FROM EXPLICIT ESTIMATES FOR THE PRIMES TO EXPLICIT ESTIMATES FOR THE MÖBIUS FUNCTION – II

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Abstract. We improve on all the results of the paper "From explicit estimates for the primes to explicit estimates for the Möbius function" [16] by the first author by incorporating the finite range computations performed since then by several authors. Thus we have

$$\Bigl|\sum_{n\le X}\mu(n)\Bigr| \le \frac{0.006688\,X}{\log X}, \qquad \text{for } X\ge 1\,798\,118,$$

$$\Bigl|\sum_{n\le X}\frac{\mu(n)}{n}\Bigr| \le \frac{0.010032}{\log X}, \qquad \text{for } X\ge 617\,990.$$

We also improve on the method described in [16] by a simple remark.

1. Introduction and results

Let μ denotes the Möbius function and Λ the von Mangoldt function. In [16], the first author exploited the identity

$$(1)\qquad 2\gamma+\sum_{n\le X}\mu(n)\log^2 n=\sum_{k\ell\le X}\mu(\ell)\bigl(\Lambda\star\Lambda(k)-\Lambda(k)\log k+2\gamma\bigr),$$

valid for any $X\ge 1$, to derive explicit estimates for the summatory function M of the Möbius function μ. In this article, we propose to update those estimates by taking into account [13] by G. Hurst, [4] by J. Büthe, [14] by R. Vanlalnagaia and [15] by the first author. We also improve on their corresponding proofs by essentially taking a closer look at them, see for instance Lemma 6.1. Moreover, we hope to have improved on the exposition with respect to [16]. In order to do so, we shall reproduce some lemmas whose proofs can still be found in [16], so that the reader may follow the argument more easily. By the same reason, more on the history of this problem and on the philosophy of the method we use can be found in the original work [16].

The summatory function of the Moebius function is a fundamental object to study averages of multiplicative functions, and the later are used for instance in sieve theory and related matters as in [22] by the second author or in [5] by E. Carneiro, A. Chirre, H. Helfgott & J. Mejía-Cordero, and in diverse subjects, as for instance in [11] by F. Götze, D. Kaliada & D. Zaporozhets.

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Below we state our results, which should be directly compared to those presented in [16] in their corresponding ranges.

Theorem 1.1. *For $X \geq 1\,798\,118$, we have*

$$\left|\sum_{n\leq X}\mu(n)\right| \leq \frac{(0.006688\,\log X - 0.039)X}{\log^2 X}.$$

This improves by almost a factor of 2 on [16, Thm. 1.1], which we recover as the next corollary by adding some simple computations to lower the range of X.

Corollary 1.1. *For $X \geq 1\,078\,853$, we have*

$$\left|\sum_{n\leq X}\mu(n)\right| \leq \frac{(0.0130\log X - 0.118)X}{\log^2 X}.$$

On the other hand, by the method outlined in §3, we also derive a bound on the logarithmic average of the Möbius function.

Corollary 1.2. *For $X \geq 617\,990$, we have*

$$\left|\sum_{n\leq X}\frac{\mu(n)}{n}\right| \leq \frac{0.010032\log X - 0.0568}{\log^2 X}.$$

As a consequence, we correctly obtain [17, Thm. 1.2].

Corollary 1.3. *For $X \geq 463\,421$, we have*

$$\left|\sum_{n\leq X}\frac{\mu(n)}{n}\right| \leq \frac{0.0144\log X - 0.1}{\log^2 X}.$$

Notation. We set

(2) $$R(X) = \psi(X) - X, \qquad r(X) = \tilde{\psi}(X) - \log X + \gamma,$$

where, by denoting Λ the von Mangoldt function,

(3) $$\psi(X) = \sum_{n\leq X}\Lambda(n), \qquad \tilde{\psi}(X) = \sum_{n\leq X}\frac{\Lambda(n)}{n}.$$

On the other hand, we consider the following summatory functions related to the Möbius function

(4) $$M(X) = \sum_{n\leq X}\mu(n), \qquad m(X) = \sum_{n\leq X}\frac{\mu(n)}{n}.$$

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2. New estimates and consequences

We recall and gather here several explicit estimates that we shall require later.

Bounds on the summatory functions of the Möbius function M ***and*** m. Let us first turn our attention to the summatory function of the Möbius function M defined in (4). The main novelty comes from the paper [13] by G. Hurst where it is proved that

$$|M(X)| \leq \sqrt{X}, \qquad X \leq 10^{16}. \tag{5}$$

Let us complete this by recalling previous estimates. In [9] by F. Dress, we find the bound

$$|M(X)| \leq 0.571\sqrt{X}, \qquad 33 \leq X \leq 10^{12} \tag{6}$$

In [10] by F. Dress & M. El Marraki, we find the bound

$$|M(X)| \leq \frac{X}{2360}, \qquad X \geq 617\,973, \tag{7}$$

(see also [8] by N. Costa-Pereira) which [6] by H. Cohen, F. Dress & M. El Marraki, (also published in [7, Thm. 5 bis]) improves as

$$|M(X)| \leq \frac{X}{4345}, \qquad X \geq 2\,160\,535. \tag{8}$$

The second novelty concerns the summatory function m, defined in (4), we find in [12, Lemma 5.10] the following result

$$|m(X)| \leq \frac{\sqrt{2}}{\sqrt{X}} \quad \text{when } X \leq 10^{14}. \tag{9}$$

Bounds on the summatory function of the squarefree numbers. We now turn towards two bounds concerning the squarefree numbers. The first one comes from [7, Thm. 3 bis].

