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ACCURATE COMPUTATIONS OF EULER PRODUCTS OVER PRIMES IN ARITHMETIC PROGRESSIONS

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Abstract: This note provides accurate truncated formulae with explicit error terms to compute Euler products over primes in arithmetic progressions of rational fractions. It further provides such a formula for the product of terms of the shape $F(1/p, 1/p^s)$ when F is a two-variable polynomial with coefficients in $\mathbb{C}$ and satisfying some restrictive conditions.

Keywords: Euler products.

1. Introduction and results

Our primary concern in this paper is to evaluate Euler products of the shape

$$\prod_{p\equiv a[q]}\left(1-\frac{1}{p^s}\right)$$

when s is a *complex* parameter satisfying $\Re s > 1$. Such computations have attracted some attention as these values occur when s is a real number as densities in number theory. D. Shanks in [11] (resp. [12], resp. [13]) has already computed accurately an Euler product over primes congruent to 1 modulo 8 (resp. to 1 modulo 4, resp. 1 modulo 8). His method has been extended by S. Ettahri, L. Surel and the present author in [6] in an algorithm that converges very fast (double exponential convergence) but this extension covers only some special values for the residue class a, or some special bundle of them; it is further limited to real values of s.

We will use logarithms, and since the logarithm of a product is not a priori the sum of the logarithms, we need to clarify things before embarking in this project. First, in this paper the log-function always corresponds to what is called *the principal branch of the logarithm.* We recognize it because its argument vanishes when we restrict it to the real line and we consider it undefined on the non-positive real numbers. The second point is contained in the next elementary proposition.

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Proposition 1. *Associate to each prime p a complex number a_p such that $|a_p| < p$ and $a_p \ll_\varepsilon p^\varepsilon$ for every $\varepsilon > 0$. We consider the Euler product defined when $\Re s > 1$ by:*

$$D(s) = \prod_{p\geqslant 2}\left(1 - \frac{a_p}{p^s}\right)^{-1}. \tag{1}$$

In this same domain we have

$$\log D(s) = \sum_{p\geqslant 2}\sum_{k\geqslant 1}\frac{a_p^k}{kp^{ks}}. \tag{2}$$

This is simply because, by using the expansion of the principal branch of the logarithm in Taylor series, namely $-\log(1-z) = \sum_{k\geqslant 1} z^k/k$ valid for any complex z inside the unit circle, we find that $C(s) = -\sum_{p\geqslant 2}\log(1 - a_p/p^s)$ verifies $\exp C(s) = D(s)$, so that $C(s)$ is indeed a candidate for $\log D(s)$. The second remark is that $D(s)$ approaches 1 when $\Re s$ goes to infinity while our choice for $\log D(s)$ indeed approaches 0 and no other multiple of $2i\pi$. These two remarks are enough justification of this proposition.

Remark 1.1. To be axiomatically correct, we should specify that our definition of $\log D(s)$ depends a priori on the chosen product representation, and thus on the choice of the coefficients $(a_p)_{p\geqslant 2}$. However, since the development in Dirichlet series is unique, we find that the coefficients in (2) are uniquely defined; this implies in particular that our definition does nor depend on the chosen product representation (as it is unique!).

We assume here that the values of the Dirichlet L-series $L(s,\chi)$ may be computed with arbitrary precision when $\Re s > 1$. Our aim is thus to reduce our computations to these ones. Here is an identity to do so.

Theorem 2. *Let a be prime to the modulus $q \geqslant 1$ and let $\widehat{G}_q$ be the group of Dirichlet characters modulo q. We have*

$$-\sum_{\substack{p\equiv a[q],\\ p\geqslant P}}\log(1 - 1/p^s) = \sum_{\ell\geqslant 1}\frac{-1}{\ell\varphi(q)}\sum_{d|\ell}\mu(d)\sum_{\chi\in\widehat{G}_q}\overline{\chi}(a)\log L_P(\ell s, \chi^d)$$

where

$$L_P(s,\chi) = \prod_{p\geqslant P}(1 - \chi(p)/p^s)^{-1}. \tag{3}$$

If finding this identity has not been immediate, checking it is only a matter of calculations that we reproduce in Section 2. A partial identity of this sort has already been used by K. Williams in [14] and more recently by A. Languasco and A. Zaccagnini in [9, 7], and [8, (2-5)] is a related formula. It is worth noticing that, with our conventions, we have the obvious

$$\log L_P(s,\chi) = \log L(s,\chi) - \sum_{p<P}\log(1 - \chi(p)/p^s).$$

This leads to the next immediate corollary.

