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AN EXPLICIT DENSITY ESTIMATE FOR DIRICHLET L-SERIES

O. RAMARÉ

ABSTRACT. We prove that, for $T \geq 2\,000$, $T \geq Q \geq 10$, and $\sigma \geq 0.52$, we have

$$\sum_{q\leq Q}\frac{q}{\varphi(q)}\sum_{\chi \bmod^* q} N(\sigma,T,\chi) \leq 20\bigl(56\,Q^5T^3\bigr)^{1-\sigma}\log^{5-2\sigma}(Q^2T)+32\,Q^2\log^2(Q^2T)$$

where $\chi \bmod^* q$ denotes a sum over all primitive Dirichlet character χ to the modulus q. Furthermore, we have

$$N(\sigma,T,\mathbb{1}) \leq 2T\log\biggl(1+\frac{9.8}{2T}(3T)^{8(1-\sigma)/3}\log^{5-2\sigma}(T)\biggr)+103(\log T)^2.$$

1. Introduction

Dirichlet L-series $L(s,\chi)=\sum_{n\geq 1}\chi(n)n^{-s}$ associated to primitive Dirichlet characters χ are one of the keys to the distribution of primes. Even the simple case $\chi = 1$ which corresponds to the Riemann zeta-function contains a lot of information on primes and on the Farey dissection. There have been many generalizations of these notions, and they all have arithmetical properties and/or applications, see [31] for instance. Investigations concerning these functions range over many directions, see [14] or [44]. We note furthermore that Dirichlet characters have been the subject of numerous studies, see [2, 4]; Dirichlet series themselves are still mysterious, see [3] and [6].

One of the main problem concerns the location of the zeroes of these functions in the strip $0 < \Re s < 1$; the Generalized Riemann Hypothesis asserts that all of those are on the line $\Re s = 1/2$. We concentrate in this paper on estimating

$$N(\sigma,T,\chi)=\sum_{\substack{\rho=\beta+i\gamma,\\ L(\rho,\chi)=0,\\ \sigma\leq\beta,|\gamma|\leq T}} 1.$$

Under the Generalised Riemann Hypothesis, this quantity vanishes when $\sigma > 1/2$ and we want to bound it from above. An upper bound is however often very powerful, one of the more striking uses of such an estimate being surely Hoheisel Theorem. In [33, Theorem 7], the authors already prove an explicit density estimates for L-functions, namely

$$\sum_{\chi \bmod q} N(\sigma,T,\chi) \leq \biggl(\frac{254\,231}{\log qT}+17\,102\biggr)(q^3T^4)^{1-\sigma}(\log qT)^{6\sigma}+16\,541(\log T)^6$$

under some size conditions on T and q we do not reproduce. [10] had in fact proved most of this result, but his bound had the restriction $\chi \neq \chi_0$, the principal character.

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This result is used in [34] to prove to show that every odd integer $\geq \exp(3\,100)$ is a sum of at most three primes.

As it turns out, I proved long ago in my M. Phil. memoir a bound in case $\chi = \chi_0$ that was better than that. This was never published but several versions circulated, at various stages of improvement. This paper will fix a version. We do so because of a regain of interest in the field (see of course [50], [27] and [28]) and more precisely [20] where these authors manage to use a density estimate from [30] to improve on the numerical bounds for the Tchebyschef-ψ function. After more than fifty years of very limited theoretical progress in this field (though there has been work on it, see [15], [16], [17], [37]), this is quite a news and announces further improvements. The second main news in this area is due to the doctoral thesis of D. Platt [39] where the Riemann hypothesis has been verified for all Dirichlet characters of conductor $q \leq Q_0 = 400\,000$ and up to the height $10^8/q$, improving in a drastic fashion on the previous works [45] and [7]. This author has also checked that the Riemann zeta function has no non-real zero off the critical line and of height bounded in absolute value by $3 \cdot 10^{10}$ (see also [53] and [23], though these results have not been the subject of any academic publications).

