t)=\begin{cases}\frac1k & \text{when } t\le k,\\ 0 & \text{otherwise.}\end{cases}$$

The function $(g\star\mathbb{1})$ is the characteristic function of those integers that are divisible by k. The proof of (2.2) splits in two cases:

(1) When $k\le\eta D$, the LHS of (2.2) is $(1/k)\sum_{n\le D/k}1/n$, while the RHS is $\frac1k(\log\frac{D}{k}+\gamma)+0-(\gamma-\log\eta)\times0+\frac1k R(D/k)$. Both sides agree by the definition of $R(D/k)$.
(2) When $\eta D<k$, the LHS of (2.2) vanishes, while the RHS is

$$\frac1k\big(\log\frac{D}{k}+\gamma\big)+\frac1k\int_{\eta D}^{k}\frac{dt}{t}-(\gamma-\log\eta)\frac1k$$

which is also verified to vanish.

The proof is complete. □

3. The Covering Remainder Lemma and proof of Theorem 1.3

Lemma 3.1 (The Covering Remainder Lemma). *Let $(g(m))_{m\ge1}$ be a sequence of complex numbers such that $\sum_{m\ge1}g(m)/m$ converges for which we define $G^\sharp(x)=\sum_{m>x}g(m)/m$. The function R is given in Lemma 2.1. Let $A_0\ge1$ be a real number and let $[A_0]$ denotes its integer part. We have*

$$\sum_{\frac{X}{A_0}<m\le X}\frac{g(m)}{m}R\left(\frac{X}{m}\right)=\gamma G^\sharp(X)+\int_1^{A_0}G^\sharp\left(\frac{X}{t}\right)\frac{dt}{t}-\sum_{1\le a\le A_0}\frac1a G^\sharp\left(\frac{X}{a}\right)\\ -G^\sharp\left(\frac{X}{A_0}\right)\left(\log\frac{A_0}{[A_0]}-R([A_0])\right)$$

When g is non-negative, this lemma does not lead to any further saving. But otherwise, the saving is interesting. In case $g(m)/m=\mu(m)/m^2$, the estimate $R(x)\ll1$ leads to a $\mathcal{O}(1/X)$ for the LHS while this lemma gives the bound $\mathcal{O}(\exp(-c\sqrt{\log X})/X)$ for some positive constant c. This would have the same effect as [20, Lemma 1]: shortening the sum that was responsible for the size of the remainder term.

Proof. We set $B = [A_0]$ to ease the typing. When a is a positive integer, $b \in [a, a+1]$, and m is inside $(X/b, X/a]$, we have

$$\log\frac{X}{m} + \gamma + R(X/m) = \sum_{n \le X/m} \frac{1}{n} = \sum_{n\le a}\frac{1}{n} = \log a + \gamma + R(a)$$

from which we infer that $R(X/m) = R(a) - \int_a^{X/m} dt/t$. This implies that

$$\sum_{\frac{X}{b} < m \le \frac{X}{a}} \frac{g(m)}{m} R(X/m) = \sum_{\frac{X}{b} < m \le \frac{X}{a}} \frac{g(m)}{m} R(a) - \int_a^b \sum_{\frac{X}{b} < m \le \frac{X}{t}} \frac{g(m)}{m}\frac{dt}{t}.$$

We sum this construction step over $a \in \{1, \cdots, B\}$ with the choice $b = \min(a+1, A_0)$. On using the notation $G^\sharp$, we get

$$\sum_{\frac{X}{A_0} < m \le X} \frac{g(m)}{m} R(X/m) = \sum_{a \le B} \left(G^\sharp\left(\frac{X}{\min(a+1, A_0)}\right) - G^\sharp\left(\frac{X}{a}\right)\right) R(a)$$
$$+ \sum_{a\le B} \int_a^{\min(a+1,A_0)} \left(G^\sharp\left(\frac{X}{t}\right) - G^\sharp\left(\frac{X}{\min(a+1, A_0)}\right)\right)\frac{dt}{t}$$

Some shuffling is called for. Here is the first step:

$$\sum_{\frac{X}{A_0} < m \le X} \frac{g(m)}{m} R(X/m) = \sum_{2\le a \le B+1} G^\sharp\left(\frac{X}{\min(a, A_0)}\right) R(a-1) - \sum_{a\le B} G^\sharp\left(\frac{X}{a}\right) R(a)$$
$$+ \int_1^{A_0} G^\sharp\left(\frac{X}{t}\right)\frac{dt}{t} - \sum_{a\le B} G^\sharp\left(\frac{X}{\min(A_0, a+1)}\right) \log\frac{\min(A_0, a+1)}{a}$$

which we rewrite in the form (we set $R(0) = 0$)

$$\sum_{\frac{X}{A_0} < m \le X} \frac{g(m)}{m} R(X/m) = \sum_{1\le a\le B} G^\sharp\left(\frac{X}{a}\right)(R(a-1) - R(a)) + G^\sharp\left(\frac{X}{A_0}\right) R(B)$$
$$+ \int_1^{A_0} G^\sharp\left(\frac{X}{t}\right)\frac{dt}{t} - \sum_{a\le B} G^\sharp\left(\frac{X}{\min(A_0, a+1)}\right) \log\frac{\min(A_0, a+1)}{a}.$$

Here is the second step:

$$\sum_{\frac{X}{A_0} < m \le X} \frac{g(m)}{m} R(X/m) = -G^\sharp(X)R(1) + \int_1^{A_0} G^\sharp\left(\frac{X}{t}\right)\frac{dt}{t}$$
$$+ \sum_{2\le a\le B} G^\sharp\left(\frac{X}{a}\right)\left(R(a-1) - R(a) - \log\frac{a}{a-1}\right)$$
$$+ G^\sharp\left(\frac{X}{A_0}\right)\left(R(B) - \log\frac{A_0}{B}\right).$$

It is then obvious to establish the preliminary formula since $R(1) = 1 - \gamma$. The lemma follows readily. □

Proof of Theorem 1.3. Theorem 1.3 is a simple application of this lemma. We start from the key expression (2.2) of previous section, select $\eta = 1$, and treat the last term via the Covering Remainder Lemma above. □

4. A convolution identity

The convolution method consists in comparing an unknown arithmetical function to a known one, here $\mathbb{1}_{(d,q)=1}\frac{\mu^2(d)d}{\varphi(d)}$ together with $\mathbb{1}$.

Lemma 4.1.

$$\sum_{\substack{k^2\ell r|d,\\ r|q,\\ (k\ell,q)=1,\\ (k,\ell)=1}} \frac{\mu(rk)\mu^2(\ell)k}{\varphi(k)\varphi(\ell)} = \mathbb{1}_{(d,q)=1}\frac{\mu^2(d)d}{\varphi(d)}.$$

Proof. Let $g(d)$ be the multiplicative function of the right-hand side. When $h \geq 2$, we check that $g(p^h) = g(p^2)$. It is then easy to verify the claimed identity. □

For the rest of this paper, we define

$$r_2(X;q) = \sum_{\substack{k^2\ell r>X,\\ r|q,\\ (k\ell,q)=(k,\ell)=1}} \frac{\mu(rk)\mu^2(\ell)}{rk\varphi(k)\ell\varphi(\ell)}, \tag{4.1}$$

as well as

$$r_1(X) = \sum_{\substack{k^2\ell r\leq X,\\ r|q,\\ (k\ell,q)=(k,\ell)=1}} \frac{\mu^2(rk\ell)k}{\varphi(k)\varphi(\ell)} \tag{4.2}$$

both quantities that we now proceed to evaluate.

5. Cancellation in $\sum_{k>K}\mu(k)/[\varphi(k)k]$. Elementary methods

Evaluating $\sum_{\substack{k>K,\\ (k,q^*)=1}}\frac{\mu(k)}{\varphi(k)k}$ is a crucial step in bounding $r_2(X;q)$. This section is devoted to rather elementary / algorithmical results, the highpoint being Lemma 5.7. The proof of Theorem 1.1 requires in fact less work, but we prepare for the proof of Theorem 1.2 and we have gathered in a single place results of a same flavor.

Lemma 5.1 ([25, Lemma 3.4])**.** *For any real number $K > 0$, one has*

$$\sum_{k\leq K}\frac{\mu^2(k)}{\varphi(k)} = \log K + c_0 + \mathcal{O}^*\big(7.3/K^{1/3}\big).$$

Note that the left-hand side is also $G_1(K)$.

Lemma 5.2. *For $K \geq 50$, we have*

$$0.946 \leq (1/K)\sum_{k\leq K}\frac{\mu^2(k)k}{\varphi(k)} \leq 1.066.$$

After having proved Theorem 1.1, we can replace 0.946 *by* 0.989*. Further assuming $K \geq 10^6$, we can replace* 0.946 *by* 0.996 *and* 1.066 *by* 1.004*.*

Proof. We simply write

$$\begin{aligned}\sum_{k\leq K}\frac{\mu^2(k)k}{\varphi(k)} &= K\sum_{k\leq K}\frac{\mu^2(k)}{\varphi(k)} - \int_1^K\sum_{k\leq t}\frac{\mu^2(k)}{\varphi(k)}dt\\ &= K(\log K + c_0 + \mathcal{O}^*(7.3/K^{1/3})) - \int_1^K(\log t + c_0 + \mathcal{O}^*(7.3/t^{1/3}))dt\\ &= K + c_0 + \mathcal{O}^*\big(\tfrac{5}{2}7.3K^{2/3}\big).\end{aligned}$$

We ran a GP-script [23] and a Sage script that checked that

$$0.98999 \le (1/K)\sum_{k\le K}\frac{\mu^2(k)k}{\varphi(k)} \le 1.066 \quad (100\le K\le 10^8). \tag{5.1}$$

If we have already proved Theorem 1.1, we can replace the error term $\frac{5}{2}7.3K^{2/3}$ by $3\times 10.8\sqrt{K}$. □

We will need later the following estimate, but it is clearer to keep it here.

Lemma 5.3. *Assuming Theorem 1.1 has been proved, we have for $K\ge 50$,*

$$0.972 \le (2/K)\sum_{\substack{k\le K,\\ (k,2)=1}}\frac{\mu^2(k)k}{\varphi(k)} \le 1.066.$$

When $K\ge 10^6$, we can replace the couple $(0.972, 1.066)$ by $(0.995, 1.005)$.

Lemma 5.4. *Assuming Theorem 1.1 has been proved, and for $K\ge 10^6$ and b within $\{210, 2\,310, 30\,030, 510\,510\}$, we have*

$$\sum_{\substack{k\le K,\\ (k,b)=1}}\frac{\mu^2(k)k}{\varphi(k)} = \frac{\varphi(b)}{bK}(1+\mathcal{O}^*(0.0001)).$$

Of course, the following factorisations hold: $210 = 2\cdot 3\cdot 5\cdot 7$, $2\,310 = 11\cdot 210$, $30\,030 = 13\cdot 2\,310$ and $510\,510 = 17\cdot 30\,030$.