Lemma 2.1. *For* $X \geq 438\,653$*, we have*

$$\sum_{n \leq X} \mu^2(n) = \frac{6}{\pi^2}X + O^*(0.02767\sqrt{X}).$$

If $X \geq 1$*, we can replace* 0.02767 *by* 0.7*.*

The second bound is a consequence of the more recent paper [18].

Lemma 2.2. *We have for* $X \geq 10^6$

$$\sum_{n \leq X} \frac{\mu^2(n)}{n} = \frac{6}{\pi^2}\log X + O^*(1.044).$$

If $X \geq 1$*, we can replace* 1.044 *by* 1.48*.*

Proof. [18, Cor. 1.2] says in particular that, for $X \geq 438\,653$, we have

$$\sum_{n \leq X} \frac{\mu^2(n)}{n} = \frac{6}{\pi^2}(\log X + \mathfrak{b}) + O^*\left(\frac{3 \times 0.02767}{\sqrt{X}}\right),$$

where $\mathfrak{b} = 1.7171\cdots$. The lemma follows immediately. □

Bounds on the usual and harmonic summatory function of Λ. Let us recall the definitions of $R(X)$ and $r(X)$ in (2). We have the following set of estimations for the function R in different ranges.

$$|R(X)| \le 0.8\sqrt{X} \quad \text{when } 1500 < X \le 10^{10}, \tag{10}$$

$$|R(X)| \le 0.94\sqrt{X} \quad \text{when } 11 < X \le 10^{19}, \tag{11}$$

$$|R(X)| \le 8 \cdot 10^{-5} \cdot X \quad \text{when } 10^8 \le X, \tag{12}$$

$$|R(X)| \le 2.58843 \cdot 10^{-5} \cdot X \quad \text{when } 21 \le \log X, \tag{13}$$

$$|R(X)| \le 1.93378 \cdot 10^{-8} \cdot X \quad \text{when } 40 \le \log X, \tag{14}$$

$$|R(X)| \le 0.0065 \frac{X}{\log X} \quad \text{when } X \ge 1\,514\,928 \tag{15}$$

$$|R(X)| \le 9 \cdot 10^{-7} \frac{X}{\log X} \quad \text{when } 10^{19} \le X, \tag{16}$$

Estimate (10) comes from [19, p. 423]. Estimate (11) comes from [4, Thm. 2]. (13) and (14) are derived from [2, Table 8] by S. Broadbent, H. Kadiri, A. Lumley, N. Ng & K. Wilk together with [3, Table 1] by J. Büthe. For (12), the same [2, Table 8] gives the bound $4.27 \cdot 10^{-5}$ provided $X \ge e^{20}$, and we extend it using (10). (15) can be found in [16, Lemma 4.2] and the last estimate (16) comes from [2, Table 15] extended to ψ via [21, Thm. 6 (5.3)-(5.4)] by J. Rosser & L. Schoenfeld.

Let us complete this series with a readily established bound

$$\max_{24\,200 \le X \le 3\cdot 10^7} \frac{|R(X)|}{\sqrt{X}} \le 0.71. \tag{17}$$

Let us further recall the second part of [14, Theorem 9] by R. Mawia (who used earlier the family name Vanlalngaia).

$$\left| r(X) - \frac{R(X)}{X} \right| \le \frac{0.05}{\sqrt{X}} + 1.75 \cdot 10^{-12} \quad \text{when } X \ge 394\,385. \tag{18}$$

We also recall the wide-ranging estimate proved in [20, Thm. 12] by J. Rosser & L. Schoenfeld.

Lemma 2.3. *The quotient $\psi(X)/X$ takes its maximum at $X = 113$; for any $X > 0$, we have*

$$\psi(X) < 1.03883X.$$

Finally, we quote [16, Lemma 5.1].

Lemma 2.4. *When $X \ge 1$, and $\sqrt{X} \ge T \ge 1$, we have*

$$\sum_{n \le T} \frac{\Lambda(n)}{n \log(X/n)} \le 1.04 \, \log\left(\frac{\log X}{\log(X/T)}\right) + \frac{1.04}{\log X}.$$

3. Strategy of proof

Call L the function $t > 0 \mapsto \log t$ and $\star$ the convolution of any two arithmetic functions. By observing that $(1/\zeta)'' = -\zeta''/\zeta^2 + (\zeta')^2/\zeta^3$, where ζ corresponds to the Riemann zeta function, we deduce the equality

$$\mu \cdot \mathrm{L}^2 = \mu \star \Lambda \star \Lambda - \mu \star \Lambda \cdot \mathrm{L},$$

whence identity (1).

We define the remainder quantity R_2^* as follows

$$R_2^*(X) = \sum_{n\leq X} \left(\Lambda\star\Lambda(n) - \Lambda(n)\log n + 2\gamma\right). \tag{19}$$

Now, for any $K\in(0,X]\cap\mathbb{Z}$, we may derive from (1) the following expression

$$\sum_{n\leq X}\mu(n)\log^2 n = -2\gamma + \sum_{\ell\leq X/K}\mu(\ell)R_2^*\left(\frac{X}{\ell}\right) + \sum_{k<K} R_2^*(k)\sum_{X/(k+1)<\ell\leq X/k}\mu(\ell). \tag{20}$$

Observe that the range in the outermost sum of the above second expression is in fact $\max\{X/K, X/(k+1)\}<\ell\leq X/k$. Nonetheless, if $X/(k+1)<X/K$, since $k<K$, we have $k<K<k+1$, giving an empty sum as K is integer.