Here is our main theorem:

Theorem 1.1. *For $T \geq 2\,000$ and $T \geq Q \geq 10$, as well as $\sigma \geq 0.52$, we have*

$$\sum_{q \leq Q} \frac{q}{\varphi(q)} \sum_{\chi \bmod^* q} N(\sigma, T, \chi) \leq 20\big(56\, Q^5 T^3\big)^{1-\sigma} \log^{5-2\sigma}(Q^2 T) + 32\, Q^2 \log^2(Q^2 T)$$

where $\chi \bmod^* q$ denotes a sum over all primitive Dirichlet character χ to the modulus q. Furthermore, we have

$$N(\sigma, T, \mathbb{1}) \leq 2T \log\left(1 + \frac{9.8}{2T}(3T)^{8(1-\sigma)/3} \log^{5-2\sigma}(T)\right) + 103(\log T)^2.$$

Our result is asymptotically better in case $Q = 1$ than Ingham's, from which we borrow most of the proof, by almost two powers of logarithm: we get the exponent $5 - 2\sigma$ instead of the classical 5. See [51, Theorem 9.19].

In case $Q = 1$, the form we have chosen for our density estimate is unusual but numerically efficient. If a simpler form is required, we can degrade the above (via $\log(1 + x) \leq x$) into

$$N(\sigma, T, \mathbb{1}) \leq 9.8(3T)^{8(1-\sigma)/3} \log^{5-2\sigma}(T) + 103(\log T)^2. \tag{1.1}$$

However the form we have chosen also implies for instance that

$$\frac{N(3/4, T, \mathbb{1})}{(3T)^{2/3}(\log T)^{7/2}} \leq \begin{cases} 1/4 & \text{when } T \leq 4.5 \cdot 10^{10}, \\ 1/2 & \text{when } T \leq 3.3 \cdot 10^{12}, \\ 1 & \text{when } T \leq 2.1 \cdot 10^{14}, \\ 9.9 & \text{when } T \geq 0. \end{cases}$$

while (1.1) would only prove the last line.

For comparison, Chen/ Liu & Wang's result does not apply here because of the exponent of T. We should however mention that, when comparing this estimate to the total number of zeroes, see Lemma 9.1, the above bound at $\sigma = 3/4$ is not more than 1/2 this total number (and this is required because of the symmetry of the zeroes with respect to $\rho \mapsto 1 - \overline{\rho}$) only when at least $T \geq 10^{16}$. This is a really small bound in such a field. The choice $17/20 = 0.85$ seems interesting. Our result yields

$$N(17/20, T, \mathbb{1}) \leq 9.9\,(3T)^{2/5}(\log T)^{33/10} \tag{1.2}$$

which is this time always not more than 0.079 times the trivial bound.

Let us compare our result in case $Q = 1$ with [30].

- When $\sigma = 17/20$, [30] yields $\frac{1}{2}N(17/20, T, \mathbb{1}) \leq 0.5561\,T + 0.7586\,\log T - 268\,658$ (the factor $\frac{1}{2}$ is required: in classical notation $N(\sigma, T)$ counts the non-trivial zeros of the Riemann zeta function with abscissa between 0 and T and *not* between $-T$ and T). The estimate (1.2) is nearly twice better.
- However when $\sigma = 4/5$, [30] yields $\frac{1}{2}N(4/5, T, \mathbb{1}) \leq 0.7269\,T + 0.9566\,\log T - 209\,795$ which is better than $9.9.5(4T)^{2/3}(\log T)^{7/2}$ when $T \leq 5.3 \cdot 10^{20}$. It is even better than the more refined bound we have given when $T \leq 4.2 \cdot 10^{12}$.
- And when $\sigma = 7/10$, [30] yields $\frac{1}{2}N(7/10, T, \mathbb{1}) \leq 1.4934\,T + 1.4609\,\log T - 136\,370$ which is smaller than our better bound at least on the range $T \leq 3.3 \cdot 10^{37}$.

