ON PRIME κ-TUPLES: SMALL VALUES OF κ

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Abstract. Given a positive integer κ and an *admissible* κ-uplet $(h_1, \cdots, h_\kappa)$, we prove that there exists infinitely many integers n such that the product $(n+h_1)\cdots(n+\kappa)$ has not more than $n(\kappa)$ prime factors, where $n(\kappa)$ is given in a table. These values are at least as good as the all the previous ones, and improve on them when $\kappa = 8$ (we have $n(8) = 28$) or $\kappa \geq 10$. A similar statement concerning the almost prime values of an integer polynomial is also proved. Incidentaly we extend the results of [14] to cover the case of more general sieve coefficients.

1. Introduction and some results

We continue here the work started in [14] on the weighted sieve, with an accent on prime κ-tuples. Let us recall rapidly the question at hand. We select a positive integer κ and κ integers $h_1 < \cdots < h_\kappa$. We assume that the κ-uplet $(h_1, \cdots, h_\kappa)$ is admissible, by which we mean that, for each prime p, the cardinality of the set $\{h_1, \cdots, h_\kappa \mod p\}$ is strictly less than p. The final goal is to prove that there exist infinitely many integers n such that $n + h_1, \cdots, n + h_\kappa$ are simultaneously prime. From a historical viewpoint, let us mention that the first appearance of this conjecture seems to be A. Polignac's memoir [1], where the author considers only the case $\kappa = 2$, namely couples of primes (p, p') with $p' = p + 2k$. In case $k = 1$, the terminology "twin" arose much later and is attributed in the first chapter of [21] to the german mathematician P. Stäckel in the late 19th century. The general conjecture known as *the prime κ-tuples conjecture* has been first stated by G.H. Hardy & J.E. Littlewood in [7].

Such results are far out of reach, so we follow a line that we trace back to A. Renyi in [15] (translated in [16]): we strive only to produce infinitely many integers n such that the total number of prime factors of

$$\Pi_{(h_1,\cdots,h_\kappa)}(n) = \prod_{1\leq i\leq \kappa}(n+h_i) \tag{1}$$

is as small as possible. This problem is readily generalized by selecting a polynomial $\mathscr{P}$ of degree κ, with integer coefficients and no fixed divisors (a condition equivalent to the admissibility hypothesis of the κ-uplet), and aiming at proving the existence of infinitely many integers such that $\mathscr{P}(n)$ has not more than h prime factors, where h is the number of prime factors of $\mathscr{P}$ in $\mathbb{Q}[X]$ (see [20] and [19]). This problem is equally out of reach.

2000 *Mathematics Subject Classification.* 11N35, 11N36, secondary : 42B10.
Key words and phrases. Weighted sieve, Selberg sieve, Prime κ-tuple.

Theorem 1.1. *Let $(h_1, \cdots, h_\kappa)$ be a κ-tuple of admissible shifts. We can find infinitely many integers n such that $\prod_{1\le i\le\kappa}(n+h_i)$ has at most $n(\kappa)$ prime factors, where $n(\kappa)$ is given in the table below.*

These values are not valid for the case of a general weighted sieve of dimension κ as we use a combinatorial trick that is numerically as efficient and sometimes more efficient that Richert logarithmic coefficients.

In a general context, we have only these coefficients at our disposal. We just specialize slightly the situation (see [5] or [3] for more details) and prove:

Theorem 1.2. *Let $\mathscr{P}$ be a polynomial of degree κ, with integer coefficients and without any fixed divisors. We can find infinitely many integers n such that $\mathscr{P}(n)$ has at most $n_R(\kappa)$ prime factors, where $n_R(\kappa)$ is given in the table below.*

In both these results, we do not count the prime factors according to their multiplicity. We may have to add some hypothesis in order to be able to do so.

We included the results from [12], [22], [18], [8], [2], [9] and [3, table 11.1]. The reader may also consult [5], [6] and [10] with benefit. It emerges from this table that the method we propose equals the best of the others for small values of κ and start showing its teeth when $\kappa = 8$.

κ	Porter	Xie	Salerno	Heath-Brown	Diamond & Halberstam	Ho & Tsang	$n_R(\kappa)$	$n(\kappa)$
1	3						2	2
2	9		6	5	5	5	5	5
3	14		10	9	8	8	9	8
4	20	14	14	13	12	12	12	12
5	27	18	18	17	16	16	16	16
6	33	23	23	21	20	20	20	20
7	40	27		26	25	24	24	24
8	46	32		32	29	29	29	28
9	53	37		39	34	33	33	33
10	60	42		45	39	38	37	37
11						44	42	42
12						48	47	46
13						53	51	51
14						58	56	56
15						63	61	60
16						69	66	65
17						74	70	70
18						80	75	75
19						85	80	80
20						91	85	85
21						97	91	90
22						103	96	96

Notice that the shown values are upper bounds for what is accessible via the method developped here, though we believe our choice of parameters to be very close to the optimal one.

Here is a corollary is a somewhat less specialized language:

Corollary 1.1. *There are infinitely many integers n such that the product*

$$n(n+2)(n+6)(n+8)(n+12)(n+18)(n+20)(n+26)$$

has at most 28 *disctinct prime factors.*

This means that each factor has on average 3.5 prime factors. The prime κ-tuple conjecture asserts that 28 could be replaced by 8, as examplified by

$$11, 13, 17, 19, 23, 29, 31, 37.$$

The main conjecture thus has $(1+o(1))\kappa$ as an ultimate goal, while the asymptotic in Theorem 1.1 is of size $(1+o(1))\kappa \operatorname{Log} \kappa$ as shown in [14]. The asymptotic is thus in a sense as "bad" as the one corresponding for other methods, but the numerical approach detailled here shows that we recover here the best of the previously known reults, and even better, even for small values of κ. Since we are not able to produce optimal choice of the parameters, we provide here numerical data so that later mathematicians may be able to come up with a fuller understanding of the situation.

The proof (and this paper in itself) relies heavily on [14] (see section 3). We only specify here that, in the course of the proof, we study a sum

$$\tilde{S}\big((a_{d^*})_{d^*}\big) = \sum_{n\le N} c(n)\beta(n)$$

$$\text{with} \quad c(n) = \sum_{d^*|\Pi_{(h_1,\cdots,h_\kappa)}(n)} a_{d^*} \quad \text{and} \quad \beta(n) = \Bigg(\sum_{d|\Pi_{(h_1,\cdots,h_\kappa)}(n)} \lambda_d\Bigg)^2$$

and show that, for proper choices of all the involved parameters, $\tilde{S}((a_{d^*})_{d^*})$ tends to infinity (read section 5 for the full argument). Once κ is fixed, there are essentially two parameters that have to be chosen in this approach: the sequence (a_{d^*}) occurring in the *sieve coefficient* $c(n)$ and the weight function w that appears in the *host sequence* $\beta(n)$. In [14], we restricted the choice of (a_{d^*}) to the sequence $a_r[P]$ of integers that are the product of exactly r prime factors, all distincts and all not more than P. Our first task here is to extend this choice to $a_r[P; g]$, the sequence that takes value $\prod_i g\big((\operatorname{Log} p_i)/\operatorname{Log} P\big)$ on products of exactly r prime factors $p_1p_2\ldots p_r$, again all distincts and all not more than P. The function g will be assumed to be regular enough. This will enable us to handle Richert's weights as detailed below.

