Comparing $L(s,\chi)$ with its truncated Euler product and generalization

O. Ramaré

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Abstract

We show that any L-function attached to a non-exceptionnal Hecke Grossencharakter Ξ may be approximated by a truncated Euler product when s lies near the line $\Re s = 1$. This leads to some refined bounds on $L(s,\Xi)$.

1 Introduction and results

For $L(1,\chi)$, see [8], [16] and [1].

We first need to fix some terminology. We select a number field $\mathbb{K}/\mathbb{Q}$ be a number field of degree d and discriminant Δ. We denote its norm by N, as a shortcut to $\mathrm{N}_{\mathbb{K}/\mathbb{Q}}$. We shall consider Hecke Grossencharakters Ξ with (finite) ideal $\mathfrak{f}$, of norm q, and associated with some finite set of infinite places. The conductor $\mathfrak{f}$ being fixed, the main Theorem of [5] tells us there exists an absolute constant $C > 0$ such that no L-function $L(s,\Xi)$ has a zero ρ in the region

$$\Re\rho \geq 1 - \frac{C}{\operatorname{Log}\max(q\Delta, q\Delta|\Im s|)} \tag{1}$$

except at most one such character; this potential exception is real valued and may have at most one real zero β in this region. We refer to this hypothetical character as the exceptional character and term the remaining ones as being non-exceptional. See also [11]. In the case of Dirichlet characters, i.e. $\mathbb{K} = \mathbb{Q}$, we know from [13] that we may take $C = 1/6.3958$.

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Theorem 1 *Let Ξ be a non-exceptional Hecke Grossencharacter with (finite) conductor $\mathfrak{f}$ of norm $q > 1$. We have*

$$L(s,\Xi) \asymp \prod_{\mathrm{N}\,\mathfrak{p} \le q\Delta|s|} \left(1 - \Xi(\mathfrak{p})/\,\mathrm{N}\,\mathfrak{p}^s\right)^{-1}$$

when $1 \ge (\Re s - 1)\operatorname{Log}(q\Delta(2+|s|)) \ge -C/2$, the constant C being the one from (1).

The restriction to non-exceptional characters can be dispensed with if we assume $|\Im s| \ge 1/\operatorname{Log}(q\Delta)$. Under the Riemann hypothesis for the implied L-function, we can restrict the above product to $p \le \operatorname{Log}\operatorname{Log}(q\Delta|s|)$. As trivial consequences, we find via (a generalization of) Mertens theorems (see (9) below) that, under these conditions

$$\frac{q/\phi(q)}{\operatorname{Log}(q\Delta|s|)} \ll |L(s,\Xi)| \ll \frac{\phi(q)}{q}\operatorname{Log}(q\Delta|s|). \tag{2}$$

The upper bound is classical in the case of Dirichlet characters but improves considerably in the general case on the one given in Theorem 5 of [17], albeit being less explicit. The factor $q/\phi(q)$ in the lower one appears to be novel, even in the case of Dirichlet characters. For instance, it supersedes the one of Corollary 2 of [11] by the factor $q/\phi(q)$ and by the fact that it is valid for any non-exceptional character. From a historical viewpoint, [14] shows that $|L(1,\chi)| \gg 1/\operatorname{Log}^5 q$ for non-real characters, and improves this in $|L(1,\chi)| \gg 1/\operatorname{Log} q$ in [15]. The proof is somewhat more delicate than expected.

Note also (by again invoking Mertens' theorems) that we can restrict the product to $p \le (q|s|)^a$ for any positive a.

Granville & Soundararajan investigated in [6] (see also [7]) the distribution of values of $L(1,\chi)$ (χ being a Dirichlet character) via an approximation by an Euler product and in particular,they show in their Proposition 1 that the Euler product may be truncated to $p \le \operatorname{Log} q$ for all but $\mathcal{O}(q^{1-2/\operatorname{Log}\operatorname{Log} q})$ characters. Note however that they aim at an exact approximation of $L(1,\Xi)$ while we only seek to recover its order of magnitude.