Let us note here that some intermediate results are of independant interest: lemma 4.3 is a complement of [42, Lemma 3.2] for evaluating averages of non-negative multiplicative functions, corollaries 6.3 and 6.4 are sharp explicit versions of [22, Theorem 3]. Lemma 6.5 is more straightforward but is indeed a numerical refinement of [36, Corollary 3]. Lemma 5.4 has been quoted earlier in [5] but this is the first published proof (as far as I know).

Acknowledgement. We take the opportunity of this paper to show how to practically use the multiprecision interval arithmetic of Sage [49]. Let Paul Zimmermann be thanked for this part. We also present some usage of Gp/Pari, and let Karim Belabas be thanked for improving some of the scripts given below.

Notation and some definitions. We follow closely Ingham's proof as given in [51], paragraph 9.16 through 9.19. We extend it to cover the case of Dirichlet characters.

We consider a real parameter $X \geq 2000$ and the following kernel that we use to "mollify" $L(s, \chi)$ (see [13] for instance)

$$M_X(s, \chi) = \sum_{n \leq X} \mu(n)\chi(n)/n^s. \tag{1.3}$$

We consider

$$\begin{cases} f_X(s, \chi) = M_X(s, \chi)L(s, \chi) - 1, \\[2ex] h_X(s, \chi) = 1 - f_X(s, \chi)^2 = L(s, \chi)M_X(s, \chi)(2 - L(s, \chi)M_X(s, \chi)), \\[2ex] g_X(s, \chi) = h_X(s, \chi)h_X(s, \overline{\chi}). \end{cases} \tag{1.4}$$

We observe that zeroes of $L(s, \chi)$ are zeroes of $h_X(s, \chi)$. We use here the fact that $M_X(s, \chi)$ is expected to be a partial inverse of $L(s, \chi)$, due to combinatorial properties of the Moebius function.

We use the shorthand

$$\sideset{}{^\star}\sum_{\substack{q \leq Q, \\ \chi \bmod^* q}} h(q, \chi) = \sum_{q \leq Q} \frac{q}{\varphi(q)} \sum_{\substack{\chi \bmod q, \\ \chi \text{ primitive}}} h(q, \chi) \tag{1.5}$$

for any arbitrary function h.

We denote by $N_1(\sigma, T, \chi)$ the number of zeroes ρ of $h_X(s,\chi)$ in the rectangle

$$\Re\rho \geq \sigma, \quad T \geq |\Im\rho| \tag{1.6}$$

to the exception of those with $\Im\rho = 0$. They are also the zeroes of $g_X(s,\chi)$ with $T \geq \Im\rho \geq 0$ and $\Re s \geq \sigma$, counted according to multiplicities. We define furthermore $N_1(\sigma, T_1, T_2, \chi) = N_1(\sigma, T_2, \chi) - N_1(\sigma, T_1, \chi)$ as well as

$$N_1(\sigma, T_1, T_2, Q) = \sum^{\star}_{\substack{q \leq Q, \\ \chi \bmod^* q}} N_1(\sigma, T_1, T_2, \chi).$$

In the course of the proof, we shall require

$$F_Q(\sigma, T) = \int_{-T}^{T} \sum^{\star}_{\substack{q \leq Q, \\ \chi \bmod^* q}} |f_X(\sigma + it, \chi)|^2 dt \tag{1.7}$$

which of course depends on the parameter X as well. The variable t ranges $[-T, T]$ and we sometimes will have results where the variable t ranges $[0, T]$. In such a case we will use the notation $\frac{1}{2}F_Q(\sigma, T)$, thanks to the symmetry induced by $\overline{f_X(\sigma + it, \chi)} = f_X(\sigma - it, \overline{\chi})$.

The remainder of the notation is standard, but here are some points: the arithmetical functions are the Moebius function μ, the number of prime factors counted without multiplicity ω, the Euler-totient function φ; the arithmetical convolution product is denoted by $\star$. The letter ψ does *not* represent the Chebyschef-ψ function but the digamma function, though ϑ *is* the Chebyschef ϑ-function. The letter p represents a prime number in summations. We use $f = \mathcal{O}^*(g)$ to say that $|f| \leq g$.