Moreover, by rearranging terms, we may express

$$\sum_{k\leq K-1} R_2^*(k)\sum_{X/(k+1)<\ell\leq X/k}\mu(\ell) = \sum_{k\leq K-1} R_2^*(k)\left[M\left(\frac{X}{k}\right) - M\left(\frac{X}{k+1}\right)\right]$$

$$= \sum_{k\leq K}(\Lambda\star\Lambda(k) - \Lambda(k)\log k + 2\gamma)M\left(\frac{X}{k}\right) - R_2^*(K)M\left(\frac{X}{K}\right). \tag{21}$$

Thus, by combining (20) and (21) together, we arrive at

$$\sum_{n\leq X}\mu(n)\log^2 n = -2\gamma + \sum_{\ell\leq X/K}\mu(\ell)R_2^*\left(\frac{X}{\ell}\right) + \sum_{k\leq K}(\Lambda\star\Lambda(k) - \Lambda(k)\log k + 2\gamma)M\left(\frac{X}{k}\right) - R_2^*(K)M\left(\frac{X}{K}\right). \tag{22}$$

Identity (22) will be used to evaluate the average of $\mu(n)\log^2 n$ in Lemma 7.3. In order to do so, we shall require estimates for $R_2^*(X)$ when $X\geq 1.8\cdot 10^9$ as well as Lemma 7.1 to bound the summation over $k\leq K$.

The estimation of R_2^* further splits in two parts, as will be shown in Lemma 6.1: we have to estimate the two auxiliary functions R_3 and R_4 that we now define. First, R_3 is defined as

$$R_3(X) = 2\sqrt{X}|\sqrt{X}r(\sqrt{X}) - R(\sqrt{X})| + R(\sqrt{X})^2 + |R(X)|\log X + \left|\int_1^X R(t)\frac{dt}{t}\right|, \tag{23}$$

with the functions r and R as defined in (2). The function R_3 will be studied in Section 4. And finally, we will require the function R_4 defined by

$$R_4(X) = \sum_{n\leq\sqrt{X}}\Lambda(n)R\left(\frac{X}{n}\right), \tag{24}$$

and which will be studied in Section 5.

Thus, upon estimating the right-hand side of (22), we will have an estimation for the sum $\sum_{n\leq X}\mu(n)\log^2 n$ within some specific range. This main bound is expressed in Lemma 7.3 and from it, we may deduce estimates for M via Lemma 3.1. Indeed, let $(f(n))_{n\in\mathbb{Z}_{>0}}$ be a sequence of complex numbers and, for any integer $k\geq 0$ and $X\geq 1$, consider the weighted summatory function

$$M_k(f,X)=\sum_{n\leq X} f(n)\log^k n. \tag{25}$$

Then, by summation by parts or quoting [16, Lemma 8.1], we have the following result.

Lemma 3.1. *For any $X_0>1$ and any $X\geq X_0$, we have*

$$M_0(f,X)-M_0(f,X_0)=\frac{M_k(f,X)}{\log^k X}-\frac{M_k(f,X_0)}{\log^k X_0}+k\int_{X_0}^{X}\frac{M_k(f,t)}{t\log^{k+1}(t)}dt.$$

Finally, upon having an estimation for $M(X)$ within some range, we are able to derive bounds for $m(X)$ by using the following result due to M. Balazard in [1, Eq. (8)]. It is much more efficient than [16, Lemma 9.1] which was used in the previous version of this work.

Lemma 3.2. *For any $X\geq 1$, we have*

$$|m(X)|\leq\frac{|M(X)|}{X}+\frac{1}{X^2}\int_1^X|M(t)|dt+\frac{8}{3X}.$$

The question being to deduce bounds for $m(X)$ from bounds for $M(X)$, the most efficient identity at the time of writing seems to be given by F. Daval as he told us in some private communication.

4. Bounding R_3

This section is devoted to bounding the function $R_3(X)$, as defined in (23). We begin with [16, Lemma 6.4] which is a direct computation of an integral over $R(t)/t$. We take advantage of the oscillating nature of $R(t)$ in this manner.

Lemma 4.1.

$$\int_1^{10^8}R(t)\frac{dt}{t}=-129.559+O^*(0.01).$$

We can next give an estimation for R_3, defined in (23).

Lemma 4.2. *We have*

$$R_3(X)\leq 0.2\cdot X^{3/4},\qquad \text{when } 1.8\cdot 10^9\leq X\leq 10^{19},$$

$$\frac{R_3(X)}{X}\leq 9\cdot 10^{-5}+\frac{1}{10\cdot X^{1/4}},\qquad \text{when } X\geq 10^{19}.$$

Proof. We consider three cases. In what follows, $K_0=1.8\cdot 10^9$ and $K=10^8$.

Case 1: Suppose that $1.8\cdot 10^9\leq X\leq 10^{10}$. By Lemma 4.1, we have

$$\left|\int_1^X R(t)\frac{dt}{t}\right|\leq 129.6+\int_{10^8}^X|R(t)|\frac{dt}{t}.$$

On the other hand, we find that

$$2\sqrt{X}|\sqrt{X}r(\sqrt{X}) - R(\sqrt{X})| \leq 0.1 \cdot X^{3/4} + 3.5 \cdot 10^{-12} X, \tag{26}$$

$$R(\sqrt{X})^2 \leq 0.8^2\sqrt{X}, \tag{27}$$

$$|R(X)| \log X \leq 0.8\sqrt{X} \log X, \tag{28}$$

$$\int_{10^8}^{X} |R(t)| \frac{dt}{t} \leq 0.8 \int_{10^8}^{X} \frac{dt}{\sqrt{t}} = 2 \cdot 0.8(\sqrt{X} - 10^4). \tag{29}$$

Inequality (26) is obtained thanks to (18) whereas (27), (28) and (29) come from estimation (10). Therefore, by combining the above bounds together, we arrive at

$$\begin{aligned}\frac{R_3(X)}{X^{3/4}} \leq & 0.1 + 3.5 \cdot 10^{-12} X^{1/4} + 0.8 \frac{\log X}{X^{1/4}} + \\ & + \frac{0.8^2 + 2 \cdot 0.8}{X^{1/4}} + \frac{129.7 - 2 \cdot 0.8 \cdot 10^4}{X^{3/4}} \quad \leq 0.2,\end{aligned} \tag{30}$$

where we have used that the function $t \mapsto \log t / t^{1/4}$ is decreasing for $\log t \geq 4$.