Proof. We proceed as in the proof of Lemma 5.2, though we have now Theorem 1.1 at our disposal. We write, with $c_2(b) = j^\star(b)b/\varphi(b)$:

$$\begin{aligned}\sum_{\substack{k\le K,\\ (k,b)=1}}\frac{\mu^2(k)k}{\varphi(k)} &= K\sum_{\substack{k\le K,\\ (k,b)=1}}\frac{\mu^2(k)}{\varphi(k)} - \int_1^K \sum_{\substack{k\le t,\\ (k,b)=1}}\frac{\mu^2(k)}{\varphi(k)}dt\\ &= K\frac{\varphi(b)}{b}(\log K + c(b) + \mathcal{O}^*(c_2(b)/K^{1/2}))\\ &\quad - \int_1^K \frac{\varphi(b)}{b}(\log t + c(b) + \mathcal{O}^*(c_2(b)/t^{1/2}))dt\\ &= \frac{\varphi(b)}{b}(K + c(b)) + \mathcal{O}^*\bigl(3\cdot j^\star(b)K^{1/2}\bigr).\end{aligned}$$

We ran a GP-script [23] that checked that claimed inequality for $K\le 10^{10}$ (As a matter of fact, it is enough to check this inequality until $K = 6\cdot 10^9$ when $q = 210$ and for all the moduli considered, until $K = 9.6\cdot 10^9$). We have

$$\sum_{\substack{k\le K,\\ (k,b)=1}}\frac{\mu^2(k)k}{\varphi(k)} = \frac{\varphi(b)}{bK}(1+\mathcal{O}^*(0.0001)) \quad (10^6\le K\le 10^{10}).$$

□

Lemma 5.5. *Assuming Theorem 1.1 has been proved, we have, when b belongs to $\{210, 2\,310, 30\,030\}$ and $K\ge K(b)$*

$$1-\epsilon_b^- \le \frac{Kb}{\varphi(b)}\sum_{\substack{k> K,\\ (k,b)=1}}\frac{\mu^2(k)}{k\varphi(k)} \le 1+\epsilon_b^+$$

where the following choices are possible:

$K(210)$	$K(2310)$	$K(30030)$	$K(2310)$	$K(30030)$
9523	*865*	*66*	*3500*	*5*
ϵ^+_{210}	ϵ^+_{2310}	ϵ^+_{30030}	ϵ^+_{2310}	ϵ^+_{30030}
0.00169	*0.0156*	*0.0911*	*0.0025*	*0.5344*
ϵ^-_{210}	ϵ^-_{2310}	ϵ^-_{30030}	ϵ^-_{2310}	ϵ^-_{30030}
0.0265	*0.0290*	*0.1073*	*0.0044*	*0.5488*

Moreover

$$\sum_{\substack{k>K,\\(k,210)=1}} \frac{\mu^2(k)}{k\varphi(k)} \le \begin{cases} 0.3/K & \textit{when } K \ge 14,\\ 0.24/K & \textit{when } K \ge 190,\end{cases}$$

and

$$\sum_{\substack{k>K,\\(k,30)=1}} \frac{\mu^2(k)}{k\varphi(k)} \le \begin{cases} 0.35/K & \textit{when } K \ge 14,\\ 0.276/K & \textit{when } K \ge 190.\end{cases}$$

It is difficult to understand the choice of $K(b)$ without reading the end of the proof of Lemma 5.6. Let we say that these values are below $2 \cdot 10^6/b$ and are chosen as small as possible without degrading the chosen value of $\epsilon_b^{\pm}$.

Proof. Summation by parts reduce the problem to using Lemma 5.4:

$$\begin{aligned}\sum_{\substack{k>K,\\(k,b)=1}} \frac{\mu^2(k)}{k\varphi(k)} &= 2\int_K^\infty \sum_{\substack{k\le t,\\(k,b)=1}} \frac{k\mu^2(k)}{k\varphi(k)}\frac{dt}{t^3} - \frac{1}{K^2}\sum_{\substack{k\le K,\\(k,b)=1}} \frac{k\mu^2(k)}{k\varphi(k)}\\ &= \frac{\varphi(b)}{b}\frac{2+\mathcal{O}^*(0.0002)-1+\mathcal{O}^*(0.0001)}{K}\end{aligned}$$

when $K \ge 10^6$. The lower bound is obtained similarly. It is easy to complete the proof by a direct computation. □

Here is an intriguing lemma.

Lemma 5.6. *Assuming Theorem 1.1 has been proved, we have, when* $K \ge 200$

$$\Bigg|\sum_{\substack{k>K,\\(k,q^*)=1}} \frac{\mu(k)}{\varphi(k)k}\Bigg| \le \frac{0.3616}{K}.$$

If q^* is assumed to be odd, we can replace 0.3616 *by* 0.3464*.*

The bound $K \ge 200$ will be reduced to $K \ge 14$ in the next lemma. It is absolutely not clear for the proof whether one can reach $o(1/K)$ when K goes to infinity, due to the presence of the coprimality condition. The proof goes by discussing the gcd of k with the product of the prime factors below 11 or 13 in the odd case. One can think that by selecting a larger bound, say B, the resulting constant would go to 0, i.e. that the RHS of the equation (5.4) below (replacing 0.026 by 0) is $o(\log B)$ uniformy for $q_0|q^* = \prod_{p\le B} p$. This seems difficult to prove. From a practical viewpoint, this lemma will have to be completed by Lemma 5.7 below.

Proof. Let $S(q^*, K)$ the sum to be estimated. When $30|q^*$, the lemma follows from the last estimate of Lemma 5.5. Let us now assume that $30 \nmid q^*$. We choose a base b within the set $\{210, 2\,310, 30\,030, 510\,510\}$, and set $\gcd(q^*, b) = b/q_0$. Notice that, when q^* is odd, then q_0 is even. We class the summation variable k in $S(q^*, x)$ according to its gcd δ with q_0. We get

$$S(q^*, K) = \sum_{\delta|q_0} \frac{\mu(\delta)}{\varphi(\delta)\delta} \sum_{\substack{\ell>K/\delta,\\(\ell,q_0q^*)=1}} \frac{\mu(\ell)}{\varphi(\ell)\ell} = \sum_{\substack{\ell>K/q_0,\\(\ell,q_0q^*)=1}} \frac{\mu(\ell)}{\varphi(\ell)\ell} \sum_{\substack{\delta|q_0,\\\delta>K/\ell}} \frac{\mu(\delta)}{\varphi(\delta)\delta}.$$

We order the divisors δ of q_0 in $1 = \delta_1 < \delta_2 < \cdots < \delta_I = q_0$ and thus

$$S(q^*, K) = \prod_{p|q_0} \frac{p^2-p-1}{p^2-p} \sum_{\substack{\ell > K,\\ (\ell, q_0 q^*)=1}} \frac{\mu(\ell)}{\varphi(\ell)\ell} + \sum_{1\le i\le I-1} \sum_{\substack{K/\delta_i \ge \ell > K/\delta_{i+1},\\ (\ell, q_0 q^*)=1}} \frac{\mu(\ell)}{\varphi(\ell)\ell} \sum_{i<j\le I} \frac{\mu(\delta_j)}{\varphi(\delta_j)\delta_j}. \tag{5.2}$$

Since $b|q_0q^*$, we have reached our main expression:

$$|S(q^*, K)| \le \prod_{p|q_0} \frac{p^2-p-1}{p^2-p} \sum_{\substack{\ell > K,\\ (\ell, b)=1}} \frac{|\mu(\ell)|}{\varphi(\ell)\ell} + \sum_{2\le i\le I} \sum_{\substack{K/\delta_{i-1} \ge \ell > K/\delta_i,\\ (\ell, b)=1}} \frac{|\mu(\ell)|}{\varphi(\ell)\ell} \left| \sum_{i\le j\le I} \frac{\mu(\delta_j)}{\varphi(\delta_j)\delta_j} \right|. \tag{5.3}$$

When $K/b \ge K(b)$, this gives us, by Lemma 5.5,

$$\frac{b}{\varphi(b)} K |S(q^*, K)| \le \prod_{p|q_0} \frac{p^2-p-1}{p^2-p}(1+\epsilon_b^+) + \sum_{1\le i\le I-1} \big((1+\epsilon_b^+)\delta_{i+1} - (1-\epsilon_b^-)\delta_i\big) \left| \sum_{i<j\le I} \frac{\mu(\delta_j)}{\varphi(\delta_j)\delta_j} \right|. \tag{5.4}$$

It is easy to run a program to check every possibilities of q_0.

```
{getlocbound(q0, epsplus, epsmoins, base, fixeddivisor = 1) =
   my(res = 0.0, aux = 1.0, divisorlist, di);
   forprime(p = 2, q0, if (q0%p==0, aux *= (1-1/p/(p-1)),));
   res += aux*(1 + epsplus);

   divisorlist = divisors(q0/fixeddivisor);
   aux = moebius(q0)/eulerphi(q0)/q0;
   forstep(i = length(divisorlist)-1, 1, -1,
           di = divisorlist[i]*fixeddivisor;
           res += ((1+epsplus)*divisorlist[i+1]
                  -(1-epsmoins)*di)* abs(aux);
           aux += moebius(di)/eulerphi(di)/di);
   return(res * eulerphi(base)/base);}

{getbound(epsplus=0.00214, epsmoins = 0.0265, base = 210, fixeddivisor = 1) =
   my(res = 0.0);
   fordiv(base/fixeddivisor, q0p,
     res = max(res, getlocbound(q0p*fixeddivisor, epsplus,
                                 epsmoins, base, fixeddivisor)));
   return(res);}
```

Anticipating on Lemma 5.7 below, we have to cover the range $K \ge 200$. We first use the above script with the ideal choice `epsplus=epsmoins=0` and found the following values:

210	2310	30030	510510
$0.3827\cdots$	$0.3560\cdots$	$0.3340\cdots$	$0.3167\cdots$

Now we have to get true values with effective `epsplus` and `epsmoins`, but it is better to be somewhat general to understand our choices. Say some values for $(\epsilon_b^\pm, K(b))$ are available in Lemma 5.5; then the value given by our script will be valid provided that $K/b \ge K(b)$. The resulting bound for K will be too large: Lemma 5.7 covers only the range $K \le 200$, so we would need to have

$bK(b) \le 200$. This is not realistic. Instead in the range $K \in [200, bK(b)]$, we compute the upper bound provided by (5.3). This enables us to extend the upper bound to $K \ge 200$. We are now blocked by a decent bound for $bK(b)$ in these computations; we found that going beyond $2 \cdot 10^6$ in general and beyond 10^7 for a specific choice would require too much work, so we used these values. Here are the maxima obtained by this additional script:

210	2310	30030
$0.3967\cdots$	$0.3686\cdots$	$0.3405\cdots$

. The worst values are obtained from $q_0 = b$ in each of the three cases. Now that this is fixed, we understand the values chosen in Lemma 5.5.

This being, *when q^* is even*, we select $b = 2\,310$, and run the same computations but only with this modulus and for all $K \le 10^7$. Note that we thus only need $2310K(2310) \ge 10^7$, i.e. $K(2310) \ge 4329$ is enough. We run `getbound(0.0025,0.0044,2310)` (where the values 0.0025 and 0.0044 are given by Lemma 5.5) and reached the bound 0.3616.