Minimal orders of magnitude. The parameters that quantify the sizes are Q and T. Most of the time, we will only require bounds on $X = Q^2T$. When $Q = 1$, we can assume that $T \geq 3 \cdot 10^{10}$, while in general, we can assume that either $Q > Q_0 = 400\,000$ or $T > 10^8/Q$. Since in that case ($Q \neq 1$) we also assume that $Q \geq 10$, this means that we can in any case assume that $X \geq 10^9$. We also consider only the case $T \geq 2000$, which implies that $X \geq 2000Q^2$ (valid also when $Q = 1$). Note however that a parameter T is often used in lemmas, and it is *not* always subject to $T \geq 2000$. The parameter X is always linked to the final choices.

Thanks. Warm thanks are due to the referee for his/her very careful reading of this paper and for the quality of his/her comments.

2. On the size of L-functions

Lemma 2.1. *Let* χ *be a primitive character of conductor* $q > 1$. *For* $-\frac{1}{2} \leq -\eta \leq \sigma \leq 1 + \eta \leq \frac{3}{2}$, *we have*

$$|L(s,\chi)| \leq \left(\frac{q|1+s|}{2\pi}\right)^{\frac{1}{2}(1+\eta-\sigma)} \zeta(1+\eta)$$

See [41, Theorem 3]. In the same paper, Theorem 4 treats in passing the case $q = 1$, where the above bound for $q = 1$ simply has to be multiplied by $3|\frac{1+s}{1-s}|$. We can treat the term $\zeta(1+\eta)$ by using the inequality (see also Lemma 5.4 below)

$$\zeta(1+\eta) \le \frac{1+\eta}{\eta} \tag{2.1}$$

valid for $\eta > 0$. Our main application has $\sigma = \Re s = \frac{1}{2}$, for which we can invoke the following result of [11, Corollary to Theorem 3], modified according to [52, Section 5]:

Lemma 2.2. *For $0 \le t \le e$, we have $|\zeta(\frac{1}{2}+it)| \le 2.657$. For $t \ge e$, we have $|\zeta(\frac{1}{2}+it)| \le 2.4\,t^{1/6}\log t$.*

The modification in question leads to the constant 2.4 instead of the initial 3.

Lemma 2.3. *Let χ be a primitive character of conductor $q \ge 1$. We have (for $T \ge 4$)*

$$\max\bigl\{|L(s,\chi)|, \Re s \ge 0, |\Im s| \le T\bigr\} \le\ 4.42(qT)^{5/8}.$$

Proof. We use Lemma 2.1 with $\eta = 1/4$ in case $q > 1$ to get the upper bound

$$\left(\frac{q(1+\sigma+T)}{2\pi}\right)^{\frac{1}{2}(\frac{5}{4}-\sigma)}\zeta(5/4)$$

In the quotient, the worse case is $\sigma = 0$. The quantity $\zeta(5/4) \le 4.6$ is trivially an upper bound in case $\Re s \ge 5/4$. In case $q = 1$, we multiply this bound by 3.001. □

Lemma 2.4. *We have, when $Q \ge 10$ and $T > 0$,*

$$\max\bigl\{|L(\tfrac{1}{2}+it,\chi)|, \chi \bmod^* q \le Q, |t| \le T\bigr\} \le 2\,(QT)^{1/4}\log(QT) + 3Q^{1/4}\log Q.$$

When $Q = 1$, we have

$$\max\bigl\{|\zeta(\tfrac{1}{2}+it)|, |t| \le T\bigr\} \le 2.4\,T^{1/6}\log(T) + 6.8.$$

The lemma is, by continuity, valid at $T = 0$ provided one understands $0^{1/4}\log 0$ and $0^{1/6}\log 0$ as being 0.