Case 2: Suppose that $10^{10} \leq X \leq 10^{19}$. Now the bounds (27), (28), (29) and (30) hold with 0.8 replaced by 0.94. Thus we obtain the estimation

$$\frac{R_3(X)}{X^{3/4}} \leq 0.18.$$

Case 3: Suppose that $X \geq 10^{19}$. We proceed similarly as before and obtain

$$\begin{aligned}\frac{R_3(X)}{X} &\leq 3.5 \cdot 10^{-12} + \frac{0.1}{X^{1/4}} + (2.59 \cdot 10^{-5})^2 + 9 \cdot 10^{-7} \\ &\quad + 8 \cdot 10^{-5} - \frac{8 \cdot 10^3 - 129.7}{X} \\ &\leq 0.000081 + \frac{0.1}{X^{1/4}} \quad \leq 0.000083,\end{aligned}$$

where we have used (13) instead of (11), since $e^{21} \leq 10^{19/2}$, then (16) instead of (11) and lastly (12) instead of (11). □

5. Bounding R_4

This section is devoted to bounding the function $R_4(X)$, as defined in (24). The next lemma, which requires some days of computations on a personnal computer, is an important step.

Lemma 5.1. *When* $1.8 \cdot 10^9 \leq X \leq 5 \cdot 10^{10}$, *we have*

$$\left| \sum_{1000 < n \leq 40\,000} \Lambda(n) R\left(\frac{X}{n}\right) \right| \leq 0.000154 \cdot X.$$

Whereas we have run the computations up to $5 \cdot 10^{10}$, we will only use the above result up to $2 \cdot 10^{10}$. Employing the larger range might improve on our final bound, but we have opted to keep the latter shorter one to ensure accuracy.

Proof. Let us set, for any two integers A_0 and A_1:

$$f(A_0, A_1, D) = \sum_{A_0 \le a \le A_1} \Lambda(a)\psi\left(\frac{X}{a}\right). \tag{31}$$

We compute and store the values of $\psi(n)$ for $n \in [10^9/A_1, \cdots, 5\cdot 10^{10}/A_0] \cap \mathbb{Z}$ with the actual choice $A_0 = 1000$ and $A_1 = 40\,000$. We go from $f(A_0, A_1, D)$ to the quantity stated in the statement by removing $X\sum_{A_0\le n\le A_1}\Lambda(n)/n$, which we simply compute directly.

Next, as X moves from the integer N to $N+1$, we may get a modification in the value of $\psi(X/a)$ only if $[N/a] < [(N+1)/a]$. This may only happen when there exists an integer c such that $N/a < c \le (N+1)/a$; this translates into $N < ac \le N+1$, which may only happen when a is a divisor of $N+1$, which must further belong to the interval $[A_0, A]$. This observation speeds up the program we use considerably and makes the computations possible on a home computer, using merely a couple of days. □

Recall now the definition (24) of $R_4(X)$.

Lemma 5.2. *We have*

$$|R_4(X)| \le 0.005 \cdot X, \qquad X \ge 1.8 \cdot 10^9.$$

This improves slightly on [16, Lemma 5.2].

Proof. We consider different cases according to the range of X.

Case 1: Suppose that $1.8 \cdot 10^9 \le X \le 9 \cdot 10^9$. We first check via a short computation that

$$\sum_{n\le 1000} \frac{\Lambda(n)}{\sqrt{n}} \le 60.51, \qquad \sum_{n\le 40000} \frac{\Lambda(n)}{\sqrt{n}} = B = 40012.8937\cdots. \tag{32}$$

Next, observe that, when $n \le 1000$, $X/n \in (1500, 10^{10})$ and when $40000 < n \le \sqrt{X}$, $X/n \in [\sqrt{X}, X/40000] \subset [24200, 3\cdot 10^7]$. Thus, by using estimations (10), (17) and also Lemma 5.1, we have the following estimation

$$|R_4(X)| \le 0.8\sqrt{X}\sum_{n\le 1000}\frac{\Lambda(n)}{\sqrt{n}} + 0.000154\cdot X + 0.71\sqrt{X}\sum_{40000<n\le\sqrt{X}}\frac{\Lambda(n)}{\sqrt{n}},$$

which, by recalling (32), Lemma 2.3 and using summation by parts, may be expressed as

$$\begin{aligned}
|R_4(X)| \le\ & 48.408\sqrt{X} + 0.000154\cdot X \\
&+ 0.71\sqrt{X}\left(\int_{40000}^{\sqrt{X}} \frac{\psi(t) - B}{2t^{3/2}}dt + \frac{\psi(\sqrt{X}) - B}{X^{1/4}}\right) \\
\le\ & 48.408\sqrt{X} + 0.000154\cdot X \\
&+ 0.71\sqrt{X}\left(\int_{40000}^{\sqrt{X}} \frac{1.04\cdot t}{2t^{3/2}}dt - \frac{B}{\sqrt{40000}} + \frac{1.04\sqrt{X}}{X^{1/4}}\right) \\
\le\ & 48.408\sqrt{X} + 0.000154\cdot X \\
&+ 0.71\sqrt{X}\left(1.04\cdot X^{1/4} - 1.04\cdot\sqrt{40000} - \frac{40012.89}{\sqrt{40000}} + \frac{1.04\sqrt{X}}{X^{1/4}}\right).
\end{aligned}$$

Therefore,

$$\frac{|R_4(X)|}{X} \leq \frac{48.408}{\sqrt{X}} + 0.000154 + 0.71\left(\frac{2.08}{X^{1/4}} - \frac{408}{X^{1/2}}\right),$$

which is not more than 0.0032. Here, we have used that the function $t \mapsto at^{-1/4} - bt^{-1/2}$ has a maximum at $t = (2b/a)^4$, taking value $a^2/4b$.