Corollary 3. *Let a be prime to the modulus $q \geqslant 1$ and let $\widehat{G}_q$ be the group of Dirichlet characters modulo q. Let further two integer parameters $P \geqslant 2$ and $L \geqslant 2$ be chosen. We have*

$$\prod_{\substack{p\geqslant P,\\ p\equiv a[q]}} \left(1 - \frac{1}{p^s}\right) = \exp\biggl(Y_P(s;q,a|L) + \mathcal{O}^*\left(\frac{1}{P^{L\Re s}}\right)\biggr)$$

where

$$Y_P(s;q,a|L) = \sum_{\ell\leqslant L} \frac{1}{\ell} \sum_{d|\ell} \mu(d) \sum_{\chi\in\widehat{G}_q} \frac{\overline{\chi}(a)}{\varphi(q)} \log L_P(\ell s, \chi^d) \tag{4}$$

and where $f = \mathcal{O}^*(g)$ means $|f| \leqslant g$.

Extension to one variable rational fractions

Once we have such an approximation, we can reuse the machinery of [6] to reach Euler products of the shape

$$\prod_{\substack{p\geqslant P,\\ p\equiv a[q]}} (1 + R(p^s))$$

where R is a rational fraction.

Theorem 4. *Let F and G be two polynomials of $\mathbb{C}[t]$. We assume that $G(0) = 1$ and that $F(0) = F'(0) = 0$. Let $\beta \geqslant 2$ be larger than the inverse of the roots of G and of $G - F$. Let $P \geqslant 2\beta$ be an integer parameter. Then, for any integer parameter $L \geqslant 2$, we have*

$$\prod_{\substack{p\geqslant P,\\ p\equiv a[q]}} \left(1 - \frac{F(1/p)}{G(1/p)}\right) = \exp\biggl(\sum_{2\leqslant j\leqslant J} \bigl(b_{G-F}(j) - b_G(j)\bigr) Y_P(j;q,a|L) + I\biggr)$$

where the integers $b_{G-F}(j)$ and $b_G(j)$ are defined in Lemma 6,

$$|I| \leqslant 8\max(\deg(G-F), \deg G)\beta^2(\beta/P)^{2L}$$

and $Y(s;q,a|L)$ is defined by (4).

We obtained in [6] an approximation that is much better but only valid for rational fractions with real coefficients and some residue classes.

One can write a similar theorem for the Euler product

$$\prod_{\substack{p\geqslant P,\\ p\in\mathcal{A}}} \left(1 - \frac{F(1/p^s)}{G(1/p^s)}\right).$$

Extension to two variables rational fractions

The general form of Euler products that one has to treat in practice is of the shape

$$\prod_{\substack{p\geqslant P,\\ p\equiv a[q]}}(1+R(p,p^s))$$

where R is a rational fraction of two variables. When s takes a specific rational value, typically 2, 3/2 or 4/3, this question reduces to the above one though each of the values of s requires a new rational fraction; this covers most of the cases when we have to compute a single special constant. In the general case however, for instance when $s=2+i$, such a trick fails. The theoretical understanding of this situation is also limited even for $q=1$. For instance, if the case of a rational fraction of one variable is covered by the theorem of T. Esterman in [5] and extended by G. Dalhquist in [1], no such result is known in the general situation. This question has been addressed in the context of enumerative algebra, for instance by M. du Sautoy and F. Grünewald in [4]. The lecture notes [3] by M. du Sautoy and L. Woodward contain material in this direction. There are several continuations of Esterman's work; for instance, one may consider Euler products of the shape $R(p^{s_1},p^{s_2})$ (with the hope of being able to specify s_1), see for instance [2] by L. Delabarre, but these results do not apply to our case.

We are able to handle some rational fractions by reducing them to the case treated in the next theorem.

