A sketch of H. Helfgott's proof of Goldbach's Ternary Conjecture

O. Ramaré

May 1, 2015

In 2014, Harald Helfgott published a preprint containing a proof that every odd integer strictly larger than 5 is a sum of three primes. The proof is still to be verified, but it is likely to be correct. The fact that this proof has now some slack goes in this direction. The aim of this appendix is to sketch it and to try to underline the major argument. Concerning the latest approximation of this result, we should mention that Ming-Chit Liu and Tianze Wang had shown in 2002 in [7] that every odd integer larger than exp(3100) is indeed a sum of three primes and that Terence Tao had shown in 2014 in [16] that every odd integer larger than 5 is a sum of at most five primes.

We stay close to the notation of Helfgott's proof to help the reader follow it, but we simplify some expressions and parameters. The proof under examination follows the path drawn by Hardy and Littlewood in [4] and Vinogradov [19] and considers when $N \geq 10^{27}$, the quantity

$$R_3(N) = \sum_{p_1+p_2+p_3=N} \eta_+\Big(\frac{p_1}{x}\Big) \log p_1 \ \eta_+\Big(\frac{p_2}{x}\Big) \log p_2 \ \eta_*\Big(\frac{p_3}{x}\Big) \log p_3 \,, \qquad (1)$$

where x is a size parameter close to $N/2$ and η_+ and η_* are two C^∞ non-negative smooth functions whose forms are chosen so as to optimize the estimates. Let us say that $\eta_+(t)$ is close to

$$t^3 \max(0,\ 2-t)^3 e^{t-(t^2+1)/2}$$

and bounded above by 1.08 while $\eta_*(t)$ also contains a $e^{-t^2/2}$-part but is more complicated and is bounded above by 1.42. Here is the main theorem.

Theorem 0.1. *If N is odd and larger than 10^{27}, we have*

$$R_3(N) \geq \frac{N^2}{5000}\,.$$

From the above theorem we infer that every odd integer $\geq 10^{27}$ is a sum of three primes. Computations run by David Platt and Harald Helfgott in [5] show that every odd integer within

$$[\,7,\ 8.875 \cdot 10^{30}\,]$$

is a sum of three odd primes, concluding the proof. The paper [6] by Habiba Kadiri and Alyssa Lumley on explicit short intervals containing primes could also lead to the same result.

The base line of the argument of Theorem 0.1 is the proof of Vinogradov that the reader will find in [18, chapter 4] and in the present book, which is devoted to a step by step proof of this theorem. We introduce, for an arbitrary function η, the trigonometric polynomial

$$S_\eta(\alpha, x) = \sum_{p\geq 3} \eta(p/x) \log p \; e(\alpha x)$$

and use the easily checked identity

$$R_3(N) = \int_0^1 S_{\eta_+}(x,\alpha)^2 S_{\eta_*}(x,\alpha) e(-N\alpha) d\alpha. \tag{2}$$

For a positive integer q, we define the following subset of $\mathbb{R}/\mathbb{Z}$:

$$\mathfrak{N}_{q,\delta,r} = \bigcup_{\substack{1\leq a\leq q\\(a,q)=1}} \left[\frac{a}{q} - \frac{\delta r}{2qx}, \frac{a}{q} + \frac{\delta r}{2qx}\right]. \tag{3}$$

We set $r_0 = 150\,000$ and $Q = (3/4)x^{2/3}$. We split the interval $[0, 1]$ in three subsets.

Majors Arcs

When

$$\alpha \in \bigcup_{q\leq r_0} \mathfrak{N}_{q,8,r_0} ,$$

the value of $S_\eta(\alpha, x)$ in (2) is controlled by the distribution of primes in progressions modulo q. From an explicit viewpoint and due to [15] and [8], we know that the clear path to such information follows by checking a partial Riemann Hypothesis, that is by showing that the L - functions attached to Dirichlet characters modulo q have no zero $\rho = \beta + i\gamma$ inside the set

$$\{1/2 < \beta < 1, \; |\gamma| \leq T_q\}$$

for some fixed T_q. Such computations are highly complex and until recently we were only able to roughly use $q \leq 60$ and $T_q = 10\,000$. David Platt's new algorithm from [9] changed all that and we can now use values of q up to 300 000 with