Case 2: Suppose that $9 \cdot 10^9 \leq X \leq 2 \cdot 10^{10}$. Let $X_1 = 2 \cdot 10^7$. Observe first that for any $U \geq \sqrt{X}$, we can write

$$|R_4(X)| \leq \sum_{2 \leq n \leq X/U} \Lambda(n)\left|R\left(\frac{X}{n}\right)\right| + \sum_{X/U < n \leq \sqrt{X}} \Lambda(n)\left|R\left(\frac{X}{n}\right)\right|. \tag{33}$$

Select now $U = X_1$. Thereupon, for the first sum above, we have $X/n \in [X_1, 10^{10}]$, thus we may use estimation (10). Concerning the above second sum, we have $X/n \in [\sqrt{X}, X_1] \subset [24200, 3 \cdot 10^7]$ so that we may use (17). This way, considering the definition (3) of $\psi(x)$, using summation by parts and then recalling Lemma 2.3, we have

$$\begin{aligned}
|R_4(X)| &\leq 0.8\sqrt{X} \sum_{n \leq X/X_1} \frac{\Lambda(n)}{\sqrt{n}} + 0.71\sqrt{X} \sum_{X/X_1 < n \leq \sqrt{X}} \frac{\Lambda(n)}{\sqrt{n}} \\
&= 0.71\sqrt{X}\left(\int_2^{\sqrt{X}} \frac{\psi(t)}{2t^{3/2}} dt + \frac{\psi(\sqrt{X})}{X^{1/4}}\right) \\
&\quad + 0.09\sqrt{X}\left(\int_2^{X/X_1} \frac{\psi(t)}{2t^{3/2}} dt + \frac{\psi(X/X_1)}{(X/X_1)^{1/2}}\right) \\
&\leq 2 \cdot 0.71 \cdot 1.04 X^{3/4} + 2 \cdot 0.09 \cdot 1.04 \frac{X}{\sqrt{X_1}},
\end{aligned}$$

which is not more than $0.0049 \cdot X$.

Case 3: Suppose that $2 \cdot 10^{10} \leq X \leq 2 \cdot 10^{19}$. Set $X_2 = 10^{10}$ and let us use expression (33) with $U = X_2$. For the arising first sum, we have $X/n \in [X_2, 10^{19}]$, so that we may use estimation (11). As for the second one, we have $X/n \in [\sqrt{X}, X_2] \subset [1500, 10^{10}]$ and we may use estimation (10). Thus, by using summation by parts and recalling Lemma 2.3, we derive

$$\begin{aligned}
|R_4(X)| &\leq 0.94X \sum_{n \leq X/X_2} \frac{\Lambda(n)}{\sqrt{n}} + 0.8\sqrt{X} \sum_{X/X_2 < n \leq \sqrt{X}} \frac{\Lambda(n)}{\sqrt{n}} \\
&= 0.8\sqrt{X}\left(\int_2^{\sqrt{X}} \frac{\psi(t)}{2t^{3/2}} dt + \frac{\psi(\sqrt{X})}{X^{1/4}}\right) \\
&\quad + 0.14\sqrt{X}\left(\int_2^{X/X_2} \frac{\psi(t)}{2t^{3/2}} dt + \frac{\psi(X/X_2)}{(X/X_2)^{1/2}}\right) \\
&\leq 2 \cdot 0.8 \cdot 1.04 \cdot X^{3/4} + 2 \cdot 0.14 \cdot 1.04 \frac{X}{\sqrt{X_2}},
\end{aligned}$$

which is not more than $0.0045 \cdot X$.

Case 4: Suppose that $X \geq 2\cdot 10^{19}$. Since $10^{19} > 1\,514\,928^2$, by estimation (15) and Lemma 2.4 with $T=\sqrt{X}$, we have

$$\frac{|R_4(X)|}{X} \leq 0.0065 \sum_{n\leq\sqrt{X}} \frac{\Lambda(n)}{n\log(X/n)} \leq 0.0065\cdot\left(0.73+\frac{1.04}{\log X}\right).$$

which implies that $|R_4(X)| \leq 0.0049\cdot X$.

Finally, on combining cases 1, 2, 3, 4 together, we derive the result. □

6. Bounding R_2^*

Recall the definition (19) of $R_2(X)$.

Lemma 6.1. *For any* $X>0$, *we have*

$$|R_2^*(X)| \leq 1+2\gamma+R_3(X)+2R_4(X),$$

where R_3 *and* R_4 *are respectively defined in* (23) *and* (24).

The reader should compare the above lemma with [16, Lemma 6.2]. The novelty here is that $|\sqrt{X}r(\sqrt{X})-R(\sqrt{X})|$ appears instead of $|r(\sqrt{X})|$; this latter quantity is well controlled by (18).