We have 32 divisors q_0 of $b = 2\,310$ to consider, but we may discard the (four) ones for which $30|2310/q_0$. This still took between three and four hours per divisor with GP. For $K \in [1000, 10^7]$ (resp. $K \in [200, 10^7]$), the worst constant was $0.3643 + \mathcal{O}^*(10^{-4})$ (resp. $0.3687 + \mathcal{O}^*(10^{-4})$) reached when $q_0 = 2310$. We can exclude this divisor since our q_0 is even, the next worst constant is $0.3530 + \mathcal{O}^*(10^{-4})$ (resp. $0.3608 + \mathcal{O}^*(10^{-4})$) reached when $q_0 = 210$. We can again exclude this divisor, the next worst constant is $0.3369 + \mathcal{O}^*(10^{-4})$ (resp. $0.3454 + \mathcal{O}^*(10^{-4})$) reached when $q_0 = 1155$.

When q^* is odd, q_0 is even, we need only cover the range $K \ge 378$. On this assumption, `getbound(0.5344,0.5487,30030,2)=0.3369...` is small, so we only have to cover the range $[378, 30030 \times 5]$ on selecting $b = 30\,030$. We ran `beprimal(378, 160000, 13)` and saw that the worst case is when $q_0 = 30030$ and $K = 817$ with value $0.3442 + \mathcal{O}^*(10^{-4})$. □

Lemma 5.7. *For any positive real number K and any modulus q^*, we have*

$$\Bigl|\sum_{\substack{k>K,\\(k,q^*)=1}} \frac{\mu(k)}{\varphi(k)k}\Bigr| \le \frac{1.26}{K}.$$

If Theorem 1.1 has been proved, and assuming $K \ge 14$, we can further reduce this 1.26 *to* 0.3616. *Assuming $K \in [14, 202]$, we can further reduce this* 1.26 *to* 0.3185.

If Theorem 1.1 has been proved, and assuming $K \ge 14$ and q^* to be odd, we can further reduce this 1.26 *to* 0.3442.

See Lemma 7.18 for a much better result when q^* is small. When K is large enough, say larger than K_0, Lemma 5.6 does the job. The present lemma completes the work for K below K_0; it relies on an algorithm of large complexity with respect to K_0.

Proof. When $K < 1$, this is trivially checked. Let $h(q) = \prod_{p|q} p(p-1)/(p^2-p-1)$ and $C = \prod_{p\ge 2} h(p)^{-1} = 0.373959\cdots$. We have

$$\sum_{\substack{k>K,\\(k,q^*)=1}} \frac{\mu(k)}{\varphi(k)k} = Ch(q^*) - \sum_{\substack{k\le K,\\(k,q^*)=1}} \frac{\mu(k)}{\varphi(k)k}.$$

- When $1 \le K < 2$, this quantity is $Ch(q^*) - 1$ which is negative, $\ge C - 1$ and this later quantity is $\ge -1.26/2$.
- When $2 \le K < 3$, and q^* is odd, this is $Ch(q^*) - \frac{1}{2}$ and $Ch(q^*) \in [h(2)^{-1}, C]$. We check that $|h(2)^{-1} - 1/2| \le 1.26/3$ and that $|C - 1/2| \le 1.26/3$.
- When $2 \le K < 3$, and q^* is even, this is $Ch(q^*) - 1$ and $Ch(q^*) \in [2C, 1]$. We check that $|2C - 1| \le 1.26/3$ and that $|1 - 1| \le 1.26/3$.

It is straightforward to write an algorithm. Let us assume we have a bound K_0 and let $q^\flat$ be the product of all the primes below K_0. Given q_1 dividing $q^\flat$, we consider all the

q^* that are such that $(q^\flat, q^*) = q_1$. We have

$$Ch(q_1) \le Ch(q^*) \le h(q_1)/h(q^\flat). \tag{5.5}$$

Note that when $q^\flat$ is large enough, the difference between C and $1/h(q^\flat)$ is small. Let us now look at the sum $S(K,q^*) = \sum_{\substack{k\le K,\\ (k,q^*)=1}} \frac{\mu(k)}{\varphi(k)k}$ which we compare to $S(K,q_1)$. We have

$$S(K,q^*) - S(K,q_1) = -\sum_{\substack{k\le K,\\ (k,q_1)=1,\\ (k,q^*)>1}} \frac{\mu(k)}{\varphi(k)k}.$$

When $K_0 \ge K$, this new sum is empty. But since if $(k,q_1)=1$ and $(k,q^*)>1$ then k has at least a prime factors $> K_0$. On assuming that $K_0^2 > K$, only one such prime p divides k. We write $k = sp$, where $s \le K/K_0$ is prime to q_1. Set $q^\sharp = \prod_{p\le c} p$ and $q_0 = (q^\sharp, q_1)$. Let

$$\Sigma^+(K,q_0) = \sum_{K_0<p\le K} \frac{1}{p(p-1)} \max\left(\sum_{\substack{s\le K/p,\\ (s,q_0)=1}} \frac{\mu(s)}{s\varphi(s)}, 0\right) \tag{5.6}$$

and let $\Sigma^-(K,q_0)$ be the corresponding sum with max instead of min. We have

$$S(K,q_1) - \Sigma^-(K,q) \le S(K) \le S(K,q_1) + \Sigma^+(K,q_0). \tag{5.7}$$

On joining (5.5) and (5.7), we reach

$$Ch(q_1) - S(K,q_1) - \Sigma^+(K,q_0) \le Ch(q^*) - S(K) \le h(q_1)/h(q^\flat) - S(K,q_1) + \Sigma^-(K,q_0).$$

Let us establish an algorithm from this inequality.

(1) We want to cover the range $K_{\min} \le K \le cK_0$, where $c\in[1,K_0)$ is a (small) real number.
(2) We define $q^\sharp = \prod_{p\le c} p$ and $q^\flat = \prod_{p\le K_0} p$. It is easy to (pre)compute all the $\Sigma^-(K,q_0)$ and $\Sigma^+(K,q_0)$ for $K\le cK_0$ and $q_0\le c$.
(3) We loop over $q^\flat$ and K.

We ran such a GP-script with up to 25 primes with $c=1$, and this takes care of $K<101$. The worst constant has been $2(1-C) = 1.252\cdots$ which is indeed not more than 1.26. When $K\ge 5$, the worst constant is 439/1008. When $K\ge 10$, the worst constant is 2/5.

On taking $c = 1.99$ and using the first 26 primes for $q^\flat$ to go to $K\le 202$, we prove that the worst constant is $0.3199\cdots$ provided that $K\ge 14$ and $K\le 202$. Reached at $K=202$ with $q_1 = 6$. Selecting c larger worsens the bound severely.

Assuming q^* to be odd, on taking $c=3.5$ and using the first 28 primes for $q^\flat$ to go to $K\le 378$, we prove that the worst constant is $0.3252\cdots$ provided that $K\ge 14$ and $K\le 378$. Reached at $K=378$ with $q_1 = 3.5.7.17.19.23.29$.

For $K\ge 50$, we proceed by integration by parts as follows:

$$\sum_{\substack{k>K,\\ (k,q^*)=1}} \frac{|\mu(k)|}{\varphi(k)k} \le 2\int_K^\infty \sum_{k\le t} \frac{|\mu(k)|k}{\varphi(k)}\frac{dt}{t^3} - \frac{1}{K^2}\sum_{k\le K}\frac{|\mu(k)|k}{\varphi(k)}$$
$$\le \frac{2\times 1.066 - 0.946}{K} \le \frac{1.186}{K}.$$

Theorem 1.1 is not required for this estimate, and when we assume Theorem 1.1, we can rely directly on Lemma 5.6. □

6. Proof of Theorem 1.1

Lemma 6.1. *We have*

$$r_1(X) \le 4.95\sqrt{X} j_1(q) \quad j_1(q) = \prod_{p|q} \frac{\sqrt{p}(p^{3/2}+p-\sqrt{p}-1)}{\sqrt{p}(p^{3/2}-\sqrt{p}+1)+1}.$$

Proof. We have

$$r_1(X) = \sum_{\substack{k^2\ell r\le X,\\ r|q,\\ (k\ell,q)=1,\\ (k,\ell)=1}} \frac{\mu^2(rk\ell)k}{\varphi(k)\varphi(\ell)} \le \sum_{\substack{a^2b^2\ell r\le X,\\ r|q,\\ (ab\ell,q)=1,\\ (ab,\ell)=1}} \frac{\mu^2(r\ell ab)}{\varphi(a)\varphi(\ell)}$$
$$\le \sum_{\substack{a^2\ell r\le X,\\ r|q,\\ (a\ell,q)=1,\\ (a,\ell)=1}} \frac{\mu^2(r\ell a)}{\varphi(a)\varphi(\ell)}\sqrt{\frac{X}{a^2\ell r}}.$$

We then simply extend the summations in a, ℓ and r:

$$\le \sqrt{X}\prod_{p|q}\frac{1+\frac{1}{\sqrt{p}}}{1+\frac{1}{\sqrt{p}(p-1)}+\frac{1}{p(p-1)}}\prod_{p\ge 2}\Big(1+\frac{1}{\sqrt{p}(p-1)\frac{1}{1+\frac{1}{p(p-1)}}}\Big)\prod_{p\ge 2}\Big(1+\frac{1}{p(p-1)}\Big)$$
$$\le 4.95\sqrt{X}\prod_{p|q}\frac{p(p-1)+\sqrt{p}(p-1)}{p(p-1)+\sqrt{p}+1} = 4.95\sqrt{X} j_1(q)$$

as required. □

Lemma 6.2. *We have*

$$|r_2(X;q)| \le \frac{3.9}{\sqrt{X}} j_2(q), \quad j_2(q) = \prod_{p|q}\frac{p^{3/2}+p-\sqrt{p}-1}{p^{3/2}-\sqrt{p}+1}.$$

The expression given for j_1 and j_2 makes it clear that $j_1 \le j_2$ so we can simplify the error term. Note however that $j_1(2) = 0.773\cdots$ which is quite smaller than $j_2(2) = 2$.