$$T_q = \frac{10^8}{q} .$$

Rather than splitting the sum

$$S_\eta\left(\frac{a}{q} + u, \; x\right) = \sum_p \eta\left(\frac{p}{x}\right) e(pu) e\left(\frac{ap}{q}\right)$$

according to the class of p modulo q, Helfgott goes back to the initial method, expresses the additive character $n \mapsto e(an/q)$ restricted to the multiplicative group $(\mathbb{Z}/q\mathbb{Z})^*$ in terms of Dirichlet characters and uses an explicit formula. This means also controlling the Mellin transform of the used gaussian smoothing, a task in which he invests a lot of effort with a successful outcome. A main term of size of order x^2 is extracted in this manner.

Minor Arcs

We now present the core of one of the main steps.

Lemma 0.2. *If*

$$\alpha = \frac{a}{q} + \frac{\tau}{qQ}$$

with $|\tau| \leq 1$, *we set* $\delta^* = 2 + |\tau x|$. *For* $q \leq x^{1/3}$, *we have*

$$S_{\eta_*}(\alpha, x) \ll \frac{x \log(\delta^* q)}{\sqrt{\delta^* \varphi(q)}} .$$

For $x^{1/3} \leq q \leq Q$, *we have*

$$S_{\eta_*}(\alpha, x) \ll x^{5/6} (\log x)^{3/2} .$$

The implied constants are small. The second part is rather classical, but the first one contains two essential novelties: when τ is small, the bound tends to zero as q goes to infinity and is not contaminated by any multiplicative factor like $\log x$, while the constant remains small. It is the first time both properties are kept, see for instance [2] and [11]. Furthermore, when τ is large, one gets *additional* saving, while earlier treatments all handled this factor as a perturbation of the case $\tau = 0$ and thus became worse when τ increased. Let us only mention that the method uses the bilinear form of [17] with $UV = x/\sqrt{q}$ when q is small. This lemma requires also an estimate of the form

$$\sum_{n \leq x} \left(\sum_{d|n} \mu(d) 1_{d \leq z} \right)^2 \ll x ,$$

where z is some power of x. With an unspecified constant, this is due to [3]. The treatment Helfgott devises relies on [13], which itself relies on [1] and [12].

The corresponding treatment of (2) follows the argument already contained in [19]:

$$\int_{\alpha \in \mathfrak{m}} S_{\eta_+}(x, \alpha)^2 S_{\eta_*}(x, \alpha) e(-N\alpha) d\alpha \ll \int_0^1 |S_{\eta_+}(x, \alpha)|^2 d\alpha \max_{\alpha \in \mathfrak{m}} |S_{\eta_*}(x, \alpha)|$$
$$\ll x^2 \log x \cdot \max_{\alpha \in \mathfrak{m}} |S_{\eta_*}(x, \alpha)|/x \qquad (4)$$

for a subset $\mathfrak{m}$ that we call minor arcs. If we select for $\mathfrak{m}$ the complement of the major arcs, we can only ensure that

$$\max_{\alpha \in \mathfrak{m}} \frac{|S_{\eta_*}(x, \alpha)|}{x}$$

is smaller than a constant, but cannot recover the loss of $\log x$. Tao in [16] uses the fact that by [7], one can assume that $\log x \leq 3100$. This bound is too large here. Helfgott introduces another subset of the circle: the intermediate arcs.

Intermediate Arcs

The key to the intermediate arcs is a lemma of the following form.

Lemma 0.3. *Let* (φ_n) *be a sequence in* $\ell_1 \cap \ell_2$ *such that* $\varphi_n = 0$ *when* n *has a prime factor which is at most* $\sqrt{x}$*. Let* $Q_0 \geq 10^5$ *be such that*

$$Q_0 \leq \frac{Q}{20000} \quad \text{with} \quad Q = \frac{\sqrt{x}}{4}\,, \quad \text{and} \quad Q_0 \leq Q^{3/5}\,.$$

We have

$$\sum_{q \leq Q_0} \int_{\mathfrak{N}_{q,8,Q_0}} \Big|\sum_{n \geq 1} \varphi_n e(n\alpha)\Big|^2 d\alpha \leq \frac{\log Q_0 + 1.36}{\log Q + 1.33} \sum_n |\varphi_n|^2$$