Proof. By summation by parts, and recalling the definition (2) of $R(X)$, we have

$$\begin{aligned}\sum_{n\leq X}\Lambda(n)\log n &= \psi(X)\log X - \int_1^X \frac{\psi(t)}{t}dt\\ &= X\log X - X + 1 + R(X)\log X - \int_1^X \frac{R(t)}{t}dt.\end{aligned}$$

On the other hand, the Dirichlet hyperbola formula yields

$$\begin{aligned}&\sum_{d_1d_2\leq X}\Lambda(d_1)\Lambda(d_2) = 2\sum_{d_1\leq\sqrt{X}}\Lambda(d_1)\psi\left(\frac{X}{d_1}\right)-\psi(\sqrt{X})^2\\ =&2X\sum_{d_1\leq\sqrt{X}}\frac{\Lambda(d_1)}{d_1} - X - 2\sqrt{X}R(\sqrt{X}) - R(\sqrt{X})^2 + 2\sum_{d_1\leq\sqrt{X}}\Lambda(d_1)R\left(\frac{X}{d_1}\right)\\ =&X\log X - 2X\gamma - X + 2Xr(\sqrt{X}) - 2\sqrt{X}R(\sqrt{X}) - R(\sqrt{X})^2 + 2R_4(X).\end{aligned}$$

Hence, by recalling the definition (19) of $R_2(X)$ and combining the above two expressions, we obtain

$$\begin{aligned}R_2^*(X) &= \sum_{n\leq X}(\Lambda\star\Lambda(n)-\Lambda(n)\log n)+2[X]\gamma = -1-2\{X\}\gamma+2R_4(X)\\ &+\left[2\sqrt{X}(\sqrt{X}r(\sqrt{X})-R(\sqrt{X}))-R(\sqrt{X})^2-R(X)\log X+\int_1^X\frac{R(t)}{t}dt\right],\end{aligned}$$

whence, by recalling the definition (23) of $R_3(X)$, we obtain the result. □

Let us recall [16, Lemma 6.3].

Lemma 6.2. *Let* X *be a real number such that* $3\leq X\leq 2.1\cdot 10^{10}$. *Then*

$$|R_2^*(X)| \leq 1.93\sqrt{X}\log X.$$

Furthermore, the study carried out in §4, §5 and §6 allows us to derive the following result.

Lemma 6.3. *When $X \geq 1.8 \cdot 10^9$, we have*
$$|R_2^*(X)| \leq 0.011 \cdot X.$$

Proof. Combine Lemma 6.1 together with Lemma 4.2 and Lemma 5.2. □

7. Proof of Theorem 1.1

We now proceed to the proof of Theorem 1.1. We start with two very specialized lemmas. The most important part of the proof lies in Lemma 7.3.

Lemma 7.1. *Let $X \geq 10^{16}$ and $K > 0$ such that $2\,160\,535 \cdot K \leq 10^{16}$. Then we have*
$$\frac{1}{X}\left|\sum_{k \leq K}(\Lambda \star \Lambda(k) - \Lambda(k)\log k + 2\gamma)M\left(\frac{X}{k}\right) - R_2^*(K)M\left(\frac{X}{K}\right)\right|$$
$$\leq \begin{cases} 0.0374 & \text{when } K = 462\,848, \\ 0.0422 & \text{when } K = 10^6, \\ 0.0579 & \text{when } K = 10^7, \\ 0.0762 & \text{when } K = 10^8. \end{cases}$$

Proof. In order to bound the above quantity, we have $X/k \geq X/K \geq 2\,160\,535$, so that estimation (8) may be applied. Thereupon, it is enough to compute numerically the following bounds
$$\sum_{k \leq K}\frac{|\Lambda \star \Lambda(k) - \Lambda(k)\log k + 2\gamma|}{k} + \frac{|R_2^*(K)|}{K} \leq 4345 \cdot \begin{cases} 0.0374, & \text{if } K = 462\,848, \\ 0.0422, & \text{if } K = 10^6, \\ 0.0579, & \text{if } K = 10^7, \\ 0.0762, & \text{if } K = 10^8. \end{cases}$$
□

Lemma 7.2. *Let X, K, K_0 be real numbers such that $0 < K < K_0 \leq X$. When $X/K_0 \geq 5 \cdot 10^6$ and $\log(K_0/K) \leq 19\,000$, we have*
$$\sum_{X/K_0 < n \leq X/K}\frac{\mu^2(n)}{\sqrt{n}} \leq \frac{12}{\pi^2}\sqrt{\frac{X}{K}}.$$

Proof. With the help of summation by parts, we find that
$$\sum_{X/K_0 < n \leq X/K}\frac{\mu^2(n)}{\sqrt{n}} = \int_{X/K_0}^{X/K}\sum_{X/K_0 < n \leq t}\mu^2(n)\frac{dt}{2t^{3/2}} + \sum_{X/K_0 < n \leq X/K}\frac{\mu^2(n)}{\sqrt{X/K}}$$
$$= \int_{X/K_0}^{X/K}\sum_{n \leq t}\mu^2(n)\frac{dt}{2t^{3/2}} + \sum_{n \leq X/K}\frac{\mu^2(n)}{\sqrt{X/K}} - \sum_{n \leq X/K_0}\frac{\mu^2(n)}{\sqrt{X/K_0}}$$
$$\leq \int_{X/K_0}^{X/K}\left(\frac{6}{\pi^2}t + 0.02767\sqrt{t}\right)\frac{dt}{2t^{3/2}} + \frac{6}{\pi^2}\sqrt{\frac{X}{K}} - \frac{6}{\pi^2}\sqrt{\frac{X}{K_0}} + 2 \cdot 0.02767$$
$$\leq \frac{12}{\pi^2}\sqrt{\frac{X}{K}} - \frac{12}{\pi^2}\sqrt{\frac{X}{K_0}} + \frac{0.02767}{2}\log\left(\frac{K_0}{K}\right) + 2 \cdot 0.02767,$$

where, since $X/K_0 \geq 5\cdot 10^6 > 438\,653$, we have used Lemma 2.1. The result is concluded by noticing that the total contribution of the above second, third and fourth terms is negative. □

Lemma 7.3. *For $X \geq 4\cdot 10^7$, we have*

$$\Bigl|\sum_{n\leq X}\mu(n)\log^2 n\Bigr| \leq (0.006688\,\log X - 0.0504)X.$$

In the above range, Lemma 7.3 improves on [16, Lemma 7.2] by almost a factor of 2.