Our main ingredient is the following Lemma of independent interest.

Lemma 1 *Under the conditions above, $|L'/L(s,\Xi)| \ll \operatorname{Log}(q\Delta(2+|s|))$.*

In this Lemma also, the restriction to non-exceptional characters can be dispensed with if we assume $|\Im s| \ge 1/\operatorname{Log}(q\Delta)$. The inequality $-\Re L'/L(s,\Xi) \le c\operatorname{Log}(q\Delta(2+|s|))$ when $\Re s > 1$ is a classical element of the proof of the zero-free region for $L(\cdot,\Xi)$ (see [4, chapter 14] for instance); by using his local method, Landau shows in [15, page 30] that $\Re L'/L(s,\Xi) \le c\,\mathrm{Log}(q\Delta(2+|s|))$. The above Lemma shows that much more is true and that only invoking a one-sided bound for the real part does nor lead to any improvement.

Under the Riemann hypothesis for $L(\cdot,\Xi)$, the upper bound becomes $\mathrm{Log}\,\mathrm{Log}(q\Delta(2+|s|))$.

Generalization

Like many properties of Dirichlet L-functions, this one generalizes to a wide class of L-functions. We shall not describe such a general context but refer the reader to chapter 5 of [12]. We work under the conditions of their Theorem 5.10: $L(f,s)$ is an L-function fo degree d such that the Rankin-Selberg convolutions $L(f\otimes f,s)$ and $L(f\otimes \overline{f},s)$ exist, the latter having a simple pole at $s=1$ while the former is entire if $f\neq\overline{f}$. We further suppose that $|\alpha_j(p)|^2\le p/2$ at the ramified primes. The notion of exceptional character is more complicated to define in a general context, since it requires a way of defining families of L-functions. Assuming that our candidate has no real zero in the classical zero-free region, we find that

$$L(f,s) \asymp \prod_{p\le \mathfrak{q}(f,s)} (1-\alpha_1(p)p^{-s})^{-1}\cdots(1-\alpha_d(p)p^{-s})^{-1} \tag{3}$$

where the analytical conductor is defined there in equation (5.7).

Notations

We need some names for our variables, and the easiest path is to keep a fixed point $s_0=\sigma_0+it_0$, which will be s in the Theorem, and a running $s=\sigma+it$. We define

$$\mathcal{L} = \mathrm{Log}(q\Delta(|s_0|+2)). \tag{4}$$

The point $s_1=\sigma_1+it_0$ with $\sigma_1 = 1+1/\mathcal{L}$ will be of special interest.

2 Some material on primes in number fields

We can use the prime number Theorem for $\mathbb{K}/\mathbb{Q}$, but we prefer to sketch an elementary approach to the classical results we need. Such material is also contained in [18]. Assume we have an asymptotic estimate:

$$\sum_{\mathrm{N}\,\mathfrak{a}\le X} 1 = c_0X + \mathcal{O}(X/\mathrm{Log}(2X)) \tag{5}$$

where $\mathfrak{a}$ ranges the integral ideals of $\mathbb{K}$. Such an estimate is linked with the fact that the Dedekind zeta function $\zeta_{\mathbb{K}}$ of $\mathbb{K}$ has a simple pole at $s=1$. In particular c_0 is the residue of this function at $s=1$. The results we seek also hold with the error term being simply $o(X)$, but our proof would require a modification. From this we deduce that

$$\sum_{\mathrm{N}\,\mathfrak{a}\le X} \operatorname{Log} \mathrm{N}\,\mathfrak{a} = c_0 X \operatorname{Log} X + \mathcal{O}(X). \tag{6}$$