The second and main point is the choice of the weight function w that is used to build $(\beta(n))$. In [14], we restricted our attention to the choice $w(t) = \max(0, 1-t)^\nu$. We investigate here more thoroughly the choice of w in case $r = 1$. We produce a general formula that leads to an extremal problem in w, but we are not able to determine a best choice (if one exists). We thus ran computations (and these require much more complicated formulae) when κ is fixed; we report here the results as well as the formulae we have used. Concerning $(\beta(n))$, the reader should look at [13, Chapter 11] and at [4].

Notation. We shall use the following four quadratic forms:

$$I_1(w,\kappa) = \int_0^1 t^{\kappa-1} w(t)^2 dt \tag{2}$$

is the norm by which we shall measure w. We shall use

$$K_1(w,\kappa) = -\int_0^1 \operatorname{Log}(1-t)\, t^{\kappa-1} w(t)^2 dt, \tag{3}$$

as well as

$$I_2(w,\kappa)=2\int_{0\le t\le t_2\le 1}\frac{(w(t_2)-w(t_1))^2}{t_2-t_1}t_1^{\kappa-1}dt_1dt_2, \tag{4}$$

$$J_2(w,\kappa)=2\int_{0\le t_1\le t_2\le 1}w'(t_1)w'(t_2)(1-t_2)t_1^{\kappa}dt_1dt_2. \tag{5}$$

2. A rather smooth sum over primes

In [14, Section 8], we proved some general Lemmas pertaining to sums over primes. It is better to generalize them some more. We consider a function ϕ that satisfies:

($H_1(\phi)$) ϕ has a finite number of bounded discontinuities.
($H_2(\phi)$) ϕ is piecewise C^1 and

$$W(\phi,t)=\max_{2\le y\le t}\biggl(|\phi(y)|+\frac{\operatorname{Log} y}{y}\int_2^y\frac{x|\phi'(x)|dx}{\operatorname{Log} x}\biggr)<\infty \tag{6}$$

In particular, ϕ is a finite linear combination of functions ϕ_k whose support is an interval, satisfies $W(\phi_k,t)<\infty$ and is C^1 on its support. The supports of all these ϕ_k are furthermore disjoint. For such a function ϕ, the conclusions of [14, Lemma 8.1, Lemma 8.3] are valid, and this Lemma thus holds true for ϕ:

Lemma 2.1. *For every $B\ge 1$, we have*

$$\sum_{P_0<p\le t}\phi(p)=\int_{P_0}^t\frac{\phi(t)dt}{\operatorname{Log} t}+\mathcal{O}_B\biggl(\frac{tW(\phi,t)}{R^B\operatorname{Log} t}\biggr)$$

and

$$\sum_{P_0<p\le t}\phi(p)/p=\int_{P_0}^t\frac{\phi(t)dt}{t\operatorname{Log} t}+\mathcal{O}_B\biggl(\frac{W(\phi,t)}{R^B\operatorname{Log} t}\biggr).$$

This lemma will replace [14, Lemma 8.3].

3. A generalization of the weighted sieve Theorem

We prove here a generalization of [14, Theorem 1.2]. This part relies heavily on [14]. We assume the function g to be have a finite number of bounded discontinuities, to be C^1 per pieces ad to have a bounded derivative. The function $\phi(t)=g((\operatorname{Log} t)/\operatorname{Log} P)$ satisfies hypotheses ($H_1(\phi)$) and ($H_2(\phi)$). Given some *admissible* κ-tuple $(h_1,\cdots,h_\kappa)$ and recalling (1), we considered, when $i\in\{1,\cdots,\kappa\}$, the quantity

$$S_i((a_{d^*})_{d^*})=\sum_{n\le N}\biggl(\sum_{d^*|n+h_i}a_{d^*}\biggr)\beta(n)\text{ with }\beta(n)=\biggl(\sum_{d|\Pi_{(h_1,\cdots,h_\kappa)}(n)}\lambda_d\biggr)^2. \tag{7}$$

which is generalized in [14, (2.5)]. There are essentially two interesting cases: looking at the condition $d^*|n+h_i$ or looking more classically at the condition $d^*|\Pi_{(h_1,\cdots,h_\kappa)}(n)$. Both cases are covered in the general sum [14, (2.5)]. In this latter case, *we denote the sum by* $\tilde S$ and hypotheses (H_6) and even (H_7) are satisfied with $\kappa'=\kappa$. The final result will be multiplied by κ'^r as is evident from [14, (10.3)].

Let us now turn our attention to the modification of a_{d^*} we mentionned above. The first remark is that the main assumptions for a long part of [14] are (H_5) and

(H_6). If the support of a_{d^*} is limited to integers having at most r prime factors and $|a_{d^*}|$ is bounded (by a constant that may depend on r), the bounds provided there for the error terms are valid here. From [14, Section 9], the sequence (a_{d^*}) is specialized. We select

$$a_r[P;g](p_1\cdots p_r) = \tilde{g}(p_1\cdots p_r) \quad \tilde{g}(p_1\cdots p_r) = \prod_{1\le i\le r} g\big((\operatorname{Log} p_i)/\operatorname{Log} P\big) \tag{8}$$

when the primes p_i are all distinct and less than P, and $a_r[P;g](d^*) = 0$ otherwise, where g is some integrable function, that is furthermore bounded in absolute value. The fact that we multiply the expression by $a_r[P](p_1\cdots p_r)$ only means that the primes p_i are distinct and all below P. We define

$$\mathscr{L}(\varepsilon,\tau) = \int_{\varepsilon/\tau}^{1} g(u)du/u \ll \operatorname{Log}(\tau/\varepsilon). \tag{9}$$

By Lemma 2.1, we get

$$\sum_{P_0<p\le P} g\big((\operatorname{Log} p)/\operatorname{Log} P\big)/p = \mathscr{L}(\varepsilon,\tau) + \mathcal{O}(1/\operatorname{Log} P_0). \tag{10}$$