Proof. The definition of r_2 is given at (4.1). We treat the summation over k by Lemma 5.7 and obtain

$$|r_2(X;q)| \le \frac{1.26}{\sqrt{X}} \sum_{\substack{\ell\ge 1,\\ r|q,\\ (\ell,q)=1}} \frac{\mu^2(r)\mu^2(\ell)}{\sqrt{r}\sqrt{\ell}\varphi(\ell)}$$
$$\le \frac{1.26}{\sqrt{X}}\prod_{p|q}\frac{\sqrt{p}(p-1)}{1+\sqrt{p}(p-1)}\frac{1+\sqrt{p}}{\sqrt{p}}\prod_{p\ge 2}\Big(1+\frac{1}{\sqrt{p}(p-1)}\Big)$$
$$\le \frac{3.87}{\sqrt{X}}\prod_{p|q}\frac{(p-1)(1+\sqrt{p})}{1+\sqrt{p}(p-1)} = \frac{3.87}{\sqrt{X}} j_2(q).$$

□

Proof of Theorem 1.1. We use Theorem 1.4 together with Lemma 6.1 and 6.2. Notice that $G^\sharp = r_2$. We thus derive that

$$G_q(D) = \sum_{\substack{k,\ell\ge 1,\\ r|q,\\ (k\ell,q)=1}} \frac{\mu^2(k\ell)\mu(rk)}{\varphi(k)\varphi(\ell)k\ell r}\Big(\log\frac{D}{k^2\ell r}+\gamma\Big)$$
$$+\mathcal{O}^*\left((2+|\gamma-\log\eta|)\frac{3.9}{\sqrt{\eta D}} j_2(q) + \frac{\gamma}{D} 4.95\sqrt{\eta D} j_1(q)\right).$$

The main term is identified via [25, Lemma 3.4]. For the error term, we discuss according to whether $2|q$ or not. When q is odd, we use $j_1(q) \le j_2(q)$ and select $\eta = e^\gamma$. This leads to the error term $\mathcal{O}^*(5.82 j_2(q)/\sqrt{D})$. When $q = 2q'$ with q' odd, we select again $\eta = e^\gamma$, leading to the error term $\mathcal{O}^*(4.95 j_2(q')/\sqrt{D})$. For the cases $q = 6, 30, 210, 2310$ and $q = 30\,030$ we again select $\eta = e^\gamma$. Our formula follows readily. □

7. Cancellation in $\sum_{k>K} \mu(k)/[\varphi(k)k]$. Analytical estimates

We continue with the problem of bounding $\sum_{\substack{k>K,\\(k,q^*)=1}} \frac{\mu(k)}{\varphi(k)k}$ but we use the results on the Moebius function recalled in Lemma 7.14. The main point is Lemma 7.18.

7.1. **Some asymptotic estimates.**

Lemma 7.1. *We have, when $L > 0$,*

$$\sum_{\ell \le L} \frac{\mu^2(\ell)}{\varphi(\ell)} \prod_{p|\ell}(1+p^{-1}) = \frac{15}{\pi^2}(\log L + c_2) + \mathcal{O}^*\big(16.4/L^{1/3}\big)$$

where

$$c_2 = \gamma - \sum_{p\ge 2} \frac{2\log p}{p^3 - p^2 + p} = 0.187\,529\cdots$$

Proof. We use [25, Lemma 3.4]. We define

$$f(\ell) = \frac{\mu^2(\ell)}{\varphi(\ell)} \prod_{p|\ell}(1+p^{-1}).$$

First note that

$$\begin{aligned}\sum_{\ell\ge 1} \frac{f(\ell)}{\ell^s} &= \prod_{p\ge 2}\left(1 + \frac{p+1}{(p-1)p^{s+1}}\right)\\ &= \zeta(s+1)\prod_{p\ge 2}\left(1 + \frac{2}{(p-1)p^{s+1}} - \frac{p+1}{(p-1)p^{2s+2}}\right) = \zeta(s+1)H(s)\end{aligned}$$

say, which gives $\ell f(\ell) = \sum_{d|\ell} g(d)$ where g is the multiplicative function defined on prime powers by

$$g(p) = \frac{2}{p-1}, \quad g(p^2) = -\frac{p+1}{p-1}, \quad g(p^k) = 0 \quad (k \ge 3).$$

We note that

$$H(0) = \prod_{p\ge 2}\left(1 + \frac{2}{(p-1)p} - \frac{p+1}{(p-1)p^2}\right) = \prod_{p\ge 2}\left(1+\frac{1}{p^2}\right) = \frac{\zeta(2)}{\zeta(4)} = \frac{15}{\pi^2}.$$

Furthermore

$$\frac{-1}{H(0)}\sum_{m\ge 1}\frac{g(m)\log m}{m} = \frac{H'(0)}{H(0)} = -\sum_{p\ge 2}\frac{2\log p}{p^3-p^2+p},$$

while

$$\overline{H}(-1/3) = \prod_{p\ge 2}\left(1 + \frac{2}{(p-1)p^{2/3}} + \frac{p+1}{(p-1)p^{4/3}}\right) \le 18.$$

□

Lemma 7.2. *We have, when $L > 0$,*

$$\sum_{\ell\le L}\mu^2(\ell)\prod_{p|\ell}\frac{p^2+p}{p^3-p^2+p+1} = d_0(\log L + c_3) + \mathcal{O}^*\big(9.7/L^{1/3}\big)$$

where $d_0 = 1.026\cdots$ and

$$c_3 = \gamma - \sum_{p\ge 2}\frac{(3p+1)\log p}{p^4 - p^3 + 3p^2 - p - 2} = 0.048\,757\cdots$$

Proof. We again use [25, Lemma 3.4]. We define

$$f(\ell)=\mu^2(\ell)\prod_{p|\ell}\frac{p^2+p}{p^3-p^2+p+1}.$$

First note that

$$\begin{aligned}\sum_{\ell\ge1}\frac{f(\ell)}{\ell^s}&=\prod_{p\ge2}\Bigl(1+\frac{p^2+p}{(p^3-p^2+p+1)p^s}\Bigr)\\&=\zeta(s+1)\prod_{p\ge2}\Bigl(1+\frac{2p^2-p-1}{(p^3-p^2+p+1)p^{s+1}}-\frac{p+1}{(p^3-p^2+p+1)p^{2s}}\Bigr)\\&=\zeta(s+1)H(s)\end{aligned}$$

say, which gives $\ell f(\ell)=\sum_{d|\ell}g(d)$ where g is the multiplicative function defined on prime powers by

$$g(p)=\frac{2p^2-p-1}{p^3-p^2+p+1},\quad g(p^2)=-\frac{p^3+p^2}{p^3-p^2+p+1},\quad g(p^k)=0\quad(k\ge3).$$

We note that

$$\begin{aligned}d_0=H(0)&=\prod_{p\ge2}\Bigl(1+\frac{2p^2-p-1}{(p^3-p^2+p+1)p}-\frac{p+1}{p^3-p^2+p+1}\Bigr)\\&=\prod_{p\ge2}\Bigl(1+\frac{p^2-2p-1}{(p^3-p^2+p+1)p}\Bigr)=1.026\cdots\end{aligned}$$

Furthermore

$$\frac{-1}{H(0)}\sum_{m\ge1}\frac{g(m)\log m}{m}=\frac{H'(0)}{H(0)}=-\sum_{p\ge2}\frac{(3p+1)\log p}{p^4-p^3+3p^2-p-2},$$

while

$$\overline{H}(-1/3)=\prod_{p\ge2}\Bigl(1+\frac{2p^2-p-1}{(p^3-p^2+p+1)p^{2/3}}+\frac{p+1}{(p^3-p^2+p+1)p^{-2/3}}\Bigr)\le9.7.$$

□

7.2. **Majorising tails of averages.**

Lemma 7.3. *When $L>0$, we have $\sum_{\ell>L}\frac{\mu^2(\ell)}{\ell\varphi(\ell)}\le1.96/L$. When $L\ge1$, we can replace 1.96 by 1.14.*

In [25, Lemma 3.10], a similar estimate is proved, but with a constant 4 instead of the 1.14 above.

Proof. Let us denote by $S(L)$ the sum we want. A summation by parts gives us

$$\begin{aligned}S(L)&=2\int_L^\infty\sum_{L<\ell\le t}\frac{\mu^2(\ell)\ell}{\varphi(\ell)}\frac{dt}{t^3}\\&=-\sum_{\ell\le L}\frac{\mu^2(\ell)\ell}{\varphi(\ell)L^2}+2\int_L^\infty\sum_{\ell\le t}\frac{\mu^2(\ell)\ell}{\varphi(\ell)}\frac{dt}{t^3}\end{aligned}$$

and this not more than $(2\times1.004-0.996)/L\le1.012/L$ when $L\ge10^6$ (as in the proof of Lemma 5.7) by Lemma 5.2. We complete by a direct verification up to 10^6. □

Lemma 7.4. *When $L\ge1$, we have $\sum_{\substack{\ell>L,\\(\ell,2)=1}}\frac{\mu^2(\ell)}{\ell\varphi(\ell)}\le0.592/L$.*

Proof. On denoting by $S(L)$ the sum we want, a summation by parts gives us

$$\begin{aligned} S(L) &= 2\int_L^\infty \sum_{\substack{L<\ell\le t,\\ (\ell,2)=1}} \frac{\mu^2(\ell)\ell}{\varphi(\ell)}\frac{dt}{t^3} \\ &= -\sum_{\substack{\ell\le L,\\ (\ell,2)=1}} \frac{\mu^2(\ell)\ell}{\varphi(\ell)L^2} + 2\int_L^\infty \sum_{\substack{\ell\le t,\\ (\ell,2)=1}} \frac{\mu^2(\ell)\ell}{\varphi(\ell)}\frac{dt}{t^3} \end{aligned}$$

and this not more than $(2\times 1.005 - 0.995)/2$ by Lemma 5.3. We complete by a direct verification up to 10^6. □

Lemma 7.5. *When $L\ge 1$, we have $\sum_{\substack{\ell>L,\\ (\ell,2)=1}} \frac{\mu^2(\ell)}{\sqrt{\ell}\varphi(\ell)} \le 1.19/\sqrt{L}$.*

Proof. We have

$$\sum_{\substack{\ell>L,\\ (\ell,2)=1}} \frac{\mu^2(\ell)}{\sqrt{\ell}\varphi(\ell)} = \sum_{\substack{\ell>L,\\ (\ell,2)=1}} \frac{\mu^2(\ell)\sqrt{Y}}{\ell\varphi(\ell)} + \int_Y^\infty \sum_{\substack{\ell>t,\\ (\ell,2)=1}} \frac{\mu^2(\ell)}{\ell\varphi(\ell)}\frac{dt}{2\sqrt{t}}$$

and an appeal to Lemma 7.4 concludes when $L\ge 1$. □

Lemma 7.6. *When $L>0$, we have $\sum_{\ell>L} \frac{\mu^2(\ell)\prod_{p|\ell}(1+p^{-1})}{\ell\varphi(\ell)} \le 2.83/L$. When $L\ge 6$, we can decrease 2.83 to 1.85, and when $L\ge 30$, we can decrease it further to 1.63.*

Proof. We proceed as in the proof of Lemma 7.3. Let us denote by $S(L)$ the sum we want to bound and $f(\ell)=\mu^2(\ell)\prod_{p|\ell}(1+p^{-1})/\varphi(\ell)$. A summation by parts gives us, via Lemma 7.1,

$$\begin{aligned} S(L) &= \int_L^\infty \sum_{L<\ell\le t} f(\ell)\frac{dt}{t^2} = -\sum_{L<\ell\le t}\frac{f(\ell)}{L} + \int_L^\infty \sum_{\ell\le t} f(\ell)\frac{dt}{t^2} \\ &= -\frac{15}{\pi^2 L}(\log L + c_2) + \int_L^\infty \frac{15}{\pi^2}(\log t + c_2)\frac{dt}{t^2} + \mathcal{O}^*\Big(\frac{16.4(1+\frac34)}{L^{4/3}}\Big) \\ &= \frac{15}{\pi^2 L} + \mathcal{O}^*\big(29/L^{4/3}\big) \end{aligned}$$

and this not more than 2.5 when $L\ge 10^5$. We complete by a direct verification up to 10^6. □