Theorem 5. *Let s be a complex number with $\Re s=\sigma>1$. Let $(a_\ell)_{\ell\leqslant k}$ be a sequence of complex numbers and $(u_\ell)_{\ell\leqslant k}$ and $(v_\ell)_{\ell\leqslant k}$ be two sequences of real numbers. We assume that $u_\ell\sigma+v_\ell>0$ and we define $A=\max(1,\max(|a_\ell|))$. Let q be a modulus, a be an invertible residue class modulo q and $P\geqslant 2kA$ and $L\geqslant k$ be two integer parameters. We have*

$$\prod_{\substack{p\geqslant P,\\ p\equiv a[q]}}\biggl(1-\sum_{1\leqslant \ell\leqslant k}\frac{a_\ell}{p^{u_\ell s+v_\ell}}\biggr)=\exp-(Z+I)$$

where

$$Z=\sum_{\substack{m_1,\ldots,m_k\geqslant 0,\\ 1\leqslant m_1+\ldots+m_k\leqslant L}}M(m_1,\ldots,m_k)\sum_{f\leqslant F}\frac{\kappa_f(\prod_{\ell\leqslant k}a_\ell^{m_\ell})}{f}Y_P\Bigl(\sum_{\ell\leqslant k}m_\ell(u_\ell s+v_\ell);q,a|L\Bigr)\quad(5)$$

where $M(m_1,m_2,\ldots,m_k)$ is defined at (18), κ_f is defined at (23), $Y(s;q,a|L)$ is defined by (4) and finally where

$$|I|\leqslant\frac{2^k\cdot A^L}{k!P^L}\biggl((L+k)^k+1+\log L+\frac{3kA}{L}\biggr).\quad(6)$$

Hence this theorem provides us with an exponentially decreasing error term. More complicated terms may be handled through this theorem by writing

$$\begin{aligned} 1+\frac{F(p,p^s)}{G(p,p^s)} &= \frac{(F+G)(p,p^s)}{p^{As+B}}\frac{p^{As+B}}{G(p,p^s)} \\ &= \left(1+\frac{(F+G)(p,p^s)-p^{As+B}}{p^{As+B}}\right)\left(1+\frac{G(p,p^s)-p^{As+B}}{p^{As+B}}\right)^{-1}. \end{aligned}$$

This would function when G has a clearly dominant monomial. It typically works for $G(p,p^s)=p^{2s}(p^2+1)$ but fails for $G(p,p^s)=p^{2s}(p+1)$. Our most important additional tool, namely Lemma 11, may be used to obtain results on analytic continuation, but since we use logarithms elsewhere, the general effect is unclear. We however provide the next example:

$$D(s)=\prod_{p\geqslant 2}\left(1+\frac{1}{p^s}-\frac{1}{p^{2s-1}}\right). \tag{7}$$

Lemma 11 gives us the decomposition

$$D(s)=\prod_{\substack{m_1,m_2\geqslant 0,\\ m_1+m_2\geqslant 1}}\prod_{p\geqslant 2}\left(1-\frac{(-1)^{m_1}}{p^{(m_1+2m_2)s-m_2}}\right)^{M(m_1,m_2)}.$$

We check that $M(1,0)=M(0,1)=1$ and that $M(m,0)=M(0,m)=0$ when $m\geqslant 2$, whence

$$D(s)=\zeta(2s-1)\frac{\zeta(2s)}{\zeta(s)}\prod_{m_1,m_2\geqslant 1}\prod_{p\geqslant 2}\left(1-\frac{(-1)^{m_1}}{p^{(m_1+2m_2)s-m_2}}\right)^{M(m_1,m_2)}. \tag{8}$$

This writing offers an analytic continuation of $D(s)$ to the domain defined by $\Re s>1/2$. This analysis can be extended to

$$\prod_{p\geqslant 2}\left(1-\frac{C_1}{p^s}-\frac{C_2}{p^{2s-1}}\right)$$

when C_1 and C_2 are *integers*. In general, Lemma 11 transfers to problem to the analytic continuation of $\prod_p(1-c/p^s)$ for some c but even the case $c=\sqrt{2}$ is difficult.

2. Proof of Theorem 2 and its corollary

Proof of Theorem 2. We have to simplify the expression

$$S=\sum_{\ell\geqslant 1}\frac{1}{\ell\varphi(q)}\sum_{d|\ell}\mu(d)\sum_{\chi\in\widehat{G}_q}\overline{\chi}(a)\sum_{p\geqslant P}\sum_{k\geqslant 1}\frac{\chi(p)^{dk}}{kp^{k\ell s}}. \tag{9}$$