We can then continue the analysis of [14, Section 9], but with $\mathscr{L}(\varepsilon,\tau)$ instead of $\operatorname{Log}(\tau/\varepsilon)$, getting

$$\begin{aligned} &Z^2 S_0^{(3)}(a)/\big(A\kappa'^r\big) = \\ &\sum_{\substack{\ell_0,\ell_1,\ell_2,\\ \omega(\ell_0\ell_1\ell_2)\le r}}^{\star} \frac{\mu(\ell_1)\mu(\ell_2)\mu^2(\ell_0\ell_1\ell_2)}{\ell_0\ell_1\ell_2}\Theta(\ell_0\ell_1,\ell_0\ell_2)\tilde{g}(\ell_0\ell_1\ell_2)\frac{\operatorname{Log}(\tau/\varepsilon)^{r-\omega(\ell_0\ell_1\ell_2)}}{(r-\omega(\ell_0\ell_1\ell_2))!} \\ &\qquad\qquad + \mathcal{O}\big(Z\operatorname{Log}(\tau/\varepsilon)^{2r}/P_0\big). \end{aligned}$$

In [14, (10.1)], we also have to add a factor $\tilde{g}(\ell_0\ell_1\ell_2)$ as above, so that, in the next formula, the term $\prod_i dx_i \prod_i dy_i \prod_i dz_i$ is to be replaced by

$$\prod_i g((\operatorname{Log} x_i)/\operatorname{Log} P)dx_i \prod_i g((\operatorname{Log} y_i)/\operatorname{Log} P)dy_i \prod_i g((\operatorname{Log} z_i)/\operatorname{Log} P)dz_i.$$

The main consequence is that in [14, (10.2)], the term $\prod_i du_i \prod_i dv_i \prod_i dw_i$ is to be replaced by

$$\prod_i g(u_i/\tau)du_i \prod_i g(v_i/\tau)dv_i \prod_i g(w_i/\tau)dw_i. \tag{11}$$

The next point in [14, Section 12], equations before and after (12.6). We reach [14, (12.7)] where we should replace $\operatorname{Log}(\tau/\varepsilon)$ by $\mathscr{L}(\varepsilon,\tau)$ as expected. Apart from these rather superficial changes, nothing is to be modified in [14, Sections 12-13]. As a conclusion, [14, (14.1)] still holds true when $\prod_i du_i \prod_i dv_i \prod_i dw_i$ is again replaced by (11), i.e.

$$\mathfrak{G}_{\kappa,r}(t_1,t_2,\tau,g)=\lim_{\epsilon\to 0}\sum_{b+c+d=r}\sum_{\substack{\mathcal{B}\subset\{1,\cdots,b\},\\ \mathcal{C}\subset\{1,\cdots,c\},\\ \mathcal{D}\subset\{1,\cdots,d\}}}\frac{(-1)^{b+|\mathcal{B}|+|\mathcal{C}|+|\mathcal{D}|}}{b!c!d!}$$
$$\int_{\substack{\epsilon\le u_1,\cdots,u_b\le\tau,\\ \epsilon\le v_1,\cdots,v_c\le\tau,\\ \epsilon\le w_1,\cdots,w_d\le\tau}}\bigl(\min(t_1-\sum_{i\in\mathcal{C}}v_i,t_2-\sum_{i\in\mathcal{D}}w_i)-\sum_{i\in\mathcal{B}}u_i\bigr)^{+\kappa}$$
$$\frac{\prod_i g(u_i/\tau)du_i\prod_i g(v_i/\tau)dv_i\prod_i g(w_i/\tau)dw_i}{\prod_i u_i\prod_i v_i\prod_i w_i}. \tag{12}$$

Here is the result we have reached:

Theorem 3.1. *Let four parameters be given: a non-negative integer r, an integer parameter κ, a parameter $\tau>0$, a function g over $[0,1]$ that has a finite number of bounded discontinuities, is piecewise C^1 and has a bounded derivative. There exists a bounded continuous function $\mathfrak{G}_{\kappa,r}(t_1,t_2,\tau,g)$ with the following property. Let $(h_1,\cdots,h_\kappa)$ be an admissible κ-tuple, let $Q\ge 1$ be a parameter and w be a function as above. We consider the sum $S(a_r[Q^\tau;g])$ from (7) when $\beta(n)$ is as above. We have*

$$\frac{\tilde{S}(a_r[Q^\tau;g])}{N\kappa^r/(\operatorname{Log}Q)^\kappa}=\mathscr{C}(h_1,\cdots,h_\kappa)\int_{0\le t_1,t_2\le 1}w'(t_1)w'(t_2)\mathfrak{G}_{\kappa,r}(t_1,t_2,\tau,g)dt_1dt_2$$
$$+\mathcal{O}\bigl(1/(\operatorname{Log}Q)^{1/9}\bigr)+\mathcal{O}(Q^{r\tau+2}N^{-1}(\operatorname{Log}Q)^\kappa)$$

where the constant $\mathscr{C}(h_1,\cdots,h_\kappa)$ is given by

$$\mathscr{C}(h_1,\cdots,h_\kappa)=\prod_{p\ge 2}\biggl(1-\frac{\#\{h_1,\cdots,h_\kappa \mod p\}}{p}\biggr)\Bigl(1-\frac{1}{p}\Bigr)^{-\kappa}.$$

The expression we find for $\mathfrak{G}_{\kappa,r}(t_1,t_2,\tau,g)$ is fairly explicit but too complicated to get even the asymptotic dependence in κ. Let us summarize here the properties we prove:

(1) $\mathfrak{G}_{\kappa,r}$ is symetrical in t_1 and t_2, i.e. $\mathfrak{G}_{\kappa,r}(t_1,t_2,\tau,g)=\mathfrak{G}_{\kappa,r}(t_2,t_1,\tau,g)$.
(2) $\mathfrak{G}_{\kappa,r}(t_1,0,\tau,g)=\mathfrak{G}_{\kappa,r}(0,t_2,\tau,g)=0$ and we extend $\mathfrak{G}_{\kappa,r}(t_1,t_2,\tau,g)$ to negative values of t_1 and/or t_2 by attributing it the value 0.
(3) $\mathfrak{G}_{\kappa,r}$ is a bounded continuous function.
(4) When $\tau>0$, we have $\mathfrak{G}_{\kappa,0}(t_1,t_2,\tau,g)=\min(t_1,t_2)^\kappa$ and

$$\int_{0\le t_1,t_2\le 1}w'(t_1)w'(t_2)\mathfrak{G}_{\kappa,0}(t_1,t_2,\tau)dt_1dt_2=\kappa\int_0^1 w(t)^2t^{\kappa-1}dt. \tag{13}$$

Note that we found no recursion formula similar to [14, (1.3)]. However, when $\mathfrak{G}_{\kappa,r}$ is twice differentiable in the domain $0\le t_1\le t_2\le\tau$, we have the exact analog of

[14, (14.2)], namely

$$\boxed{\begin{aligned}&\int_{0\le t_1,t_2\le 1} w'(t_1)w'(t_2)\mathfrak{G}_{\kappa,r}(t_1,t_2,\tau,g)dt_1dt_2\\ &\qquad = \int_0^1 w(t_1)^2\frac{d\mathfrak{G}_{\kappa,r}(t_1,1,\tau,g)}{dt_1}dt_1\\ &\qquad\quad -\tfrac12\int_{0\le t_1,t_2\le 1}(w(t_2)-w(t_1))^2\frac{d^2\mathfrak{G}_{\kappa,r}(t_1,t_2,\tau,g)}{dt_1dt_2}dt_1dt_2.\end{aligned}}\tag{14}$$

4. Handling Richert's logarithmic coefficients

Richert introduced in [17] logarithmic sieve coefficients; they have been used in our context by [18]. As far I can see, they do not lead to asymptotically smaller values for $n(\kappa)$ or $n_R(\kappa)$. They are however numerically efficient.