Lemma 7.7. *When $L>0$, we have $\sum_{\ell>L} \frac{\mu^2(\ell)\prod_{p|\ell}(1+p^{-1})}{\ell^2\varphi(\ell)} \le 2.02/L^2$.*

Proof. We denote by $S(L)$ the sum to be bounded and $f(\ell)=\mu^2(\ell)\prod_{p|\ell}(1+p^{-1})/\varphi(\ell)$. A summation by parts gives us, via Lemma 7.1,

$$\begin{aligned} S(L) &= 2\int_L^\infty \sum_{L<\ell\le t} f(\ell)\frac{dt}{t^3} = -\sum_{L<\ell\le t}\frac{f(\ell)}{L^2} + 3\int_L^\infty \sum_{\ell\le t} f(\ell)\frac{dt}{t^3} \\ &= -\frac{15}{\pi^2 L^2}(\log L + c_2) + 3\int_L^\infty \frac{15}{\pi^2}(\log t + c_2)\frac{dt}{t^3} + \mathcal{O}^*\Big(\frac{16.4(1+\frac37)}{L^{7/3}}\Big) \\ &= \frac{15}{2\pi^2 L^2} + \mathcal{O}^*\big(24/L^{7/3}\big) \end{aligned}$$

and this not more than 2 when $L\ge 10^4$. We complete by a direct verification up to 10^6. □

Lemma 7.8. *When $L>0$, we have $\sum_{\ell>L} \frac{\mu^2(\ell)\prod_{p|\ell}(1+p^{-1})}{\sqrt{\ell}\varphi(\ell)} \le 4.43/\sqrt{L}$. This is ≤ 3.80 when $L\ge 2$, and ≤ 3.12 when $L\ge 100$.*

Proof. We denote by $S(L)$ the sum we want to bound and $f(\ell) = \mu^2(\ell)\prod_{p|\ell}(1+p^{-1})/\varphi(\ell)$. A summation by parts gives us, via Lemma 7.1,

$$\begin{aligned} S(L) &= \tfrac{3}{2}\int_L^\infty \sum_{L<\ell\le t} f(\ell)\frac{dt}{t^{3/2}} = -\sum_{L<\ell\le t}\frac{f(\ell)}{\sqrt{L}} + \tfrac{3}{2}\int_L^\infty \sum_{\ell\le t} f(\ell)\frac{dt}{t^{3/2}} \\ &= -\frac{15}{\pi^2\sqrt{L}}(\log L + c_2) + \tfrac{3}{2}\int_L^\infty \frac{15}{\pi^2}(\log t + c_2)\frac{dt}{t^{3/2}} + \mathcal{O}^*\left(\frac{16.4(1+\frac{3}{2})}{L^{2/3}}\right) \\ &= \frac{30}{\pi^2\sqrt{L}} + \mathcal{O}^*\big(41/L^{2/3}\big) \end{aligned}$$

and this not more than 2 when $L \ge 10^4$. We complete by a direct verification up to 10^6. □

Lemma 7.9. *When $L > 0$, we have $\sum_{\ell>L}\frac{\mu^2(\ell)}{\ell^2}\prod_{p|\ell}\frac{p^2+p}{p^3-p^2+p+1} \le 1.25/L^2$. When $L \ge 20$, we can decrease the constant 1.25 to 0.66.*

Proof. We denote by $S(L)$ the sum we need and $f(\ell) = \mu^2(\ell)\prod_{p|\ell}\frac{p^2+p}{p^3-p^2+p+1}$. A summation by parts gives us, via Lemma 7.2,

$$\begin{aligned} S(L) &= 2\int_L^\infty \sum_{L<\ell\le t} f(\ell)\frac{dt}{t^3} = -\sum_{L<\ell\le t}\frac{f(\ell)}{L^2} + 3\int_L^\infty \sum_{\ell\le t} f(\ell)\frac{dt}{t^3} \\ &= -\frac{d_0}{L^2}(\log L + c_3) + 2\int_L^\infty d_0(\log t + c_3)\frac{dt}{t^3} + \mathcal{O}^*\left(\frac{9.7(1+\frac{3}{7})}{L^{7/3}}\right) \\ &= \frac{d_0}{2L^2} + \mathcal{O}^*\big(13.9/L^{7/3}\big) \end{aligned}$$

and this not more than 2 when $L \ge 10^4$. We complete by a direct verification up to 10^6. □

Lemma 7.10. *When $K > 0$, we have $\sum_{k>K}\frac{\mu^2(k)}{k^2\varphi(k)} \le 0.606/K^2$.*

Proof. A summation by parts gives us

$$\begin{aligned} \sum_{k>K}\frac{\mu^2(k)}{k^2\varphi(k)} &= 3\int_K^\infty \sum_{K<\ell\le t}\frac{\mu^2(k)k}{\varphi(k)}\frac{dt}{t^4} \\ &= -\sum_{k\le K}\frac{\mu^2(k)k}{\varphi(k)K^3} + 3\int_K^\infty \sum_{k\le t}\frac{\mu^2(k)k}{\varphi(k)}\frac{dt}{t^4} \end{aligned}$$

and this not more than $(\frac{3}{2}\times 1.004 - 0.996)/K \le 0.51/K$ when $K \ge 10^6$ (as in the proof of Lemma 5.7) by Lemma 5.2. We complete by a direct verification up to 10^6. □

Lemma 7.11. *When $X > 0$, we have*

$$\sum_{a^2b^3>X}\frac{\mu^2(ab)}{a\varphi(a)b^2\varphi(b)} \le \frac{2.45}{\sqrt{X}}.$$

Proof. We split the sum in two:

$$\sum_{a^2b^3>X}\frac{\mu^2(ab)}{a\varphi(a)b^2\varphi(b)} \le \sum_{b^3>X,a\ge1}\frac{\mu^2(ab)}{a\varphi(a)b^2\varphi(b)} + \sum_{b^3\le X}\sum_{\substack{a^2>X/b^3,\\ (a,b)=1}}\frac{\mu^2(ab)}{a\varphi(a)b^2\varphi(b)}.$$

In the second sum, we forget the condition $(a, b) = 1$ when b is odd, and degrade it to $(a, 2) = 1$ when b is even. We get via Lemma 7.3 and 7.4

$$\sum_{a^2b^3>X} \frac{\mu^2(ab)}{a\varphi(a)b^2\varphi(b)} \le 1.96 \sum_{b^3>X} \frac{\mu^2(b)}{b^2\varphi(b)} + 1.14 \sum_{\substack{b^3\le X,\\ (b,2)=1}} \frac{\mu^2(b)}{b^2\varphi(b)}\sqrt{\frac{b^3}{X}} + 0.592 \sum_{\substack{b^3\le X/8,\\ (b,2)=1}} \frac{\mu^2(b)}{4b^2\varphi(b)}\sqrt{\frac{b^3}{X}}$$

and, via Lemma 7.10

$$\sum_{\substack{a^2b^3>X,\\ (ab,q^*)=1}} \frac{\mu^2(ab)}{a\varphi(a)b^2\varphi(b)} \le 1.96 \times \frac{0.606}{X^{2/3}} + \frac{1.14 + 0.592/4}{\sqrt{X}} \sum_{\substack{b^3\le X,\\ (b,2)=1}} \frac{\mu^2(b)}{b^{1/2}\varphi(b)}$$

getting the upper bound

$$\frac{1.19}{X^{2/3}} + \frac{2.32}{\sqrt{X}}$$

We check directly that our quantity is below $2.45/\sqrt{X}$ when $1 \le X \le 10^6$. Assuming that $X \ge 5000$, we can reduce the 2.45 to 2.2. We would have to be more careful in the above analysis. □

Lemma 7.12. *When $X > 0$, we have*

$$\sum_{a^2b^3>X} \frac{\mu^2(ab)\prod_{p|ab}(1+p^{-1})}{a\varphi(a)b^2\varphi(b)} \le \frac{6.21}{\sqrt{X}}.$$

Proof. We proceed as in Lemma 7.11 and split the sum in two:

$$\sum_{a^2b^3>X} \frac{\mu^2(ab)\prod_{p|ab}(1+p^{-1})}{a\varphi(a)b^2\varphi(b)} \le \sum_{b^3>X/6^2, a\ge 1} \frac{\mu^2(ab)\prod_{p|ab}(1+p^{-1})}{a\varphi(a)b^2\varphi(b)} + \sum_{b^3\le X/6^2} \sum_{a^2>X/b^3} \frac{\mu^2(ab)\prod_{p|ab}(1+p^{-1})}{a\varphi(a)b^2\varphi(b)}$$

and thus via Lemma 7.6 and on using $\prod_{p\ge 2}(1+\frac{p+1}{p^2(p-1)}) \le 2.42$,

$$\sum_{a^2b^3>X} \frac{\mu^2(ab)\mu^2(ab)\prod_{p|ab}(1+p^{-1})}{a\varphi(a)b^2\varphi(b)} \le 2.42 \sum_{b^3>X/6^2} \mu^2(b)\prod_{p|b}\frac{p+1}{p(p^3-p^2+p+1)} + 1.85 \sum_{b^3\le X} \frac{\mu^2(b)\prod_{p|b}(1+p^{-1})}{b^2\varphi(b)}\sqrt{\frac{b^3}{X}}.$$

Lemma 7.9 gives us

$$\sum_{\substack{a^2b^3>X,\\ (ab,q^*)=1}} \frac{\mu^2(ab)\mu^2(ab)\prod_{p|ab}(1+p^{-1})}{a\varphi(a)b^2\varphi(b)} \le 2.42 \times \frac{0.66\cdot 6^{4/3}}{X^{2/3}} + \frac{1.85}{\sqrt{X}} \prod_{p\ge 2}\Big(1+\frac{1}{\sqrt{p}(p-1)}\Big)$$

getting the upper bound

$$\frac{5.69}{\sqrt{X}} + \frac{17.5}{X^{2/3}}.$$

We check directly that our quantity is below $4.4/\sqrt{X}$ when $1 \le X \le 1.5\cdot 10^9$. □

7.3. **External lemmas concerning the Moebius function.** Here is part of [7, Lemma 10.2]. See also [28, Theorem 1.1].

Lemma 7.13. *We have, when $q^* \ge 1$ and $x > 0$,*
$$\Big|\sum_{\substack{d\le x,\\ (d,q^*)=1}}\frac{\mu(d)}{d}\Big| \le 1.$$

Here is part of [29, Theorem 1.12, 1.13].