Proof. We consider different cases.

Case 1: Suppose that $X\geq 10^{16}$. Let $K_0 = 2\cdot 10^9$ and $K = 10^8$. We deduce from (22) that

(34)

$$\sum_{n\leq X}\mu(n)\log^2 n = -2\gamma + \sum_{\ell\leq X/K_0}\mu(\ell)R_2^*\Bigl(\frac{X}{\ell}\Bigr) + \sum_{X/K_0<\ell\leq X/K}\mu(\ell)R_2^*\Bigl(\frac{X}{\ell}\Bigr) + \sum_{k\leq K}(\Lambda\star\Lambda(k) - \Lambda(k)\log k + 2\gamma)M\Bigl(\frac{X}{k}\Bigr) - R_2^*(K)M\Bigl(\frac{X}{K}\Bigr).$$

for any $K\leq K_0$.

Now, by Lemma 6.2, we have $|R_2^*(k)|\leq 1.93\sqrt{k}\log k$ when $3\leq k\leq K_0$. Moreover, as $X/K_0\geq 5\cdot 10^6$ and $\log(K_0/K) < 19\,000$, we can estimate the third term in (34) with the help of Lemma 7.2 as

$$\Bigl|\sum_{X/K_0<\ell\leq X/K}\mu(\ell)R_2^*\Bigl(\frac{X}{\ell}\Bigr)\Bigr| \leq 1.93\sqrt{X}\log(K_0)\sum_{X/K_0<\ell\leq X/K}\frac{\mu^2(\ell)}{\sqrt{\ell}}$$
$$\leq 1.93\sqrt{X}\log(K_0)\frac{12}{\pi^2}\sqrt{\frac{X}{K}} \leq \frac{2.35}{\sqrt{K}}X\log K_0.$$

Furthermore, by combining Lemma 6.3 and Lemma 2.2, we have

$$\Bigl|\sum_{\ell\leq X/K_0}\mu(\ell)R_2^*\Bigl(\frac{X}{\ell}\Bigr)\Bigr| \leq 0.011\cdot X\sum_{\ell\leq X/K_0}\frac{\mu^2(\ell)}{\ell}$$
$$\leq 0.011\cdot X\Bigl(\frac{6}{\pi^2}\log\Bigl(\frac{X}{K_0}\Bigr) + 1.044\Bigr),$$

since $X/K_0\geq 10^6$.

Finally, by Lemma 7.1, we may bound the last two terms of (34). All in all, we have

$$\frac{1}{X}\Bigl|\sum_{n\leq X}\mu(n)\log^2 n\Bigr| \leq \frac{2\gamma}{X} + 0.011\Bigl(\frac{6}{\pi^2}\log\Bigl(\frac{X}{K_0}\Bigr) + 1.044\Bigr) + \frac{2.35}{\sqrt{K}}\log(K_0) + 0.0762$$
$$\leq 0.006688\cdot\log X - 0.0504$$

Case 2: Suppose that $X\in[4\cdot 10^7, 10^{16})$. By summation by parts, we write

$$\sum_{n\le X}\mu(n)\log^2 n = M(X)\log^2 X - \int_1^X M(t)\frac{2\log t}{t}dt,$$

so that estimation (5) gives

$$\begin{aligned}\left|\sum_{n\le X}\mu(n)\log^2 n\right| &\le \sqrt{X}\log^2 X + \int_1^X \frac{2\log t}{\sqrt{t}}dt\\ &= \sqrt{X}\log^2 X + 4\sqrt{X}\log X - 8\sqrt{X} + 8\\ &\le (\log X + 4)\sqrt{X}\log X.\end{aligned}$$

We readily check that $0.006688\log X - 0.0504 \ge (\log X)(\log X + 4)/\sqrt{X}$ when $X \ge 4\cdot 10^7$. □

Proof of Theorem 1.1. We consider different cases.

Case 1: Suppose that $X \ge X_1 = 10^{16}$. We use Lemma 3.1 with $f = \mu$ and $k = 2$ and $X_0 = 4\cdot 10^7$. Thus, $M(\cdot) = M_0(\mu,\cdot)$ and, by Lemma 7.3, we derive

$$\begin{aligned}|M(X)| \le \frac{0.006688\ \log X - 0.0504}{\log^2 X}X &+ \left|M(X_0) - \frac{M_2(\mu,X_0)}{\log^2 X_0}\right|\\ &+ 2\int_{X_0}^X \frac{0.006688\log t - 0.0504}{\log^3 t}dt.\end{aligned}\tag{35}$$