We handle these logarithmic coefficients by selecting $g(x) = x$. We restrict our attention to the case $r = 1$, as it will be enough for the sequel. Our starting point is equation (12). We get

$$\mathfrak{G}_{\kappa,1}(t_1,t_2,\tau,x)\tau = \sum_{b+c+d=1}\sum_{\substack{\mathcal{B}\subset\{1,\cdots,b\},\\ \mathcal{C}\subset\{1,\cdots,c\},\\ \mathcal{D}\subset\{1,\cdots,d\}}}\frac{(-1)^{b+|\mathcal{B}|+|\mathcal{C}|+|\mathcal{D}|}}{b!c!d!}\int_{\substack{0\le u_1,\cdots,u_b\le\tau,\\ 0\le v_1,\cdots,v_c\le\tau,\\ 0\le w_1,\cdots,w_d\le\tau}} \Big(\min\big(t_1-\sum_{i\in\mathcal{C}}v_i, t_2-\sum_{i\in\mathcal{D}}w_i\big)-\sum_{i\in\mathcal{B}}u_i\Big)^{+\kappa}\prod_i du_i\prod_i dv_i\prod_i dw_i. \tag{15}$$

We unfold it and get, assuming that $t_1\le t_2$,

$$\mathfrak{G}_{\kappa,1}(t_1,t_2,\tau,x) = t_1^\kappa + \int_0^\tau\Big(\min(t_1-u,t_2-u)^{+\kappa}-\min(t_1-u,t_2)^{+\kappa}-\min(t_1,t_2-u)^{+\kappa}\Big)\frac{du}{\tau}$$

where $\min(a,b)^{+\kappa}$ is to be understood as $\max(0,\min(a,b))^\kappa$. This expression simplifies in

$$\mathfrak{G}_{\kappa,1}(t_1,t_2,\tau,x) = \begin{cases}0 & \text{when } \tau\le t_2-t_1,\\ t_1^\kappa-\frac{t_1^\kappa(t_2-t_1)}{\tau}-\frac{t_1^{\kappa+1}-(t_2-\min(\tau,t_2))^{\kappa+1}}{(\kappa+1)\tau} & \text{when } t_2-t_1\le\tau.\end{cases}$$

We shall only use the case $\tau\ge 1$ (i.e. $P\ge Q$), in which we have access to the shorter form

$$\mathfrak{G}_{\kappa,1}(t_1,t_2,\tau,x) = t_1^\kappa-\frac{t_1^\kappa t_2}{\tau}+\frac{\kappa t_1^{\kappa+1}}{(\kappa+1)\tau},\quad (0\le t_1\le t_2\le 1\le\tau). \tag{16}$$

We readily deduce from this expression that

$$\int_{0\le t_1,t_2\le 1} w'(t_1)w'(t_2)\mathfrak{G}_{\kappa,1}(t_1,t_2,\tau,x)dt_1dt_2 = \kappa\int_0^1 w^2(t)t^{\kappa-1}dt + \frac{\kappa}{\tau}\int_0^1 w^2(t)t^\kappa dt - \frac{2}{\tau}\int_{0\le t_1\le t_2\le 1} w'(t_1)w'(t_2)t_1^\kappa t_2 dt_1dt_2.$$

We prefer to write it in the form:

$$\int_{0\le t_1,t_2\le 1} w'(t_1)w'(t_2)\mathfrak{G}_{\kappa,1}(t_1,t_2,\tau,x)dt_1dt_2 = \kappa\int_0^1 w^2(t)t^{\kappa-1}dt$$
$$+\frac{\kappa}{\tau}\int_0^1 w^2(t)t^{\kappa}dt - \frac{\kappa}{\tau}\int_0^1 w^2(t)t^{\kappa-1}dt + \frac{2}{\tau}\int_{0\le t_1\le t_2\le 1} w'(t_1)w'(t_2)t_1^{\kappa}(1-t_2)dt_1dt_2.$$

5. Using Richert's logarithmic coefficients

We can assume without loss of generality that $h_i \le 0$ for every i. We use the sieve coefficients given by:

$$c(n) = b - \sum_{\substack{p|m,\\ p\le P}} \frac{\operatorname{Log}(P/p)}{\operatorname{Log} P} \tag{17}$$

where $m = \Pi_{(h_1,h_2,\cdots,h_\kappa)}(n)$. To infer results from these sieve coefficients, we notice that

$$\sum_{\substack{p|m,\\ p\le P}} \operatorname{Log}(P/p) \ge \sum_{p|m} \operatorname{Log} P - \operatorname{Log} m \ge \omega(m)\operatorname{Log} P - \operatorname{Log} m$$

yielding (this is to get the exact inequality $\operatorname{Log} m \le \kappa \operatorname{Log} N$ that we have assumed that $h_i \le 0$ for every i):

$$c(n) \le b - \omega\big(\Pi_{(h_1,\cdots,h_\kappa)}(n)\big) + \kappa\frac{\operatorname{Log} N}{\operatorname{Log} P}. \tag{18}$$

We consider the quantity

$$\Big(b\tilde{S}(a_0[Q^\tau]) - \tilde{S}(a_1[Q^\tau]) + \tilde{S}(a_1[Q^\tau;x])\Big)\Big/\Big(\kappa^2 N\mathscr{C}(h_1,\cdots,h_\kappa)/(\operatorname{Log} Q)^\kappa\Big). \tag{19}$$

This quantity equals (recall [14, (18.2)])

$$\begin{aligned}\frac{b}{\kappa}I_1(w,\kappa) - \operatorname{Log}\tau\, I_1(w,\kappa) - K_1(w,\kappa) - I_2(w,\kappa)\\ + I_1(w,\kappa) + \frac{I_1(w,\kappa+1) - I_1(\kappa,w)}{\tau} + \frac{J_2(w,\kappa)}{\tau\kappa}\\ + \mathcal{O}\big(1/(\operatorname{Log} Q)^{1/9}\big) + \mathcal{O}(Q^{\tau+2}N^{-1}(\operatorname{Log} Q)^\kappa).\end{aligned} \tag{20}$$

Let $\varepsilon > 0$ be given. The main term is strictly positive when

$$\frac{b}{\kappa} \ge \varepsilon + \operatorname{Log}\tau - 1 + \frac{1}{\tau} + \frac{K_1(w,\kappa)+I_2(w,\kappa)}{I_1(w,\kappa)} - \frac{I_1(w,\kappa+1) + (1/\kappa)J_2(w,\kappa)}{\tau I_1(w,\kappa)}. \tag{21}$$