We readily check that, when $h \geqslant 1$ and p are fixed, we have

$$\begin{aligned}\sum_{k\ell=h}\sum_{d|\ell}\mu(d)\sum_{\chi\in\widehat{G}_q}\overline{\chi}(a)\chi(p)^{dk} &= \sum_{k|h}\sum_{dk|h}\mu(d)\sum_{\chi\in\widehat{G}_q}\overline{\chi}(a)\chi(p)^{dk}\\ &= \sum_{g|h}\sum_{d|g}\mu(d)\sum_{\chi\in\widehat{G}_q}\overline{\chi}(a)\chi(p)^{g}\\ &= \sum_{\chi\in\widehat{G}_q}\overline{\chi}(a)\chi(p) = \varphi(q)1_{p\equiv a[q]}\end{aligned}$$

and the theorem follows directly. ■

Proof of Corollary 3. A moment thought discloses that

$$|\log L_P(s,\chi)| \leqslant \log\zeta_P(\sigma)$$

where $\sigma = \Re s$. We have furthermore

$$\log\zeta_P(\sigma) \leqslant \sum_{n\geqslant P}\frac{1}{n^\sigma} \leqslant \int_P^\infty \frac{dt}{t^\sigma} = \frac{1}{(\sigma-1)P^{\sigma-1}}.$$

by our assumptions. We next check that

$$\left|\sum_{\ell>L}\frac{1}{\ell\varphi(q)}\sum_{d|\ell}\mu(d)\sum_{\chi\in\widehat{G}_q}\overline{\chi}(a)\log L_P(\ell s,\chi^d)\right| \leqslant \sum_{\ell>L}\frac{2^{\omega(\ell)}}{\ell}\frac{P}{(\ell\sigma-1)P^{\ell\sigma}}.$$

Here $\omega(\ell)$ denotes the number of prime factors of ℓ (without multiplicity). We use the simplistic bounds $2^{\omega(\ell)} \leqslant \ell$ and $\ell\sigma - 1 \geqslant 2$. This yields the upper bound $\frac{P}{2P^{L\sigma}(P^\sigma-1)}$ which is no more than $1/P^{L\sigma}$. We finally recall that $e^x - 1 \leqslant \frac{8}{7}x$ when $x \in [0, 1/4]$ as the function $(e^x-1)/x$ is non-decreasing (its expansion in power series has non-negative coefficients). ■

3. Proof of Theorem 4

We first need to extend [6, Lemma 16] to cover the case of polynomials with complex coefficients. The ancestor of this Lemma is [10, Lemma 1].

Lemma 6. *Let $H(t) = 1 + a_1 t + \ldots + a_\delta t^\delta \in \mathbb{C}[t]$ be a polynomial of degree δ. Let $\alpha_1, \ldots, \alpha_\delta$ be the inverses of its roots. Put $s_H(k) = \alpha_1^k + \ldots + \alpha_\delta^k$. The $s_H(k)$ satisfy the Newton-Girard recursion*

$$s_F(k) + a_1 s_F(k-1) + \ldots + a_{k-1}s_F(1) + ka_k = 0, \tag{10}$$

where we have defined $a_{\delta+1} = a_{\delta+2} = \ldots = 0$. We define

$$b_H(k) = \frac{1}{k}\sum_{d|k}\mu(k/d)s_H(d). \tag{11}$$

Lemma 7. *Let F and G be two polynomials of $\mathbb{C}[t]$. We assume that $G(0) = 1$ and that $F(0) = 0$. Let $\beta \geqslant 1$ be larger than the inverse of the roots of G and of $G - F$. When z is a complex number such that $|z| < \beta$ and $|1 - (F/G)(z)| < 1$. We have*

$$\log\left(1 - \frac{F(z)}{G(z)}\right) = \sum_{j\geqslant 1}\bigl(b_{G-F}(j) - b_G(j)\bigr)\log(1 - z^j). \tag{12}$$

Proof. We adapt the proof of [10, Lemma 1]. We write $(G-F)(t) = \prod_i(1-\alpha_i t)$. We have

$$\frac{(G-F)'(t)}{(G-F)(t)} = \sum_i \frac{\alpha_i t}{1-\alpha_i t} = \sum_{k\geqslant 1} s_{G-F}(k)t^{k-1}.$$