Lemma 7.14. *We have, when $x > q^* \ge 1$,*
$$\Big|\sum_{\substack{d\le x,\\ (d,q^*)=1}}\frac{\mu(d)}{d}\Big| \le \kappa_1(x/q^*)\prod_{p|q^*}(1+p^{-1})/\log(x/q^*).$$
where
$$\kappa_1(t)=\begin{cases}4/5 & \text{when } 1<t<296,\\ 1/2 & \text{when } 296\le t<687,\\ 5/16 & \text{when } 687\le t<882,\\ 5/38 & \text{when } 882\le t<11\,811,\\ 1/7 & \text{when } 11\,811\le t.\end{cases}$$

7.4. **Taking care of the $1/\log(X/\ell)$ factor.** In several instances, the usage of Lemma 7.14 leads to expressions of the shape
$$S_\kappa(f;U,c)=\sum_{u\le U/c}\frac{\kappa(U/u)f(u)}{\log(U/u)}$$
where c is uniformly taken as $=50$. This value is however irrelevant in the general discussion. The function κ being either the function κ_1 defined in Lemma 7.14 or the constant function 1 and f is some non-negative arithmetical function such that $\sum_{u\ge1}f(u)$ converges. Since κ is a decreasing step function with jumps at $c=a_1<a_2<\cdots<a_I$, we have
$$S_\kappa(f;U,c)=\kappa(a_I^+)S(f;U,a_I)+\sum_{1\le i\le I-1}\kappa(a_i^+)\big(S(f;U,a_i)-S(f;U,a_{i+1})\big)$$
where
$$S(f;U,a)=\sum_{u\le U/a}\frac{f(u)}{\log(U/u)}.$$
This gives
$$S_\kappa(f;U,c)=\kappa(c^+)S(f;U,c)-\sum_{2\le i\le I}S(f;U,a_i)\big(\kappa(a_{i-1}^+)-\kappa(a_i^+)\big). \tag{7.1}$$
We first handle $S(f;U,a)$ by an integration by parts:
$$S(f;U,a)=\sum_{u\le U/a}\frac{f(u)}{\log a}-\int_a^U\sum_{u\le U/t}f(u)\frac{dt}{t(\log t)^2}. \tag{7.2}$$
This expression is usable to compute $S(f;U,a)$ for bounded values of U. Let us introduce the notation $C(f)=\sum_{u\ge1}f(u)$ and $G(f;t)=\sum_{u\le t}f(t)$ as well as
$$S_0(f;U,a)=\int_a^U\sum_{U/t<u}f(u)\frac{dt}{t(\log t)^2}. \tag{7.3}$$
We have
$$S(f;U,a)=\sum_{u\le U/a}\frac{f(u)}{\log a}-C(f)\Big(\frac{1}{\log a}-\frac{1}{\log U}\Big)+S_0(f;U,a).$$

i.e.

$$S(f;U,a) = -\sum_{u>U/a}\frac{f(u)}{\log U} + \frac{C(f)}{\log U} + S_0(f;U,a). \tag{7.4}$$

In order to evaluate $S(f;U,a)$ for large U, we use some upper bound for $\sum_{U/t<u} f(u)$. In this process, the numerical difficulty comes from the fact that we need such an evaluation when U/t can be as small as 1, and this required uniformity is a drag on the constants. For instance, when we have a bound of the shape $\sum_{x<u} f(u) \ll 1/\sqrt{x}$, the term $S_0(f;U,a)$ is $\ll 1/(\log x)^2$, but this happens only when x is rather large. It is much better to handle the values when x (i.e. initially U/t) is small by direct computations, when f is indeed directly computable (this is not the case for instance when $f(u)$ vanishes as soon as u is not co-prime to some parameter q). We have, for an integer parameter $H \geq 1$:

$$0 \leq S_0(f;U,a) - \sum_{1\leq h\leq \min(H,U/c)} \int_{\max(a,U/(h+1))}^{U/h} \big(C(f) - G(f;U/t)\big)\frac{dt}{t(\log t)^2} \\ \leq \int_{a\leq t\leq U/(H+1)} \sum_{U/t<u} f(u)\frac{dt}{t(\log t)^2}.$$

We have used the notation $\int_{a\leq t\leq b}$ and not $\int_a^b$: in the first case the integral has value 0 when $b < a$ while the second one is usually understood as $-\int_b^a$. This upper bound develops into:

$$0 \leq S_0(f;U,a) - \sum_{1\leq h\leq \min(H,U/a)} \Big(C(f) - \sum_{u\leq h} f(u)\Big)\left(\frac{1}{\log\frac{U}{h}} - \frac{1}{\log\max(a,\frac{U}{h+1})}\right) \\ \leq \int_{a\leq t\leq U/(H+1)} \sum_{U/t<u} f(u)\frac{dt}{t(\log t)^2}. \tag{7.5}$$

The very last problem consists in estimating the first summand in (7.4). We use

$$\sum_{u>U/a} f(u) = \begin{cases} C(f) - \sum_{u\leq U/a} f(u) & \text{when } U/a \leq H+1, \\ \mathcal{O}^*(\sum_{u>H+1} f(u)) & \text{when } U/a \geq H+1. \end{cases} \tag{7.6}$$

The cutting point $a\cdot(H+1)$ in the conditions $U \leq a\cdot(H+1)$ and $U > a\cdot(H+1)$ is somewhat arbitrary. It turns out that the small values of U are going to have a predominant role; for such values, the partial sum $\sum_{u\leq U/a} f(u)$ may be noticeably smaller than $C(f)$. In case, there is too large a jump between around $c(H+1)$, it can be dampened by increasing the value of H.

If we gather the above, here is how we proceed:

- We split $S_\kappa(f;U,c)$ according to (7.1).
- We compute a lower bound $S^{(1)}(f;U,a)$ and an upper bound $S^{(2)}(f;U,a)$ for $S(f;U,a)$ on following (7.4), (7.5) and (7.6) and a given bound for $\sum_{U/t<u} f(u)$.
- We produce form these bounds an upper bound for $S_\kappa(f;U,c)$. This bound is a "simple" function that we simply study. Practically it is enough to compute and plot it a large enough range; It vanishes at infinity, and this infinity is close enough that the remaining range can be covered by computations.

This is what we do in the next two lemmas.

Lemma 7.15. *When $x \geq 1800$, we have*

$$(\log x)\sum_{a^2b^3\leq x/50} \frac{\mu^2(ab)\kappa_1(x/(a^2b^3))\prod_{p|ab}(1+1/p)}{a\varphi(a)b^2\varphi(b)\log(x/a^2b^3)} \leq 1.89.$$

Proof. We use equation (7.5) with

$$f(\ell)=\begin{cases}\frac{\mu^2(ab)\prod_{p|ab}(1+1/p)}{a\varphi(a)b^2\varphi(b)} & \text{when } \ell=a^2b^3 \text{ with } \mu^2(ab)=1,\\ 0 & \text{else.}\end{cases}$$

and $c=50$. We have

$$C(f)=\prod_{p\ge2}\biggl(1+\frac{1+1/p}{p(p-1)}+\frac{1+1/p}{p^2(p-1)}\biggr).$$

We select $H=10^6$ in the method above. The bound for the tail is provided by Lemma 7.12. The lengthy discussion above results in the following simple script:

```
{getval30(n)=
   my(res=1.0, p, fac = factor(n)~);
   if(n==1, return(1),);
   for(k = 1, length(fac),
      p = fac[1,k];
      if(fac[2, k] == 2, res *= (p+1)/p^2/(p-1),
         if(fac[2, k] == 3, res *= (p+1)/p^3/(p-1),
            return(0))));
   return(res);}

cc8 = prodeuler(p = 2,1000000,1.0+(p+1)^2/p^3/(p-1));

{upper8inner(x, c, H, notail=0)=
   my(res = 0.0, aux = 0.0);
   for(h = 1, min(x/c, H),
      aux += getval30(h);
      res += (cc8 - aux)*(-1/log(x/h)+1/log(max(c,x/(h+1)))));
   if(x <= c*(H+1), res += aux/log(x), res += cc8/log(x));
   if(notail==1, return(res - 6.21/sqrt(H+1)/log(x)),);
   if((x <= c*(H+1)), return(res),
      res += 6.21/sqrt(x)*intnum( t = c, x/(H+1), 1/sqrt(t)/log(t)^2));
   return(res);}

{upper8lower(x, c=50, H=300) = upper8inner(x, c, H, 1);}
{upper8upper(x, c=50, H=300) = upper8inner(x, c, H, 0);}

{upper8(x, H=300)=
  4/5*upper8upper(x, 50, H)
  -(4/5-1/2)*upper8lower(x,296,H)
  -(1/2-5/16)*upper8lower(x,687,H)
  -(5/16-5/38)*upper8lower(x,882,H)
  -(5/38-1/7)*upper8lower(x,11811,H);}

{bupper8(x, H=400)= upper8(x,H)*log(x);}
```

□

Lemma 7.16. *When $x\ge50$, we have*

$$(\log x)\sum_{\ell\le x/50}\frac{\mu^2(\ell)\prod_{p|\ell}(1+p^{-1})}{\sqrt{\ell}\varphi(\ell)\log(x/\ell)}\le 5.02.$$

Proof. We use equation (7.5) with $f(\ell) = \mu^2(\ell)\prod_{p|\ell}(1+p^{-1})/(\sqrt{\ell}\varphi(\ell))$ and $c = 50$. We have

$$C(f) = \prod_{p\geq 2}\left(1 + \frac{p+1}{p^{3/2}(p-1)}\right).$$

We use Lemma 7.8 and select $H = 10^6$ in the method above. The bound for the tail is provided by Lemma 7.8. □

7.5. **Usage.**

Lemma 7.17. *We have, when $K/q^* \geq 1800$,*

$$\left|\sum_{\substack{k>K,\\(k,q^*)=1}} \frac{\mu(k)}{\varphi(k)}\right| \leq \frac{1.89}{\log(K/q^*)}\prod_{p|q^*}\left(1+\frac{1}{p}\right) + 17.4\left(\frac{q^*}{K}\right)^{1/2}.$$

Proof. Let us denote by $S(K)$ the sum we want to bound. We check by multiplicativity that

$$\frac{\mu(k)}{\varphi(k)} = \sum_{dab^2=k}\frac{\mu(d)}{d}\frac{\mu^2(abd)}{a\varphi(a)b^2\varphi(b)}. \tag{7.7}$$

This identity leads to the following expression for $S(K)$:

$$S(K) = \sum_{\substack{a,b\geq 1,\\(ab,q^*)=1}} \frac{\mu^2(ab)}{a\varphi(a)b^2\varphi(b)} \sum_{\substack{d>K/(ab^2),\\(d,q^*ab)=1}}\frac{\mu(d)}{d}.$$

Since $\sum_{\substack{d\geq 1,\\(d,q^*ab)=1}}\mu(d)/d = 0$, we also have

$$S(K) = -\sum_{\substack{a,b\geq 1,\\(ab,q^*)=1}} \frac{\mu^2(ab)}{a\varphi(a)b^2\varphi(b)} \sum_{\substack{d\leq K/(ab^2),\\(d,q^*ab)=1}}\frac{\mu(d)}{d}.$$

When $q^*a^2b^3 \leq K/50$, we can use Lemma 7.14, while otherwise we rely on Lemma 7.13. We get:

$$|S(K)| \leq \sum_{\substack{q^*a^2b^3\leq K/50,\\(ab,q^*)=1}} \frac{\kappa_1(K/(q^*a^2b^3))\mu^2(ab)}{a\varphi(a)b^2\varphi(b)\log(K/(q^*a^2b^3))}\prod_{p|q^*ab}(1+p^{-1})$$
$$+ \sum_{\substack{q^*a^2b^3>K/50,\\(ab,q^*)=1}}\frac{\mu^2(ab)}{a\varphi(a)b^2\varphi(b)}.$$

We use Lemma 7.15 with $x = K/q^*$: the first sum is not more than $1.89\prod_{p|q^*}(1+p^{-1})/\log(K/q^*)$. This together with Lemma 7.11 (and noting that $2.45\sqrt{50} \leq 17.4$) yields

$$|S(K)| \leq \prod_{p|q^*}\frac{p+1}{p}\frac{1.89}{\log(K/q^*)} + 17.4\left(\frac{q^*}{K}\right)^{1/2}.$$

□

Lemma 7.18. *We have, when $K > q^*$ and $K \geq 14$,*

$$\left|\sum_{\substack{k>K,\\(k,q^*)=1}}\frac{\mu(k)}{k\varphi(k)}\right| \leq \frac{1.89\prod_{p|q^*}(1+p^{-1})}{K\log(K/q^*)} + \frac{5}{K}\left(\frac{q^*}{K}\right)^{1/2} + \frac{6.6}{K}\left(\frac{q^*}{K}\right)^{1/2}\mathbb{1}_{K\geq 1970q^*}.$$

The last statement means that the third summand is to be added only when K/q^* is larger than 1 970.