We further take $Q = N^\theta$ with $1/\theta = \tau + 2 + \varepsilon\theta$. This ensures in particular that the error term in (20). We ensure in this way the existence of infinitely many integers n for which $c(n) > 0$. On using (18), we get

$$\omega\big(\Pi_{(h_1,\cdots,h_\kappa)}(n)\big) < b + \kappa\frac{\operatorname{Log} N}{\operatorname{Log} P} = b + 1 + \frac{2}{\tau} + \varepsilon.$$

After some time, we get

$$\begin{aligned}\omega\big(\Pi_{(h_1,\cdots,h_\kappa)}(n)\big)/\kappa < \operatorname{Log}\tau + \frac{3}{\tau} + 2\varepsilon\\ + \frac{K_1(w,\kappa)+I_2(w,\kappa)}{I_1(w,\kappa)} - \frac{I_1(w,\kappa+1) + (1/\kappa)J_2(w,\kappa)}{\tau I_1(w,\kappa)}.\end{aligned}$$

We let ε go to zero and get finally

$$(22)\quad \omega\big(\Pi_{(h_1,\cdots,h_\kappa)}(n)\big)/\kappa \le \operatorname{Log}\tau + \frac{3}{\tau} + \frac{K_1(w,\kappa)+I_2(w,\kappa)}{I_1(w,\kappa)} - \frac{I_1(w,\kappa+1)+(1/\kappa)J_2(w,\kappa)}{\tau I_1(w,\kappa)}.$$

We have thus reached a minimisation problem in w. We do not know how to solve it. We decided in [14] to select an explicit family of weights and carry out the formal computations. We adopt here the viewpoint of numerical investigations.

6. ... Or not!

When using Richert's logarithmic coefficients as in the preceding section, we do not use that fact that $\Pi_{(h_1,h_2,\cdots,h_\kappa)}(n)$ is split. The simpler approach we propose here uses this fact. We again assume without loss of generality that $h_i \le 0$ for every i.

We use the coefficients historically introduced in [11] and which are

$$c(n) = b - \sum_{\substack{p|\Pi_{(h_1,h_2,\cdots,h_\kappa)}(n),\\ p\le P}} 1.$$

Each factor $n+h_i$ can have at most

$$\left\lfloor \frac{\operatorname{Log} N}{\operatorname{Log} P}\right\rfloor$$

prime factors larger than P. We consider the sum

$$(23)\qquad \big(b\tilde{S}(a_0[Q^\tau]) - \tilde{S}(a_1[Q^\tau])\big)/\big(\kappa^2 N\mathscr{C}(h_1,\cdots,h_\kappa)/(\operatorname{Log} Q)^\kappa\big).$$

This quantity equals (recall [14, (18.2)])

$$(24)\quad \frac{b}{\kappa}I_1(w,\kappa) - \operatorname{Log}\tau\, I_1(w,\kappa) - K_1(w,\kappa) - I_2(w,\kappa) + \mathcal{O}\big(1/(\operatorname{Log} Q)^{1/9}\big) + \mathcal{O}(Q^{\tau+2}N^{-1}(\operatorname{Log} Q)^\kappa).$$

We again take $Q = N^\theta$ with $1/\theta = \tau + 2 + \varepsilon\theta$. This ensures in particular that the error term above is indeed negligible

This time, we detect integers having at most

$$\lfloor b\rfloor + \kappa\left\lfloor \frac{\operatorname{Log} N}{\operatorname{Log} P}\right\rfloor = \lfloor b\rfloor + \kappa\left\lfloor 1 + \frac{2}{\tau} + \varepsilon\right\rfloor$$

prime factors provided that

$$(25)\qquad \frac{b}{\kappa} \ge \operatorname{Log}\tau + \varepsilon + \frac{K_1(w,\kappa)+I_2(w,\kappa)}{I_1(w,\kappa)}.$$

We let ε go to zero and finally get

$$(26)\qquad \omega\big(\Pi_{(h_1,\cdots,h_\kappa)}(n)\big) \le \left\lfloor \kappa\operatorname{Log}\tau + \kappa\frac{K_1(w,\kappa)+I_2(w,\kappa)}{I_1(w,\kappa)}\right\rfloor + \kappa\left\lfloor 1 + \frac{2}{\tau}\right\rfloor.$$

As shown in the main table, this simple trick enables us to reach at a lesser cost (and sometimes improve on) the values obtained by using Richert sieve coefficients.

Let us state explicitely the optimization problem we have reached:

Determine the minimum of

$$-\int_0^1 \mathrm{Log}(1-t)\, t^{\kappa-1} w(t)^2 dt + 2\int_{0\le t_1\le t_2\le 1} \frac{(w(t_2)-w(t_1))^2}{t_2-t_1} t_1^{\kappa-1} dt_1 dt_2$$

under the condition $\int_0^1 t^{\kappa-1} w(t)^2 dt = 1$.

7. The bilinear forms associated with I_2 and J_2

We deploy I_2 and J_2 defined respectively in (4) and (5) in:

$$I_2^*(w_1,w_2,\kappa) = 2\int_{0\le t\le t_2\le 1} \frac{(w_1(t_2)-w_1(t_1))(w_2(t_2)-w_2(t_1))}{t_2-t_1} t_1^{\kappa-1} dt_1 dt_2, \tag{27}$$

and

$$J_2^*(w_1,w_2,\kappa) = 2\int_{0\le t_1\le t_2\le 1} w_1'(t_1) w_2'(t_2)(1-t_2) t_1^{\kappa} dt_1 dt_2. \tag{28}$$

We want here an expression of J_2^* that avoids the notion of derivative. We write

$$\begin{aligned}
\tfrac{1}{2} J_2^*(w_1,w_2,\kappa) &= \int_0^1 w_1'(t_1)\int_{t_1}^1 (w_2(t_2)-w_2(t_1)) dt_2 t_1^{\kappa} dt_1 \\
&= \int_0^1 \int_0^{t_2} (w_2(t_2)-w_2(t_1)) w_1'(t_1) t_1^{\kappa} dt_1 dt_2 \\
&= \int_0^1 \int_0^{t_2} (w_2(t_2)-w_2(t_1))(w_1(t_2)-w_1(t_1)) t_1^{\kappa} dt_1 dt_2.
\end{aligned}$$

8. Extending the class of possible weights

Inequalities (22) and (26) a priori require w to be continuous, piecewise differentiable with a bounded derivative and such that $w(1)=0$. This class is almost enough for our purpose, aside from the last condition, but extending it is not difficult. The factor $I_2(w,\kappa)$ is the main trouble. One can use for w any $\mathrm{L}^2([0,1], t^{\kappa-1} - t^{\kappa-1}\mathrm{Log}(1-t))$ weight such that $I_2(w,\kappa) < \infty$. Indeed, instead of $I_2(w,\kappa)$, we may consider

$$I_2(w,\kappa,\eta) = \int_0^1 \int_t^1 \frac{(w(u)-w(t))^2}{u-t+\eta} du t^{\kappa-1} dt$$

and replace $I_2(w,\kappa)$ by $I_2(w,\kappa,1/\mathrm{Log}\,P)$ in (20) and (24) above (that's in fact the quantity that appears in the proof!). It is then straightforward to approximate w by C^1 functions. We finally notice that $I_2(w,\kappa,\eta) \le I_2(w,\kappa)$ when $\eta > 0$. The Lebesgue Theorem of dominated convergence then applies.