This series is absolutely convergent in any disc $|t| \leqslant b < 1/\beta$ where $\beta = \max_j(1/|\alpha_j|)$. We may also decompose $(G-F)'(t)/(G-F)(t)$ in Lambert series as

$$\frac{(G-F)'(t)}{(G-F)(t)} = \sum_{j\geqslant 1} b_{G-F}(j)\frac{jt^{j-1}}{1-t^j}$$

as some series shuffling in any disc of radius $b < \min(1, 1/\beta)$ shows. The comparison of the coefficients justifies the formula (11). We may do the same for G instead of $G - F$ (or use the case $F = 0$). We find that

$$\frac{G'-F'}{G-F} - \frac{G'}{G} = \frac{-(F'G-FG')}{G(G-F)} = \frac{-(F'G-FG')}{G^2}\sum_{k\geqslant 0}\left(\frac{F}{G}\right)^k$$

and by formal integration, this gives us the identity

$$-\sum_{k\geqslant 1}\frac{(F/G)(t)^k}{k} = -\sum_{j\geqslant 1}\bigl(b_{G-F}(j) - b_G(j)\bigr)\log(1-t^j).$$

This readily extends into an equality between analytic function in the domain where $|(F/G)(z) - 1| < 1$ and $|z| < \beta$. The lemma follows readily. ∎

Here is now [6, Lemma 17], though for polynomials with complex coefficients.

Lemma 8. *We use the hypotheses and notation of Lemma 6. Let $\beta \geqslant 2$ be larger than the inverse of the modulus of all the roots of $H(t)$. We have*

$$|b_H(k)| \leqslant 2\deg H \cdot \beta^k/k.$$

And we finally recall [6, Lemma 18] that yields easy upper estimates for the inverse of the modulus of all the roots of $F(t)$ in terms of its coefficients.

Lemma 9. *Let $H(X) = 1 + a_1X + \ldots + a_\delta X^\delta$ be a polynomial of degree δ. Let ρ be one of its roots. Then either $|\rho| \geqslant 1$ or $1/|\rho| \leqslant |a_1| + |a_2| + \ldots + |a_\delta|$.*

Proof of Theorem 4. The proof requires several steps. We start from Lemma 7, i.e. from the identity

$$\log\left(1-\frac{F(z)}{G(z)}\right)=\sum_{j\geqslant 2}\bigl(b_{G-F}(j)-b_G(j)\bigr)\log(1-z^j), \tag{13}$$

in the domain $|z|<\beta$ and $|1-(F/G)(z)|<1$. The fact that the term with $j=1$ vanishes comes from our assumption that $F(0)=F'(0)=0$. To control the rate of convergence, we notice that, by Lemma 8, we know that $|b_{G-F}(j)-b_G(j)|\leqslant 4\max(\deg(G-F),\deg G)\beta^j/j$. Therefore, for any bound J, we have

$$\sum_{j\geqslant J+1}|t^j||b_{G-F}(j)-b_G(j)|\leqslant 4\max(\deg(G-F),\deg G)\frac{|t\beta|^{J+1}}{(1-|t\beta|)(J+1)}, \tag{14}$$

if $|t|<1/\beta$. Furthermore, we deduce that $|\log(1-z)/z|\leqslant \log(1-1/2)/(1/2)\leqslant 3/2$ when $|z|\leqslant 1/2$ by looking at the Taylor expansion. Thus we have

$$\log\left(1-\frac{F(z)}{G(z)}\right)=\sum_{2\leqslant j\leqslant J}\bigl(b_{G-F}(j)-b_G(j)\bigr)\log(1-z^j)+I_1 \tag{15}$$

where $|I_1|\leqslant 6\max(\deg(G-F),\deg G)|z\beta|^{J+1}/(1-|z\beta|)$. Now that we have the expansion (15) at our disposal for each prime p, we may combine them. We readily get

$$\sum_{\substack{p\geqslant P,\\ p\equiv a[q]}}\log\left(1-\frac{F(1/p)}{G(1/p)}\right)=\sum_{2\leqslant j\leqslant J}\bigl(b_{G-F}(j)-b_G(j)\bigr)\sum_{\substack{p\geqslant P,\\ p\equiv a[q]}}\log(1-1/p^j)+I_2,$$

where I_2 satisfies

$$\begin{aligned}|I_2|&\leqslant 6\max(\deg(G-F),\deg G)\sum_{p\geqslant P}\frac{\beta^{J+1}}{(1-\beta/P)(J+1)}\frac{1}{p^{J+1}}\\
&\leqslant\frac{6\max(\deg(G-F),\deg G)\beta^{J+1}}{(1-\beta/P)(J+1)}\left(\frac{1}{P^{J+1}}+\int_P^\infty\frac{dt}{t^{J+1}}\right)\\
&\leqslant\frac{6\max(\deg(G-F),\deg G)(\beta/P)^J\beta}{(1-\beta/P)(J+1)}\left(\frac{1}{P}+\frac{1}{J}\right),\end{aligned}$$