This lemma is qualitatively much better than Lemma 5.7 when q^* is small: it indeed uses the oscillations of the Moebius function.

Proof. When $K \geq 14$ and $K/q^* \leq 1970$, this is a consequence of Lemma 5.7. Otherwise, we use an integration by parts to get

$$\sum_{\substack{k>K,\\(k,q^*)=1}} \frac{\mu(k)}{k\varphi(k)} = \int_K^\infty \sum_{\substack{k>t,\\(k,q^*)=1}} \frac{\mu(k)}{\varphi(k)} \frac{dt}{t^2}.$$

The lemma follows readily after invoking Lemma 7.17. □

8. Bounding $r_2(X;1)$

8.1. Some asymptotic estimates.

Lemma 8.1. *We have, when* $L>0$, $\sum_{\ell>L} \frac{\mu^2(\ell)}{\sqrt{\ell}\varphi(\ell)} \leq 3.08/\sqrt{L}$. *When* $L \geq 1$, *we can replace* 3.08 *by* 2.28.

Proof. We have

$$\sum_{\ell>L} \frac{\mu^2(\ell)}{\sqrt{\ell}\varphi(\ell)} = \sum_{\ell>L} \frac{\mu^2(\ell)\sqrt{Y}}{\ell\varphi(\ell)} + \int_Y^\infty \sum_{\ell>t} \frac{\mu^2(\ell)}{\ell\varphi(\ell)} \frac{dt}{2\sqrt{t}}$$

and an appeal to Lemma 7.3 concludes when $L \geq 1$. Else a direct computation of $\prod_{p\geq 2}(1+\frac{1}{\sqrt{p}(p-1)})$ is enough. □

Lemma 8.2. *We have, when* $A>0$, $\sum_{a\leq A} \mu^2(a)a^{1/4}/\varphi(a) \leq 4\,A^{1/4}$.

Proof. We use Theorem 1.1 and an integration by parts:

$$\begin{aligned}\sum_{a\leq A} \mu^2(a)a^{1/4}/\varphi(a) = A^{1/4}\Big(\log A + c_0 + \mathcal{O}^*\Big(\frac{5.9}{\sqrt{A}}\Big)\Big) - \frac{1}{4}\int_1^A \big(\log t + c_0\big)\frac{dt}{t^{3/4}}\\ - \frac{1}{4}\int_1^\infty (G(t) - \log t - c_0)\frac{dt}{t^{3/4}} + \mathcal{O}^*\Big(5.9\int_A^\infty \frac{dt}{t^{5/4}}\Big)\end{aligned}$$

i.e.

$$\sum_{a\leq A} \mu^2(a)a^{1/4}/\varphi(a) = 4A^{1/4} + c_1 + \mathcal{O}^*\Big(\frac{30}{A^{1/4}}\Big)$$

where

$$c_1 = -4 + c_0 - \frac{1}{4}\int_1^\infty (G(t) - \log t - c_0)\frac{dt}{t^{3/4}} = -1.424\cdots$$

We compute this constant directly. □

8.2. Main engine.

Most of the remainder term is controlled by $r_2(X;q)$ from (4.1). The p-factor of the Dirichlet series associated with its summand reads, with the notation $u=1/p$ and $v=1/p^s$,

$$\begin{aligned}D_p(s) &= 1 - \frac{v^2}{1-u} + \frac{uv}{1-u} = 1 - v^2 + \frac{(1-v)uv}{1-u}\\ &= (1-v^2)\Big(1 + \frac{uv}{(1-u)(1+v)}\Big)\end{aligned}$$

and thus $D(s) = \prod_{p\geq 2} D_p(s)$ is given by

$$D(s) = \frac{1}{\zeta(2s)}\prod_{p\geq 2}\Big(1 + \frac{1}{(p-1)(p^s+1)}\Big). \tag{8.1}$$

The second series is absolutely convergent (in the sense of Godement, i.e. that $\sum_{p\geq 2} |1/[(p-1)(p^s+1)]| < \infty$) when $\Re s > 0$. Finding an extension of it beyond this line, or showing that this line is a natural boundary is open. The main part of $D(s)$ comes from $1/\zeta(2s)$, which implies in particular that $r_2(X;q) \ll_{q,\varepsilon} 1/X^{3/4+\varepsilon}$ under the Riemann Hypothesis.

Numerical computations (see Lemma 8.4) concerning $|r_2(X;q)/X^{3/4}|$ when $q=1$ exhibit an oscillating behavior that is as mysterious as the one of $|\sqrt{X}\sum_{n>X}\mu(n)/n|$.

Lemma 8.3. *We have, when* $X \ge (700\,q)^3$

$$|r_2(X;q)| \le \frac{\prod_{p|q}(1+p^{-1/2})(1+p^{-1})}{\sqrt{X}\log(X/q^3)}\Big(6.4 + \frac{18.3\,\log(X/q^3)}{X^{1/6}}\sqrt{q}\Big).$$

Proof. Recall (4.1):

$$r_2(X;q) = \sum_{\substack{k^2\ell r>X,\\ r|q,\\ (k\ell,q)=(k,\ell)=1}} \frac{\mu(rk)\mu^2(\ell)}{rk\varphi(k)\ell\varphi(\ell)}.$$

We split the summation on ℓ according to whether $\ell q \le Y = X^{1/3}/50 \ge 14q$ or not, say in $\Sigma(\ell q\le Y) + \Sigma(\ell q > Y)$. For the second sum, we use Lemma 5.7 and 8.1, and get

$$\begin{aligned}\Sigma(\ell q > Y) &\le \sum_{\substack{\ell>Y/q,\\ (\ell,q)=1}} \frac{\mu^2(\ell)}{\ell\varphi(\ell)}\sum_{r|q}\frac{\mu^2(r)}{r}0.3616\sqrt{\frac{r\ell}{X}}\\ &\le 0.3616\times 2.28\frac{\prod_{p|q}(\sqrt{p}+1)}{\sqrt{XY}} \le \frac{0.825\prod_{p|q}(\sqrt{p}+1)}{\sqrt{XY}}.\end{aligned}$$

As for the first sum, we use Lemma 7.18 with $q^* = \ell q$ and $K = \sqrt{X/(\ell r)}$, as well as $\theta = 50/(1970)^{2/3}$, and get

$$\begin{aligned}\big|\Sigma(\ell q \le Y)\big| \le& \sum_{\substack{\ell\le Y/q,\\ (\ell,q)=1}} \frac{\mu^2(\ell)}{\ell\varphi(\ell)}\sum_{r|q}\frac{\mu^2(r)}{r}\prod_{p|q\ell}\Big(1+\frac{1}{p}\Big)\frac{1.89\sqrt{r\ell}}{\sqrt{X}\frac{3}{2}\log(X^{1/3}/(q\ell))}\\ &+\frac{5\,q^{1/2}}{X^{3/4}}\sum_{r|q}\mu^2(r)r^{-1/4}\sum_{\substack{\ell\le 50\,Y/(rq^2)^{1/3},\\ (\ell,q)=1}}\frac{\mu^2(\ell)\ell^{5/4}}{\ell\varphi(\ell)}\\ &+\frac{6.6\,q^{1/2}}{X^{3/4}}\sum_{r|q}\mu^2(r)r^{-1/4}\sum_{\substack{\ell\le \theta Y/(rq^2)^{1/3},\\ (\ell,q)=1}}\frac{\mu^2(\ell)\ell^{5/4}}{\ell\varphi(\ell)}\end{aligned}$$

i.e., with Lemma 7.16, and Lemma 8.2

$$\begin{aligned}\big|\Sigma(\ell q \le Y)\big| &\le \frac{1.89\times 5.02\times\frac{2}{3}}{\sqrt{X}\log(X^{1/3}/q)}\prod_{p|q}(1+p^{-1/2})(1+p^{-1}) + \frac{5\cdot 50^{1/4}+6.6\theta^{1/4}}{X^{3/4}}\prod_{p|q}(p^{1/3}+1)4Y^{1/4}\\ &\le \frac{6.33}{\sqrt{X}\log(X^{1/3}/q)}\prod_{p|q}(1+p^{-1/2})(1+p^{-1}) + \frac{12.46}{X^{2/3}}\prod_{p|q}(1+p^{1/4})\end{aligned}$$

On putting the pieces together, we reach

$$|r_2(X;q)| \le \frac{6.33}{\sqrt{X}\log(X/q^3)}\prod_{p|q}(1+p^{-1/2})(1+p^{-1}) + \frac{0.825\times 50^{1/2}+12.46}{X^{2/3}}\prod_{p|q}(\sqrt{p}+1)$$

which one simplifies into

$$|r_2(X;q)| \le \frac{\prod_{p|q}(1+p^{-1/2})(1+p^{-1})}{\sqrt{X}\log(X/q^3)}\Big(6.33 + \frac{18.3\log(X/q^3)}{X^{1/6}}\prod_{p|q}\sqrt{p}\Big).$$

□

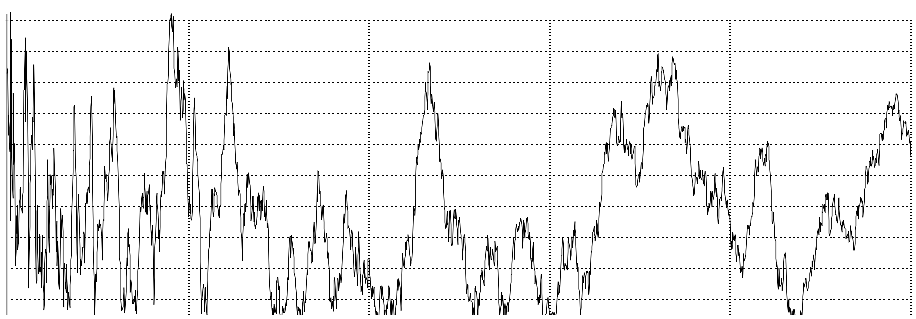
Lemma 8.4. *When* $10^6 \le X \le 1.3\cdot 10^{11}$, *we have*

$$|r_2(X;1)| \le 1.03/X^{3/4}.$$

The quantity $|r_2(X;1)|X^{3/4}$ is larger than 1.02 for some X inside $[8\cdot 10^7, 9\cdot 10^7]$ and some X inside $[358\cdot 10^7, 359\cdot 10^7]$.