The class obtained is a vector space and exact computations contained in the next chapter shows that it contains piecewise affine (and even not necessarily continuous) function. As a matter of fact, computations tend to show that there are optimal functions, and that these are very regular. It is theoretically more satisfactory to have a larger class, and may help in studying the problem.

9. Piecewise affine weights

Since we are not able to get an explicit solution, if it exists, to the optimization problem arising from (22) or (26), we do some direct optimization for small values of κ. We look only at continuous and locally affine functions. Note that they are Lipschitz so the definition of I_2 does not present any problem. Note furthermore that the functions we shall choose at the end are non-increasing, indeed ensuring that $|\tilde{\lambda}_d| \le 1$, thanks to [14, Lemma 4.1].

Our strategy runs as follows: we split the interval $[0,1]$ is a finite number of intervals, say $0 = a_1 < a_2 < \cdots < a_I = 1$ and fix a value of τ. Given a $(I+1)$-tuple of real numbers $(\alpha_0, \alpha_1, \cdots, \alpha_I)$, we consider the function w that is affine on each (a_i, a_{i+1}) and takes value α_i at a_i. If we ignore the integer parts in (22) and (26), the upper bound is a quadratic form in the parameter $(\alpha_0, \alpha_1, \cdots, \alpha_I)$ divided by another quadratic form (namely $I_1(w,\kappa)$). We thus optimize the quadratic form of the numerator under the condition $I_1(w,\kappa) = 1$; this is a classical problem. We determine an extremal point w and then recompute the upper bound with this function, but this time by inserting the integer parts.

The object of this section is to work out formulae for the quadratic forms that appear.

Note that in what follows we consider chunks of two functions, one with parameters a, α, b, β and a second one with parameters a', α', b', β'. We will shorten the first by calling it w and the latter by calling it w', *which does not have anything to do with the derivative!*

We select functions

$$w_{a,\alpha,b,\beta} = \begin{cases} 0 & \text{when } t \notin [a,b], \\ \frac{(\beta-\alpha)t+\alpha b-\beta a}{b-a} = \gamma t + \eta & \text{when } t \in [a,b]. \end{cases} \tag{29}$$

from which we build

$$w = \sum_{1 \le i \le I} w_{a_i,\alpha_i,b_i,\beta_i} \tag{30}$$

with $0 = a_1 < b_1 = a_2 < b_2 = a_3 < \cdots < b_I = 1$ and $\beta_i = \alpha_{i+1}$.

Formulae for I_1 and K_1. We immediately see from (2) that

$$I_1(w_{a,\alpha,b,\beta}, \kappa) = \gamma^2 \frac{b^{\kappa+2} - a^{\kappa+2}}{\kappa+2} + 2\gamma\eta \frac{b^{\kappa+1} - a^{\kappa+1}}{\kappa+1} + \eta^2 \frac{b^{\kappa} - a^{\kappa}}{\kappa}. \tag{31}$$

The computation of K_1 is equaly trivial but the wwiting will be easier by setting

$$\Sigma(x,\kappa) = -\int_0^1 t^{\kappa-1} \operatorname{Log}(1-t)dt = \sum_{\ell \ge 1} \frac{x^{\kappa+\ell}}{\ell(\ell+\kappa)}. \tag{32}$$

From (3), we readily find that

$$\begin{aligned} K_1(w_{a,\alpha,b,\beta}, \kappa) = \gamma^2(\Sigma(b,\kappa+2) - \Sigma(a,\kappa+2)) \\ + 2\gamma\eta(\Sigma(b,\kappa+1) - \Sigma(a,\kappa+1)) + \eta^2(\Sigma(b,\kappa) - \Sigma(a,\kappa)). \end{aligned}$$

Formulae for I_2. This quantity is rather intricate to evaluate. Let us first assume that $0 \le a \le b \le a' \le b' \le 1$. We find in this case that

$$\begin{aligned}
\tfrac12 I_2(w_{a,\alpha,b,\beta}, w_{a',\alpha',b',\beta'},\kappa) &= \int_a^{b'}\int_t^{b'} \frac{(w'(u)-w'(t))(w(u)-w(t))}{u-t}du t^{\kappa-1}dt \\
&= \int_a^{b}\int_{a'}^{b'} \frac{(w'(u)-w'(t))(w(u)-w(t))}{u-t}du t^{\kappa-1}dt \\
&+ \int_b^{b'}\int_t^{b'} \frac{(w'(u)-w'(t))(w(u)-w(t))}{u-t}du t^{\kappa-1}dt \\
&= -\int_a^{b}\int_{a'}^{b'} \frac{w'(u)w(t)}{u-t}du t^{\kappa-1}dt \\
&= -\int_a^{b}\Bigl(\gamma'(b'-a') + (\gamma' t+\eta')\operatorname{Log}\frac{b'-t}{a'-t}\Bigr) w(t)t^{\kappa-1}dt
\end{aligned}$$

so that we have reached the formula

$$\begin{aligned}
\tfrac12 I_2(w_{a,\alpha,b,\beta}, w_{a',\alpha',b',\beta'},\kappa) &= -\gamma'(b'-a')\left(\frac{\gamma(b^{\kappa+1}-a^{\kappa+1})}{\kappa+1} + \frac{\eta(b^\kappa-a^\kappa)}{\kappa}\right) \\
&- \gamma'\gamma\mathfrak{p}(\kappa+2,a,b,a'b') - (\gamma\eta'+\eta\gamma')\mathfrak{p}(\kappa,a,b,a'b') - \eta'\eta\mathfrak{p}(\kappa,a,b,a'b')
\end{aligned}$$

where, to render the writing less cumbersome, we have used the auxiliary function $\mathfrak{p}$ defined by:

$$\begin{aligned}
\mathfrak{p}(\kappa,a,b,a'b') &= \int_a^b t^{\kappa-1}\operatorname{Log}\frac{b'-t}{a'-t}dt \\
&= \frac{b^\kappa-a^\kappa}{\kappa}\operatorname{Log}\frac{b'}{a'} - b'^\kappa\Bigl(\Sigma\Bigl(\kappa,\frac{b}{b'}\Bigr) - \Sigma\Bigl(\kappa,\frac{a}{b'}\Bigr)\Bigr) \\
&\qquad + a'^\kappa\Bigl(\Sigma\Bigl(\kappa,\frac{b}{a'}\Bigr) - \Sigma\Bigl(\kappa,\frac{a}{a'}\Bigr)\Bigr).
\end{aligned} \tag{33}$$