since $P\geqslant 2$ and $J\geqslant 3$. We now approximate each sum over p by using Corollary 3 and obtain

$$\sum_{\substack{p\geqslant P,\\ p\equiv a[q]}}\log\left(1-\frac{F(1/p)}{G(1/p)}\right)=\sum_{2\leqslant j\leqslant J}\bigl(b_{G-F}(j)-b_G(j)\bigr)Y_P(j;q,a|L)+I_3$$

where I_3 satisfies

$$\begin{aligned}|I_3| &\leqslant \sum_{2\leqslant j\leqslant J} |b_{G-F}(j) - b_G(j)| \frac{1}{P^{Lj}} + |I_2| \\ &\leqslant \sum_{2\leqslant j\leqslant J} 4\max(\deg F, \deg G) \frac{\beta^j}{j} \frac{1}{P^{Lj}} + |I_2|.\end{aligned}$$

Therefore (and since $r \geqslant 2$)

$$\frac{|I_3|}{2\max(\deg F, \deg G)} \leqslant \frac{\beta^2(\beta/P)^{2L}}{1-\beta/P} + \frac{3(\beta/P)^J\beta}{(1-\beta/P)(J+1)}\left(\frac{1}{P} + \frac{1}{J}\right), \quad (16)$$

and the choice $J = 2L$ ends the proof. ■

4. Proof of Theorem 5

Lemma 10. *We have* $\binom{dN'}{dm'_1,\cdots,dm'_k} \geqslant \binom{N'}{m'_1,\cdots,m'_k}^d$.

Proof. The coefficient $\binom{dN'}{dm'_1,\cdots,dm'_k}$ is the number of partitions of a set of dN' elements in parts of $dm'_1, \cdots, dm'_k$ elements. The product partitions are partitions. ■

In [15] Witt proved a generalization of the Necklace Identity which we present in the next lemma.

Lemma 11. *For $k \geqslant 1$, we have*

$$1 - \sum_{i=1}^{k} z_i = \prod_{\substack{m_1,\ldots,m_k\geqslant 0,\\ m_1+\ldots+m_k\geqslant 1}} (1 - z_1^{m_1}\cdots z_k^{m_k})^{M(m_1,\ldots,m_k)}, \quad (17)$$

where the integer $M(m_1,\ldots,m_k)$ is defined by

$$M(m_1,\ldots,m_k) = \frac{1}{N} \sum_{d|\gcd(m_1,m_2,\ldots,m_k)} \mu(d) \frac{(N/d)!}{(m_1/d)!\cdots(m_k/d)!} \quad (18)$$

with $N = m_1 + \ldots + m_k$. We have $M(m_1,\ldots,m_k) \leqslant k^N/N$.

Proof. Only the bound needs to be proved as the identity may be found in [15]. Each occuring multinomial is not more than $\binom{N}{m_1,\cdots,m_k}$ by Lemma 10. The multinomial Theorem concludes. ■

Proof of Theorem 5. Let Π be the product to be computed. By employing Lemma 11, we find that

$$1 - \sum_{1\leqslant \ell\leqslant k} \frac{a_\ell}{p^{u_\ell s+v_\ell}} = \prod_{\substack{m_1,\ldots,m_k\geqslant 0,\\ m_1+\ldots+m_k\geqslant 1}} \left(1 - \frac{c(m_1,m_2,\ldots,m_k)}{p^{\sum_{\ell\leqslant k} m_\ell(u_\ell s+v_\ell)}}\right)^{M(m_1,\ldots,m_k)},$$

with $c(m_1,\dots,m_k)$ given by

$$c(m_1,m_2,\dots,m_k)=\prod_{\ell\leqslant k}a_\ell^{m_\ell}. \tag{19}$$

Each coefficient $c(m_1,\dots,m_k)$ is not more, in absolute value, than A^N, where $m_1+\ldots+m_k=N$. Note that, for each ℓ, we have $\Re(u_\ell s+v_\ell)>1$, so that $\Re\sum_{\ell\leqslant k}m_\ell(u_\ell s+v_\ell)\geqslant m_1+\ldots+m_k=N$. It thus seems like a good idea to truncate the infinite product in (20) whether $m_1+\cdots+m_k=N\leqslant N_0$ or not for some parameter $N_0\geqslant k$ that we will choose later. We readily find that, when $p\geqslant 2A$,