Writing $\zeta_{\mathbb{K}}'/\zeta_{\mathbb{K}}(s) = \sum_{\mathfrak{a}} \Lambda_{\mathbb{K}}(\mathfrak{a})/\mathrm{N}\,\mathfrak{a}^s$ we find that $\sum_{\mathfrak{b}|\mathfrak{a}} \Lambda_{\mathbb{K}}(\mathfrak{b}) = \operatorname{Log} \mathrm{N}\,\mathfrak{a}$ and plugging this into (6), we get

$$c_0 X \operatorname{Log} X + \mathcal{O}(X) = \sum_{\mathrm{N}\,\mathfrak{b}\le X} \Lambda_{\mathbb{K}}(\mathfrak{b}) \sum_{\mathrm{N}\,\mathfrak{c}\le X/\,\mathrm{N}\,\mathfrak{b}} 1 = X \sum_{\mathrm{N}\,\mathfrak{b}\le X} \frac{\Lambda_{\mathbb{K}}(\mathfrak{b})}{\mathrm{N}\,\mathfrak{b}}$$

by appealing to (5), from which we infer

$$\sum_{\mathrm{N}\,\mathfrak{b}\le X} \frac{\Lambda_{\mathbb{K}}(\mathfrak{b})}{\mathrm{N}\,\mathfrak{b}} = \operatorname{Log} X + \mathcal{O}(1). \tag{7}$$

Using the expression of $\zeta_{\mathbb{K}}$ as an Euler product, we find that $\Lambda_{\mathbb{K}}(\mathfrak{b})$ is zero except when $\mathfrak{b}$ is a power of a prime $\mathfrak{p}$, at which point it takes the value $\operatorname{Log} \mathrm{N}\,\mathfrak{p}$. This finally leads us to the estimate

$$\sum_{\mathrm{N}\,\mathfrak{p}\le X} \frac{\operatorname{Log} \mathrm{N}\,\mathfrak{p}}{\mathrm{N}\,\mathfrak{p}} = \operatorname{Log} X + \mathcal{O}(1). \tag{8}$$

We infer $\sum_{\mathrm{N}\,\mathfrak{p}\le X} 1/\mathrm{N}\,\mathfrak{p} = \operatorname{Log}\operatorname{Log} X + \mathcal{O}(1)$ and thus

$$\prod_{\mathrm{N}\,\mathfrak{p}\le X} (1 - 1/\,\mathrm{N}\,\mathfrak{p}) \asymp 1/\operatorname{Log} X \tag{9}$$

which is enough for our purpose. This is not what is referred to as Mertens' Theorem, since we do not have a proper asymptotic, but these estimates are enough for our purpose. We refer the reader to [3] for related material on explicit Mertens' Theorem in abelian number fields.

3 Proof of Lemma 1

We start from Linnik's density lemma which the reader may find in [5, Lemma 7] or in [2, chapter 6] in case of Dirichlet characters. We define

$n(1+it,r)$ to be the number of zeros ρ of $L(s,\chi)$ in the disc $|\rho-i-it|\le r$. We have

$$\frac{L'}{L}(s,\Xi)=\frac{-\delta_\Xi}{s-1}+\sum_{|\rho-1-it_0|\le 1/3}\frac{1}{s-\rho}+\mathcal{O}(\mathcal{L})\quad (|s-1-it_0|\le 1/4),\qquad (10)$$

where δ_Ξ is 1 if Ξ is principal, and 0 otherwise. This is for instance Lemma 6 of [5]; In case of Dirichlet characters, this is (4) of chapter 16 of [4], and in a general context (5.28) of [12]. These two last proofs relie on a global representation of L'/L, while Fogel's one follows the local method of Landau. The latest refinements of this method may be found in [10] and [9].