We next have to handle the case when $w_{a,\alpha,b,\beta} = w_{a',\alpha',b',\beta'}$. In that case

$$\begin{aligned}
\tfrac12 I_2(w_{a,\alpha,b,\beta}, w_{a,\alpha,b,\beta},\kappa) &= \int_0^{a}\int_a^{b} \frac{w(u)^2}{u-t}du t^{\kappa-1}dt \\
+\int_a^b\int_t^1 &\frac{(w(u)-w(t))(w(u)-w(t))}{u-t}du t^{\kappa-1}dt + \int_b^1\int_t^1 \frac{(w(u)-w(t))w(u)}{u-t}du t^{\kappa-1}dt \\
&= W + \int_a^b\int_t^b \frac{(w(u)-w(t))(w(u)-w(t))}{u-t}du t^{\kappa-1}dt + \int_a^b\int_b^1 \frac{w(t)w(t)}{u-t}du t^{\kappa-1}dt \\
&= W + \gamma^2\int_a^b \frac{(b-t)^2}{2}t^{\kappa-1}dt + \int_a^b w(t)^2 t^{\kappa-1}\operatorname{Log}\frac{1-t}{b-t}dt \\
&= W + \frac{\gamma^2}{2}\left(\frac{b^2(b^\kappa-a^\kappa)}{\kappa} - 2\frac{b(b^{\kappa+1}-a^{\kappa+1})}{\kappa+1} + \frac{b^{\kappa+2}-a^{\kappa+2}}{\kappa+2}\right) \\
&+ \gamma^2\mathfrak{p}(\kappa+2,a,b,b,1) + 2\gamma\eta\mathfrak{p}(\kappa+1,a,b,b,1) + \eta^2\mathfrak{p}(\kappa,a,b,b,1)
\end{aligned}$$

where W is defined and computed now:

$$\begin{aligned}
W &= \int_0^a \int_a^b \frac{w(u)^2}{u-t} du t^{\kappa-1} dt \\
&= \int_0^a \left(\gamma^2 \frac{b^2-a^2}{2} + (\gamma^2 t + 2\gamma\eta)(b-a) + (\gamma t + eta)^2 \operatorname{Log} \frac{b-t}{a-t}\right) t^{\kappa-1} dt \\
&= \left(\gamma^2 \frac{b^2-a^2}{2}\right) \frac{a^\kappa}{\kappa} + \gamma^2 (b-a) \frac{a^{\kappa+1}}{\kappa+1} + 2\gamma\eta(b-a)\frac{a^\kappa}{\kappa} \\
&\quad + \gamma^2 \mathfrak{p}(\kappa+2, 0, a, a, b) + 2\eta\gamma \mathfrak{p}(\kappa+1, 0, a, a, b) + \eta^2 \mathfrak{p}(\kappa, 0, a, a, b).
\end{aligned}$$

Formulae for J_2. The quantity J_2 is again rather intricate to evaluete explicitly. We separate two cases. In the first case we assume that $0 \le a \le b \le a' \le b' \le 1$. We get:

$$\begin{aligned}
\tfrac{1}{2} J_2(w_{a,\alpha,b,\beta}, w_{a',\alpha',b',\beta'}, \kappa) &= \int_a^{b'} \int_t^{b'} (w'(u) - w'(t))(w(u) - w(t)) du t^{\kappa-1} dt \\
&= \left(\int_a^b \int_{a'}^{b'} + \int_b^{b'} \int_t^{b'}\right) (w'(u) - w'(t))(w(u) - w(t)) du t^{\kappa-1} dt \\
&= -\int_a^b \int_{a'}^{b'} w'(u) w(t) du t^{\kappa-1} dt \\
&= -\left(\gamma' \frac{b'^2 - a'^2}{2} + \eta'(b'-a')\right)\left(\gamma \frac{b^{\kappa+1} - a^{\kappa+1}}{\kappa+1} + \eta \frac{b^\kappa - a^\kappa}{\kappa}\right).
\end{aligned}$$

This settles the case $a \neq a'$. We next have to handle the case when $w_{a,\alpha,b,\beta} = w_{a',\alpha',b',\beta'}$. In that case

$$\begin{aligned}
&\tfrac{1}{2} J_2(w_{a,\alpha,b,\beta}, w_{a,\alpha,b,\beta}, \kappa) = \int_0^a \int_a^b w(u)^2 du t^{\kappa-1} dt \\
&+ \int_a^b \int_t^1 (w(u)-w(t))(w(u)-w(t)) du t^{\kappa-1} dt + \int_b^1 \int_t^1 (w(u)-w(t)) w(u) du t^{\kappa-1} dt
\end{aligned}$$

so that

$$\begin{aligned}
\tfrac{1}{2} J_2(w_{a,\alpha,b,\beta}, w_{a,\alpha,b,\beta}, \kappa) &= \frac{a^\kappa}{\kappa}\left(\gamma^2 \frac{b^3-a^3}{3} + 2\gamma\eta \frac{b^2-a^2}{2} + \eta^2(b-a)\right) \\
+ \int_a^b \int_t^b (w(u) - w(t))(w(u) - w(t)) du t^{\kappa-1} dt &+ \int_a^b \int_b^1 w(t) w(t) du t^{\kappa-1} dt \\
&= \frac{a^\kappa}{\kappa}\left(\gamma^2 \frac{b^3-a^3}{3} + 2\gamma\eta \frac{b^2-a^2}{2} + \eta^2(b-a)\right) \\
&\quad + \gamma^2 \int_a^b \frac{(b-t)^3}{3} t^{\kappa-1} dt + (1-b) \int_a^b w(t)^2 t^{\kappa-1} dt
\end{aligned}$$

which results in the following expression

$$\tfrac{1}{2}J_2(w_{a,\alpha,b,\beta},w_{a,\alpha,b,\beta},\kappa)=\frac{a^\kappa}{\kappa}\left(\gamma^2\frac{b^3-a^3}{3}+2\gamma\eta\frac{b^2-a^2}{2}+\eta^2(b-a)\right)$$
$$+\frac{\gamma^2}{3}\left(\frac{b^3(b^\kappa-a^\kappa)}{\kappa}-3\frac{b^2(b^{\kappa+1}-a^{\kappa+1})}{\kappa+1}+3\frac{b(b^{\kappa+2}-a^{\kappa+2})}{\kappa+2}-\frac{b^{\kappa+3}-a^{\kappa+3}}{\kappa+3}\right)$$
$$+(1-b)I_1(w_{a,\alpha,b,\beta},w_{a,\alpha,b,\beta},\kappa).$$