This is essentially a circle method-version of [14, Theorem 5] or more explicitly of [10, Theorem 5.3]. As in (4), the norm $\sum_n |\varphi_n|^2$ in our case is of size $x \log x$, potentially loosing a $\log x$, but it is recovered by $1/\log Q$. Applying summation by parts and using Lemma 0.2, we get something of the type

$$\int_{\alpha \in \mathfrak{m}'} S_{\eta_+}(x,\alpha)^2 S_{\eta_*}(x,\alpha) e(-N\alpha) d\alpha \ll x \cdot \max_{\alpha \in \mathfrak{m}'} (\log q) |S_{\eta_*}(x,\alpha)|/x$$

where $\alpha = (a/q) + \tau/(qQ)$ and $\mathfrak{m}' = \bigcup_{r_0 < q \leq r_1} \mathfrak{N}_{q,8,q}$ with r_1 being approximately $x^{4/15}/8$.

This is not quite enough in order to conclude: we need the arc $\mathfrak{N}_{q,8,r_1}$ rather than the arc $\mathfrak{N}_{q,8,q}$, but it is another place where the enhanced effect of δ^* in Lemma 0.2 features!

Note. There are several other interesting techniques in this proof that cannot fit in this short sketch.

References

[1] M. Balazard. Elementary remarks on Möbius' function. *Proceedings of the Steklov Intitute of Mathematics*, 276, 2012.

[2] H. Daboussi and J. Rivat. Explicit upper bounds for exponential sums over primes. *Math. Comp.*, 70(233):431–447, 2001.

[3] F. Dress, H. Iwaniec, and G. Tenenbaum. Sur une somme liée à la fonction de Möbius. *J. Reine Angew. Math.*, 340:53–58, 1983.

[4] G.H. Hardy and J.E. Littlewood. Some problems of "Partitio Numerorum" III. On the expression of a number as a sum of primes. *Acta Math.*, 44:1–70, 1922.

[5] H. A. Helfgott and David J. Platt. Numerical verification of the ternary Goldbach conjecture up to $8.875 \cdot 10^{30}$. *Exp. Math.*, 22(4):406–409, 2013.

[6] H. Kadiri and A. Lumley. Short effective intervals containing primes. *Submitted*, 2014. arXiv:1407:7902.

[7] Ming-Chit Liu and Tianze Wang. On the Vinogradov bound in the three primes Goldbach conjecture. *Acta Arith.*, 105(2):133–175, 2002.

[8] K.S. McCurley. Explicit estimates for the error term in the prime number theorem for arithmetic progressions. *Math. Comp.*, 42:265–285, 1984.

[9] D.J. Platt. *Numerical computations concerning the GRH.* PhD thesis, 2013. `http://arxiv.org/abs/1305.3087`.

[10] O. Ramaré. *Arithmetical aspects of the large sieve inequality*, volume 1 of *Harish-Chandra Research Institute Lecture Notes*. Hindustan Book Agency, New Delhi, 2009. With the collaboration of D. S. Ramana.

[11] O. Ramaré. On Bombieri's asymptotic sieve. *J. Number Theory*, 130(5):1155–1189, 2010.

[12] O. Ramaré. From explicit estimates for the primes to explicit estimates for the Moebius function. *Acta Arith.*, 157(4):365–379, 2013.

[13] O. Ramaré. Explicit estimates on several summatory functions involving the Moebius function. *Math. Comp.*, page 32pp, 2014.

[14] O. Ramaré and I.M. Ruzsa. Additive properties of dense subsets of sifted sequences. *J. Théorie N. Bordeaux*, 13:559–581, 2001.

[15] J.B. Rosser. Explicit bounds for some functions of prime numbers. *American Journal of Math.*, 63:211–232, 1941.

[16] T. Tao. Every odd number greater than 1 is the sum of at most five primes. *Math. Comp.*, 83(286):997–1038, 2014.

[17] R.C. Vaughan. Sommes trigonométriques sur les nombres premiers. *C. R. Acad. Sci. Paris Sér. A-B*, 285(16):A981–A983, 1977.

[18] R.C. Vaughan. *The Hardy-Littlewood method*, volume 80 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1981.

[19] I.M. Vinogradov. Representation of an odd number as a sum of three primes. *Dokl. Akad. Nauk SSSR*, 15:291–294, 1937.