One of the consequences of (10) is Linnik's density lemma:

$$n(1+it,r)\ll r\mathfrak{L}+1.\qquad (11)$$

Apply (10) to $s=\sigma+it$ with $\sigma\ge 1-2C/\mathcal{L}$ and to s_1 and substract. For any zero ρ in the summation above, we have $|s-\rho|\ge|1+it_0-\rho|/2$ and thus

$$\begin{aligned}\left|\frac{L'}{L}(s,\Xi)-\frac{L'}{L}(s_1,\Xi)\right| &\le \sum_{|\rho-1-it_0|\le 1/3}\frac{4|\sigma-\sigma_1|}{|1+it_0-\rho|^2}+\mathcal{O}(\mathcal{L})\\ &\le |\sigma-\sigma_1|\sum_{0\le k\le \operatorname{Log}\mathcal{L}}\ \sum_{r=2^k\le|\rho-1-it_0|\mathcal{L}\le 2^{k+1}}\frac{4}{r^2}+\mathcal{O}(\mathcal{L})\\ &\le |\sigma-\sigma_1|\sum_{0\le k\le \operatorname{Log}\mathcal{L}}\frac{2n(1+it_0,r)}{r^2}+\mathcal{O}(\mathcal{L})\\ &\ll |\sigma-\sigma_1|\sum_{0\le k\le \operatorname{Log}\mathcal{L}}\left(\frac{\mathcal{L}}{r}+\frac{1}{r^2}\right)+\mathcal{O}(\mathcal{L})\ll|\sigma-\sigma_1|\mathcal{L}^2+\mathcal{L}.\end{aligned}$$

Notice furthermore that $|L'/L(s_1,\Xi)|\le-\zeta'/\zeta(\sigma_1)\ll\mathcal{L}$, so that, when $\sigma\le 1+\mathcal{L}$, the above inequality reduces to

$$\left|\frac{L'}{L}(s,\Xi)\right|\ll\mathcal{L}.\qquad (12)$$

This ends the proof in case of non-exceptional characters. In case of an exceptional character, we simply consider separately in (10) its contribution, namely $1/(s-\beta)$ which is again $\mathcal{O}(\mathcal{L})$. Under the Riemann hypothesis, we simply invoke Theorem 5.17 of [12].

4 Proof of the Theorem

Define $R = q\Delta|s_0|$. We check that

$$\left|\frac{L'}{L}(s,\Xi) + \sum_{\mathrm{N}\,\mathfrak{p}\leq R} \frac{\Xi(\mathfrak{p})\,\mathrm{Log}\,\mathrm{N}\,\mathfrak{p}}{\mathrm{N}\,\mathfrak{p}^s - \Xi(\mathfrak{p})}\right| \ll \mathcal{L} + \mathrm{Log}\,R \ll \mathcal{L} \tag{13}$$

when $s = \sigma + it_0$ and $1 \geq (\sigma - 1)\mathcal{L} \geq -C/2$. We integrate (12) between s_1 and s_0 and find that

$$|\,\mathrm{Log}\,L_R(s_0,\Xi) - \mathrm{Log}\,L_R(s_1,\Xi)| \ll 1 \tag{14}$$

with $L_R(s,\Xi) = \prod_{\mathrm{N}\,\mathfrak{p}>R}(1 - \Xi(\mathfrak{p})/\,\mathrm{N}\,\mathfrak{p}^s)^{-1}$. Next we note that

$$\begin{aligned} |L_R(s_1,\Xi)| &\leq \prod_{\mathrm{N}\,\mathfrak{p}>R}(1 - \mathrm{N}\,\mathfrak{p}^{-\sigma_1})^{-1} \leq \exp \sum_{\mathrm{N}\,\mathfrak{p}>R} \mathrm{N}\,\mathfrak{p}^{-\sigma_1} \\ &\ll \exp \int_R^\infty \frac{dt}{t^{\sigma_1}\,\mathrm{Log}\,t} = \exp \int_{R^{\sigma_1-1}}^\infty \frac{dv}{v^2\,\mathrm{Log}\,v} \end{aligned}$$

by setting $v = t^{\sigma_1 - 1}$, and where we have invoked (8). The last quantity is bounded since so is R^{σ_1-1}. Considering only real parts in (14), the Theorem readily follows.

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