$$\begin{aligned}\left|\log\prod_{\substack{m_1,\dots,m_k\geqslant 0,\\ m_1+\ldots+m_k>N_0}}\left(1-\frac{c(m_1,m_2,\dots,m_k)}{p^{\sum_{\ell\leqslant k}m_\ell(u_\ell s+v_\ell)}}\right)^{M(m_1,\dots,m_k)}\right|&\\ \leqslant\frac{3}{2}\sum_{\substack{m_1,\dots,m_k\geqslant 0,\\ m_1+\ldots+m_k>N_0}}M(m_1,\dots,m_k)\frac{A^N}{p^N}&\\ \leqslant\frac{3}{2}\sum_{N>N_0}\binom{N+k}{k}\frac{(kA)^N}{Np^N}&\end{aligned}$$

as the number of solutions to $m_1+\ldots+m_k=N$ is the N-th coefficient of the power series $1/(1-z)^k$ which happens to be equal to $(1/k!)\frac{d}{dz^k}1/(1-z)$. We next check that, with $N=N_0+n+1$, we have $(n+1+N_0+k)\leqslant(N_0+n+1)^2$ since $N_0\geqslant k$, and thus

$$\frac{\binom{N+k}{k}}{N\binom{n+k}{k}}=\frac{(n+1+N_0+k)(n+N_0+k)\cdots(n+N_0+2)}{(n+k)(n+k-1)\cdots(n+1)\cdot(N_0+n+1)}\leqslant\binom{N_0+k}{k}.$$

Hence, when $p\geqslant 2kA$, we have

$$\begin{aligned}\sum_{N>N_0}\binom{N+k}{k}\frac{(kA)^N}{Np^N}&=\frac{(kA)^{N_0+1}}{p^{N_0+1}}\binom{N_0+k}{k}\sum_{n\geqslant 0}\binom{n+k}{k}\frac{(kA)^n}{p^n}\\ &\leqslant\binom{N_0+k}{k}\frac{(kA)^{N_0+1}}{p^{N_0+1}}\frac{1}{(1-1/2)^k}.\end{aligned}$$

On summing over p, this yields

$$\Pi=I_1\prod_{\substack{m_1,\dots,m_k\geqslant 0,\\ 1\leqslant m_1+\ldots+m_k\leqslant N_0}}\prod_{\substack{p\geqslant P,\\ p\equiv a[q]}}\left(1-\frac{c(m_1,m_2,\dots,m_k)}{p^{\sum_{\ell\leqslant k}m_\ell(u_\ell s+v_\ell)}}\right)^{M(m_1,\dots,m_k)}, \tag{20}$$

where

$$|\log I_1|\leqslant 2^k\frac{3}{2}\binom{N_0+k}{k}\frac{(kA)^{N_0+1}}{P^{N_0}}\left(\frac{1}{P}+\frac{1}{N_0}\right). \tag{21}$$

We next note the following identity

$$\sum_{k\geqslant 1}\frac{d^k}{kp^{kw}}=\sum_{f\geqslant 1}\frac{\kappa_f(d)}{f}\sum_{g\geqslant 1}\frac{1}{gp^{fgw}} \tag{22}$$

where

$$\kappa_f(d)=\begin{cases} c & \text{when } f=1,\\ c^f-c^{f-1} & \text{when } f>1.\end{cases} \tag{23}$$

We truncate identity (22) at $f\leqslant F$ where F is an integer, getting

$$\sum_{k\geqslant 1}\frac{d^k}{kp^{kw}}=\sum_{f\leqslant F}\frac{\kappa_f(d)}{f}\sum_{g\geqslant 1}\frac{1}{gp^{fgw}}+\mathcal{O}^*\biggl(-\sum_{f>F}\frac{\max(1,|d|)^f}{f}\log\Bigl(1-p^{-f\Re w}\Bigr)\biggr).$$

We next use $-\log(1-x)\leqslant 3x/2$ when $0\leqslant x\leqslant 1/2$. We assume that $p^{\Re w}\leqslant 1/2$ and $p^{\Re w}\geqslant 2\max(1,|d|)$ to get

$$-\sum_{f>F}\frac{\max(1,|d|)^f}{f}\log\Bigl(1-p^{-f\Re w}\Bigr)\leqslant\frac{3}{2}\sum_{f>F}\frac{\max(1,|d|)^f}{fp^{f\Re w}}\leqslant\frac{3\max(1,|d|)^{F+1}}{(F+1)p^{(F+1)\Re w}}.$$