A computer calculation may handle the above second term. Subsequently, by the use Pari/Gp, we obtain

$$\left|M(X_0) - \frac{M_2(\mu,X_0)}{\log^2 X_0}\right| \le 7.01.\tag{36}$$

Hence, on combining (35) and (36), we derive

$$\begin{aligned}|M(X)| \le& \frac{0.006688\ \log X - 0.0504}{\log^2 X}X + 7.01 + \int_{X_0}^X\left(\frac{0.013376}{\log^2 t} - \frac{0.1008}{\log^3 t}\right)dt\\ \le& \frac{0.006688\ \log X - 0.0504}{\log^2 X}X + 7.01\\ &+ 0.013376\left(\frac{X}{\log^2 X} - \frac{X_0}{\log^2 X_0}\right) - (0.1008 - 0.026752)\int_{X_0}^X \frac{dt}{\log^3 t}\end{aligned}\tag{37}$$

where we have used the identity $(\mathrm{Id}/\log^2)' = 1/\log^2\cdot(1 - 2/\log)$. Further, the bound

$$\frac{X}{\log^3 X} - \frac{X_0}{\log^3 X_0} = \int_{X_0}^X\left(\frac{t}{\log^3 t}\right)' dt \le \int_{X_0}^X \frac{dt}{\log^3 t}$$

leads to a simplification on (35) as

$$|M(X)| \le \frac{0.006688 \log X - 0.0504}{\log^2 X} X + \left(7.01 - \frac{0.013376 X_0}{\log^2 X_0} + \frac{0.074048 X_0}{\log^3 X_0}\right) + \frac{0.013376 X}{\log^2 X} - \frac{0.074048 X}{\log^3 X}$$

$$\le \frac{0.006688}{\log X} X + \frac{X}{\log^2 X}\left(-0.0504 + 0.013376 - \frac{0.074048}{\log X_1} - \frac{1186.93 \log^2 X_1}{X_1}\right)$$

$$\le \frac{0.006688 \log X - 0.039}{\log^2 X} X. \tag{38}$$

Case 2: Suppose that $X \in [X_2, 10^{16}]$, where $X_2 = 1.5 \cdot 10^7$. Then

$$\frac{1}{\sqrt{X}} \frac{0.006688 \log X - 0.039}{\log^2 X} X \ge \left(0.006688 - \frac{0.039}{\log(X_2)}\right) \frac{\sqrt{X}}{\log X}$$
$$\ge \left(0.006688 - \frac{0.039}{\log X_2}\right) \frac{\sqrt{X_2}}{\log X_2} \ge 1.$$

Therefore, by using (5), the bound (38) is valid in the range $[1.5 \cdot 10^7, 10^{16}]$.
Case 3: Suppose that $X \le 1.5 \cdot 10^7$. We conclude by computer verification by relying on Pari/Gp that the bound (38) holds in the range $[T, 1.5 \cdot 10^7)$, where $T = 1\,798\,118$. □

8. Corollaries

Proof of Corollary 1.2. We consider three cases.
Case 1: Suppose that $X \ge 10^{14}$. Let $T = 1\,798\,118$ and $X_0 = 10^{14}$. By a numerical calculation, we obtain

$$\int_1^T |M(t)| dt \le 216378740 \tag{39}$$

Let $A = 216378740$. Now, by Theorem 1.1, Lemma 3.2 and (39), we derive

$$|m(X)| \le \frac{0.006688 \log X - 0.039}{\log^2 X} + \frac{1}{X^2}\int_T^X \frac{(0.006688 \log t - 0.039)t}{\log^2 t} dt + \frac{1}{X^2}\int_1^T |M(t)| dt + \frac{8}{3X},$$
$$\le \frac{0.006688 \log X - 0.039}{\log^2 X} + \frac{1}{X^2}\int_T^X \frac{0.006688\, t dt}{\log t} - \frac{1}{X^2}\int_T^X \frac{0.039\, t dt}{\log^2 t} + \frac{A}{X^2} + \frac{8}{3X}.$$

Moreover, by using $(\mathrm{Id}^2/\log)' = \mathrm{Id}/\log \cdot (2 - 1/\log)$ and then the bound

$$\frac{X^2}{2\log^2 X} - \frac{T^2}{2\log^2 T} = \frac{1}{2}\int_T^X \left(\frac{t^2}{\log^2 t}\right)' dt \le \int_T^X \frac{t dt}{\log^2 t},$$

we derive

$$\begin{aligned}|m(X)| \leq & \frac{0.010032 \log X - 0.039}{\log^2 X} - \frac{0.003344}{X^2}\frac{T^2}{\log T} \\ & - \frac{1}{X^2}\int_T^X \frac{0.035656\, t dt}{\log^2 t} + \frac{A}{X^2} + \frac{8}{3X} \\ \leq & \frac{0.010032 \log X - 0.039}{\log^2 X} - \frac{0.003344}{X^2}\frac{T^2}{\log T} \\ & - \frac{0.017828}{X^2}\left(\frac{X^2}{\log^2 X} - \frac{T^2}{\log^2 T}\right) + \frac{A}{X^2} + \frac{8}{3X}.\end{aligned}$$

Now, by rearranging terms, we obtain

$$|m(X)| \leq \frac{0.010032}{\log X} + \frac{1}{\log^2 X}\left(- 0.039 - 0.017828 + \frac{A \log^2 X_0}{X_0^2} + \frac{8 \log^2 X_0}{3X_0}\right)$$

$$- \frac{1}{X^2}\left(0.003344\frac{T^2}{\log T} - 0.017828\frac{T^2}{\log^2 T}\right) \tag{40}$$

$$\leq \frac{0.010032 \log X - 0.0568}{\log^2 X}, \tag{41}$$

where we have used that the expression (40) in the above estimation is negative.

Case 2: Suppose that $X \in [X_1, 10^{14})$, *where* $X_1 = 1.5\cdot 10^7$. By (9), we extend the simplified bound $|m(X)| \log^2 X \leq 0.01 \log X - 0.057$ to any $X \geq X_1$.

Case 3: Suppose that $X < 1.5 \cdot 10^7$. We verify numerically that the bound (41) is valid for any $X \geq 617\,990$, whence the result. □

Proof of Corollary 1.3. Recall Corollary 1.2. We note that $0.01004 \log X - 0.056 \leq 0.0144 \log X - 0.1$ when $X \geq 617\,990$. Then, we inspect numerically that the result also holds for $X \in [463\,421, 617\,990)$. □

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