10. Results

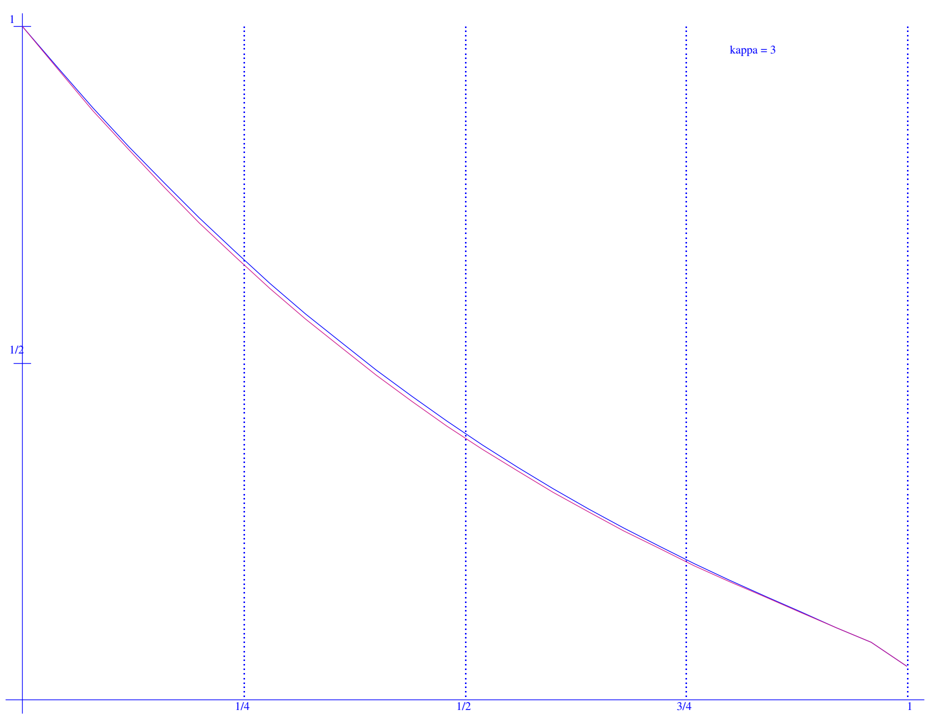
We ran a Pari/GP-script to get the results below. The process has been to optimize the resulting quadratic form under the constraint $I_1(\kappa,w)=1$; this is a classical problem. We then found rational coefficients close enough to the optimal ones obtained and recomputed the resulting $n(\kappa)$ (resp. $n_R(\kappa)$).

We do not give all the parameters we used to get the main table, but detailled three examples.

Case $\kappa=1$. This first example is here only for the reader to check formulae and compare his/her results with ours. We reach $n(1)=2$ with $1/\tau=0.15$ and the simplest affine function w that takes the values

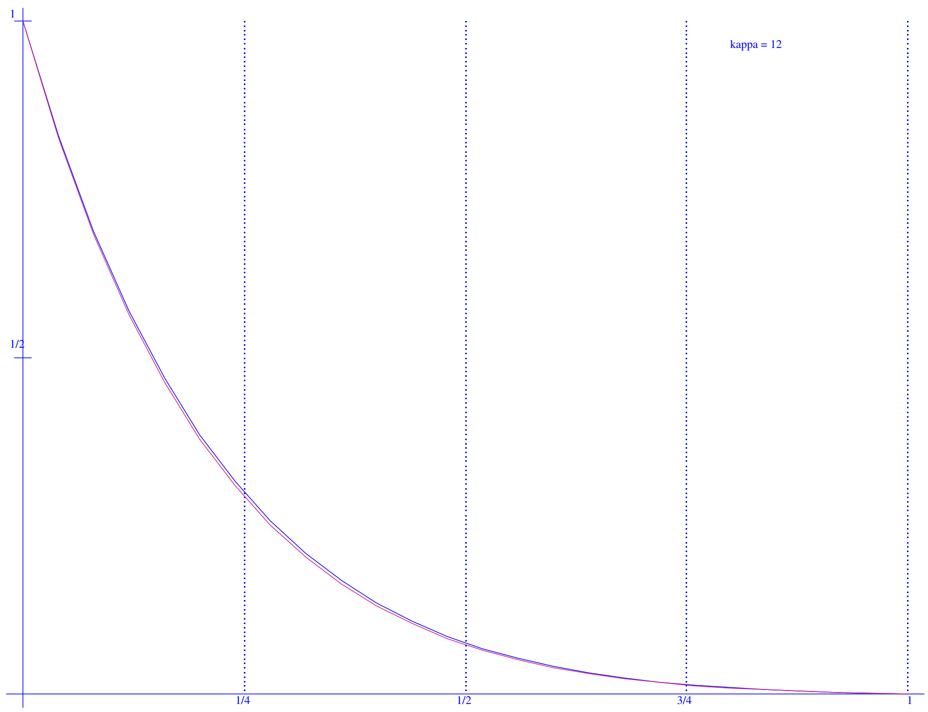
$$w(0)=1,\quad w(1)=133/500.$$

Case $\kappa=3$. We reach $n(3)=8$ and $n_R(3)=9$. On splitting the interval $[0,1]$ in 25 sub-intervals of length $1/25$, here is a plot of the function we used for $n(3)$ and $n_R(3)$: The function corresponding to $n(3)$ is slightly below the one used for $n_R(3)$,



but both are extremely close one to another. Two things appear numerically, when one increasing the number of pieces into which we decompose the unit interval: first the optimal functions seem to converge towards a given function; secondly, the value $\kappa w(1)$ does not approach zero.

Case $\kappa = 12$. We reach $n(12) = 46$ and $n_R(12) = 47$. On splitting again the interval $[0, 1]$ in 25 sub-intervals of length $1/25$, here is a plot of the function we used for $n(12)$ and $n_R(12)$: The function corresponding to $n(12)$ is slightly below the one



used for $n_R(12)$, but both are extremely close one to another. Two things appear numerically, when one increasing the number of pieces into which we decompose the unit interval: first the optimal functions seem to converge towards a value; secondly, the value $w(1)$ does not approach zero.

In order to enable checking, here is a simple function that leads to $n_R(12) = 47$: it is the function that is affine on each of the interval $[0, 1/3]$, $[1/3, 2/3]$ and $[2/3, 1]$ and that takes the four values:

t	0	1/3	2/3	1
$w(t)$	1	0.1998	0.029	0.0005

$(1/\tau = 0.38)$.

To reach $n(3) = 46$, we use the function w that is affine on each of the interval $[i/5, (i + 1)/5]$, when $i \in \{0, \cdots, 4\}$ and that takes the values:

t	0	1/5	2/5	3/5	4/5	1
$w(t)$	1	0.37037	0.13157	0.04051	0.00952	0.00051

and with the choice $1/\tau = 0.48$.

General remarks. We gather here some comments on the numerical results. When κ increased, the optimal function normalized by $w(0) = 1$ tends to dip faster when κ increases. We investigated the slope at $t = 0$, and its seems to be

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