We have reached

$$\begin{aligned}&\prod_{\substack{p\geqslant P,\\ p\equiv a[q]}}\biggl(1-\frac{c(m_1,m_2,\ldots,m_k)}{p^{\sum_{\ell\leqslant k}m_\ell(u_\ell s+v_\ell)}}\biggr)\\ &\qquad=\exp-\biggl\{\sum_{f\leqslant F}\frac{\kappa_f(c(m_1,m_2,\ldots,m_k))}{f}\sum_{\substack{p\geqslant P,\\ p\equiv a[q]}}\log\Bigl(1-p^{-f\sum_{\ell\leqslant k}m_\ell(u_\ell s+v_\ell)}\Bigr)\\ &\qquad\quad+\mathcal{O}^*\biggl(\frac{3\max(1,|c(m_1,m_2,\ldots,m_k)|)^{F+1}}{(F+1)P^{(F+1)\sum_{\ell\leqslant k}m_\ell(u_\ell\sigma+v_\ell)}}\biggl(1+\frac{P}{F\sum_{\ell\leqslant k}m_\ell(u_\ell\sigma+v_\ell)}\biggr)\biggr)\biggr\}\end{aligned}$$

which simplifies info

$$\begin{aligned}&\prod_{\substack{p\geqslant P,\\ p\equiv a[q]}}\biggl(1-\frac{c(m_1,m_2,\ldots,m_k)}{p^{\sum_{\ell\leqslant k}m_\ell(u_\ell s+v_\ell)}}\biggr)\\ &\qquad=\exp-\biggl\{\sum_{f\leqslant F}\frac{\kappa_f(c(m_1,m_2,\ldots,m_k))}{f}\sum_{\substack{p\geqslant P,\\ p\equiv a[q]}}\log\Bigl(1-p^{-f\sum_{\ell\leqslant k}m_\ell(u_\ell s+v_\ell)}\Bigr)\\ &\qquad\quad+\frac{3A^{N(F+1)}}{(F+1)P^{(F+1)N}}\biggl(1+\frac{P}{FN}\biggr)\biggr\}.\end{aligned}$$

We approximate the sum of the logs by Corollary 3 and get

$$\prod_{\substack{p\geqslant P,\\ p\equiv a[q]}}\biggl(1-\frac{c(m_1,m_2,\ldots,m_k)}{p^{\sum_{\ell\leqslant k}m_\ell(u_\ell s+v_\ell)}}\biggr)$$
$$=\exp-\biggl\{\sum_{f\leqslant F}\frac{\kappa_f(c(m_1,m_2,\ldots,m_k))}{f}Y_P\Bigl(\sum_{\ell\leqslant k}m_\ell(u_\ell s+v_\ell);q,a|L\Bigr)$$
$$+\mathcal{O}^*\biggl(\frac{A^{NF}(1+\log F)}{P^{LN}}+\frac{3A^{N(F+1)}}{(F+1)P^{(F+1)N}}\Bigl(1+\frac{P}{FN}\Bigr)\biggr)\biggr\}.$$

We then raise that to the power $M(m_1,m_2,\ldots,m_k)$ and sum over the m_i's, getting, on recalling (5),

$$\Pi/I_1=\exp-Z+\mathcal{O}^*\biggl(\sum_{\substack{m_1,\ldots,m_k\geqslant 0,\\ 1\leqslant m_1+\ldots+m_k\leqslant N_0}}\frac{M(m_1,\ldots,m_k)A^{NF}(1+\log F)}{P^{LN}}$$
$$+\sum_{\substack{m_1,\ldots,m_k\geqslant 0,\\ 1\leqslant m_1+\ldots+m_k\leqslant N_0}}\frac{3M(m_1,\ldots,m_k)A^{N(F+1)}}{(F+1)P^{(F+1)N}}\Bigl(1+\frac{P}{FN}\Bigr)\biggr).$$

We now take $F=L$. The error term is bounded above by (since $P\geqslant 2kA$)

$$\frac{kA^L}{P^L}\biggl(\frac{2^k}{k!}(1+\log L)+\frac{3\cdot 2^kA}{k!(L+1)P}\Bigl(1+\frac{P}{L}\Bigr)\biggr).$$

We select $N_0=L$ and we gather our estimates to end the proof. ∎

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