Proof. We wrote the script `Cr2.gp`, which we converted into a C-program with garbage collecting via `gp2c -g Cr2.gp > Cr2.gp.c`. We then started `gp -s600000000 -p1000000000`, installed the relevant function and let the program run. A very similar script was developped in Perl and used interval arithmetic and much more RAM memory. The C-program stopped at $1.1\cdot 10^{11}$ while the Perl script went up to $1.3\cdot 10^{11}$ in about three months. We describe the algorithm used, the times indicated are relevant to the C-program and are of course much larger for its Perl counterpart.

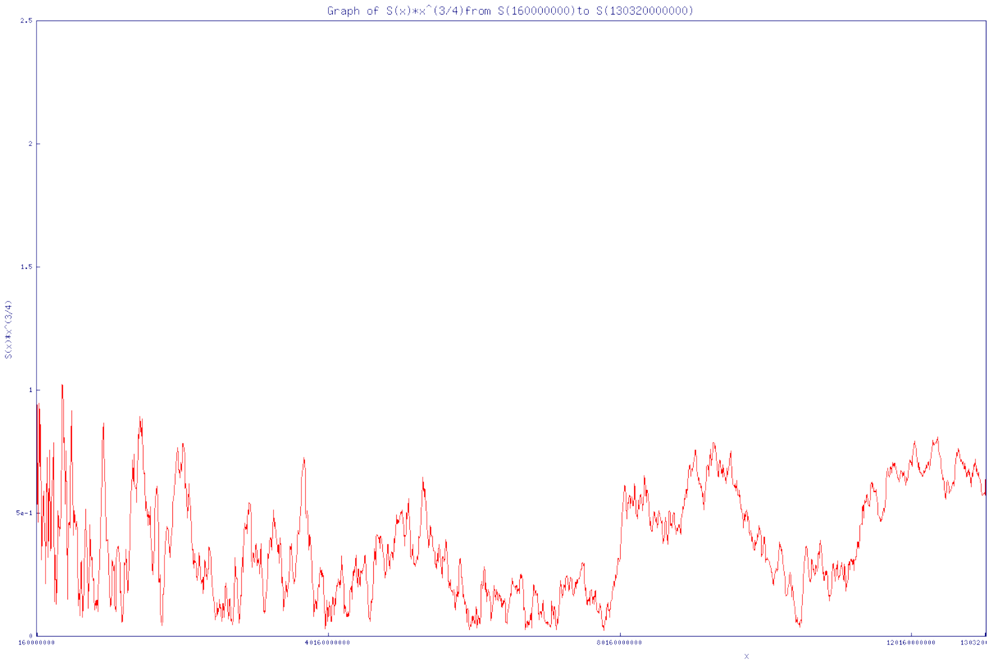
We did part of these computations twice. The first batch went up to $2\cdot 10^{10}$ by intervals of length 10^7, while the second batch went up to $1.1\cdot 10^{11}$ by intervals of $2\cdot 10^7$. The computation took about sixty-five hours in the first case and about ten days in the second one. Let us give some details concerning the general process. We treated every interval $I = [k\cdot 10^7, (k+1)\cdot 10^7]$ as a whole. A loop over primes $\le \sqrt{k\cdot 10^7}$ detected integers from I divisible by such a prime, a second loop over primes $\le \sqrt{k\cdot 10^7}$ detected integers from I divisible by a square of such a prime, a third loop over primes $\le (k\cdot 10^7)^{1/3}$ discarded integers from I divisible by the cube of such a prime. Since we store the product $P(n)$ of all the divisors of the previous shape for a given n, the integers that are such that $P(n) < n$ are of the shape $n = P(n)\cdot n/P(n)$ where $n/P(n)$ is a prime co-prime with $n/P(n)$. In this manner we avoided decomposing every integers in prime factors. Each interval of 10^7 integers took about 120 seconds to proceed: closer to 115 at the beginning of the range and closer to 125 at the end of it. The limitation of this method is the live-memory size; with blocks of size 10^7 we used a bit less than 6 Gigaoctets. (For the second batch: about 145 seconds at the beginning of the range and about 160 at the end, and 12 Gigaoctets of RAM).



This drawing plots an approximation of

$$\rho(t) = \max_{t < u/10^7 \le t+1} |r_2(u;1)\cdot u^{3/4}|$$

where t ranges the integers from $[1, 1999]$. The points are simply joined by segments. The bottom horizontal line is $x = 0$, the top one is $x = 1$. One can see that the behavior seems to vary less when the variable becomes larger. However a plot in logarithmic scale would be somewhat less regular. Here is the plot of the value from 10^7 to $1.3\cdot 10^{11}$.



Here is a sample of approximate values obtained, to enable checking.

t	$\rho(t)$	t	$\rho(t)$	t	$\rho(t)$	t	$\rho(t)$	t	$\rho(t)$	t	$\rho(t)$
1	0.8272	16	0.6010	31	0.3915	46	0.4391	61	0.4649	76	0.2475
2	0.8435	17	0.2590	32	0.3759	47	0.1331	62	0.3243	77	0.3051
3	0.6276	18	0.2065	33	0.4291	48	0.2502	63	0.1487	78	0.1468
4	0.6920	19	0.2516	34	0.4437	49	0.3323	64	0.1577	79	0.1169
5	0.6525	20	0.3682	35	0.4742	50	0.4734	65	0.3828	80	0.0629
6	0.6044	21	0.3110	36	0.7747	51	0.4998	66	0.4008	81	0.0898
7	0.5746	22	0.4012	37	0.7761	52	0.5777	67	0.2505	82	0.2136
8	1.0249	23	0.2566	38	0.8418	53	0.6271	68	0.1798	83	0.2601
9	0.9370	24	0.2861	39	0.9435	54	0.5811	69	0.1893	84	0.2455
10	0.5984	25	0.2757	40	0.6719	55	0.6851	70	0.3084	85	0.1508
11	0.4799	26	0.4099	41	0.7976	56	0.6705	71	0.3009	86	0.2774
12	0.4245	27	0.4210	42	0.9013	57	0.7157	72	0.1862	87	0.4153
13	0.7654	28	0.4534	43	0.7998	58	0.8566	73	0.2029	88	0.4321
14	0.6972	29	0.4599	44	0.6974	59	0.7082	74	0.1363	89	0.5492
15	0.5337	30	0.3937	45	0.6805	60	0.6620	75	0.2491	90	0.3114

□

Lemma 8.5. *When* $49 \le X$, *we have*

$$|r_2(X;1)| \le 0.508/\sqrt{X}.$$

Proof. We use Lemma 8.3 when $X \ge 1.1 \cdot 10^{11}$, Lemma 8.4 when $10^6 \le X \le 1.1 \cdot 10^{11}$ and direct computation otherwise. □

Lemma 8.6. *When* $2 \le X$, *we have*

$$|r_2(X;1)| \le \frac{13}{\sqrt{X}\log X}.$$

Proof. We use Lemma 8.3 when $X \ge 1.3 \cdot 10^{11}$, Lemma 8.4 when $10^6 \le X \le 1.1 \cdot 10^{11}$ and direct computation otherwise. □

9. Proof of Theorem 1.2

We prove the first estimate. We use Theorem 1.3 with $\mathfrak{R}'$. The function $G^\sharp$ is $r_2(X;1)$ defined in (4.1) and bounded in Lemma 8.5, while $\sum_{m\le D}|g(m)|$ is $r_1(X)$ defined in (4.2) and bounded in Lemma 6.1 with $q=1$. When $A_0 \in [1,2)$, we see that is is best to use A_0 around 5.526, with value $3.9800\cdots$. We readily check that

$$G_1(D) = \log D + c_0 + \mathcal{O}^*\big(1/\sqrt{D}\big), \qquad (2 \le D \le 100)$$

and that

$$G_1(D) = \log D + c_0 + \mathcal{O}^*\big(1.5/\sqrt{D}\big), \qquad (1 \le D \le 100).$$

The proof of the theorem is complete.

The second estimate is slightly more difficult to prove, though most of the work has been done. We select $A_0 = \log D$ in Theorem 1.3 together with Lemma 6.1 (with $q = 1$) and Lemma 8.6. We have (in the notation of Theorem 1.3):

(9.1)

$$\Re \leq \left| \sum_{1\leq a\leq A_0} \frac{13\sqrt{a}}{a\sqrt{D}\log(D/a)} + \frac{13\sqrt{A_0}}{\sqrt{D}\log(D/A_0)}\left(\log\frac{A_0}{A_0-1} - \frac{6/11}{A_0-1}\right)\right| + \frac{\frac{6}{11}4.95}{\sqrt{DA_0}}.$$

We use $0 \leq \log\frac{A_0}{A_0-1} \leq 1/(A_0-1)$ and (we single out the term with $a = 1$) the inequality $\sum_{2\leq a\leq A_0} 1/\sqrt{a} \leq \int_1^{A_0} dt/\sqrt{t} \leq \frac{1}{2}(\sqrt{A_0}-1)$. This yields

$$\Re\sqrt{D} \leq \frac{13}{\log D} + \frac{13.2}{2}\frac{\sqrt{A_0}-1}{\log(D/A_0)} + \frac{13\sqrt{A_0}}{\log(D/A_0)}\frac{5/11}{A_0-1} + \frac{\frac{6}{11}4.95}{\sqrt{A_0}}.$$

We have to add $13\sqrt{D}\int_{D/A_0}^{\infty} dt/[t^{3/2}\log t]$. This quantity is not more than $26/\log(D/A_0)$. Some numerical analysis shows that, when $D \geq \exp(21.25)$,

$$\frac{13}{\log D} + \frac{13}{2}\frac{\sqrt{A_0}-1}{\log(D/A_0)} + \frac{13\sqrt{A_0}}{\log(D/A_0)}\frac{5/11}{A_0-1} + \frac{\frac{6}{11}4.95}{\sqrt{A_0}} + \frac{26}{\log(D/A_0)} \leq \frac{18.4}{\sqrt{A_0}} = \frac{18.4}{\sqrt{\log D}}$$

When $D \leq \exp(21.25)$, the wanted estimate follows from the first part of Theorem 1.2.

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Laboratoire Paul Painlevé, CNRS / Université Lille 1, 59 655 Villeneuve d'Ascq, cedex, France
E-mail address: ramare@math.univ-lille1.fr

Harish-Chandra Research Institute, Chhatnag Road, Jhusi, Allahabad 211 019, India
E-mail address: akhileshp.clt@gmail.com