Proof. We use Lemma 2.1 with $\eta = 1/\log(QT)$ in case $q > 1$ and get the upper bound

$$e^{1/2}\left(\frac{q(\frac{3}{2}+T)}{2\pi}\right)^{1/4}(\log(QT)+1) \le 2\,(QT)^{1/4}\log(QT)$$

for $QT \ge 5$. When $QT \le 5$, then we take $\eta = 1/\log Q$ and numerically check that

$$\left(1+\frac{1}{\log 2}\right)e^{1/2}\left(\frac{\frac{3}{2}+T}{2\pi}\right)^{\frac{1}{4}}Q^{1/4}\log Q - 2\,(QT)^{1/4}\log Q \le 1.7Q^{1/4}\log Q$$

when $T \ge 0$. As for the remaining case $QT \le 5$ and $T \le 1$, we add the maximum of $-2T^{1/4}\log T$ divided by $\log 10$ (this is $8/(e\log 10)$) to the coefficient of $Q^{1/4}\log Q$. This readily extends to encompass case $q = 1$ and this concludes the first half of the lemma.

Let us turn to the estimate concerning solely the Riemann zeta-function. We first check that $\min_{0\le t\le 3}(14.4\,T^{1/6}\log(T^{1/6}) + 7.96) \ge 2.657$ since the minimum is reached when $T^{1/6} = 1/e$. One can in fact be more precise by relying on explicit computations of $\zeta(1/2+it)$ on the very restricted range $t \in [0,3]$. This hints at the property $|\zeta(\frac{1}{2}+it)| \le 2.4\,T^{1/6}\log(T) + 6.78$. The RHS is more than 2.657 if $t \ge 0.07$, so the only the range $[0, 0.07]$ needs to be covered. It is then not necessary to give more details. □

3. Some arithmetical lemmas

Here is a lemma from [12]:

Lemma 3.1. *We have, for* $D \geq 1\,004$
$$\sum_{d \leq D} \mu^2(d) = \frac{6D}{\pi^2} + \mathcal{O}^*\bigl(0.1333\sqrt{D}\bigr).$$
In particular, this is not more than $0.62D$ *when* $D \geq 1700$.

We shall require explicit computations that involve sums over primes (we convert products in sums via the logarithm). We shall truncate these sums and here is a handy lemma to control the error term.

Lemma 3.2. *Let* f *be a* C^1 *non-negative, non-increasing function over* $[P, \infty[$, *where* $P \geq 3\,600\,000$ *is a real number and such that* $\lim_{t\to\infty} tf(t) = 0$. *We have*
$$\sum_{p \geq P} f(p) \log p \leq (1+\epsilon)\int_P^\infty f(t)dt + \epsilon P f(P) + Pf(P)/(5\log^2 P)$$
with $\epsilon = 1/914$. *When we can only ensure* $P \geq 2$, *then a similar inequality holds, simply replacing the last* $1/5$ *by a* 4.

On the value of ϵ*:* we rely on [47, (5.1*)] because it is easily accessible. However on using [17, Proposition 5.1], one has access to $\epsilon = 1/36260$. And on using [20, Theorem 1.1] together with a finite range verification, we may expect to have access to $\epsilon = 1/70000$. These bounds are all in the process of being published (though P. Dusart's thesis [15] is published, it is not easily accessible).

Proof. A summation by parts tells us that
$$\sum_{p \geq P} f(p) \log p = -\int_P^\infty f'(t)\vartheta(t)dt - \vartheta(P)f(P)$$
where $\vartheta(x) = \sum_{p \leq x} \log p$. At this level, we recall two results. One is (a weakening of) [47, (5.1*)] and reads
$$\vartheta(x) - x \leq x/914 \quad (x > 0)$$
The second one is [17, Theorem 5.2], or also the third inequality of [18, page 54] (these result may also be found in [15]):
$$|\vartheta(x) - x| \leq 0.2\, x/(\log^2 x) \quad (x \geq 3\,600\,000).$$
The lemma follows readily on applying these estimates. □

Lemma 3.3. *We have*
$$\sum_{d \leq D} \mu^2(d)\frac{\varphi(d)}{d^2} = a \log D + b + \mathcal{O}^*(0.174)$$
with $a = \prod_{p \geq 2}(p^3 - 2p + 1)/p^3 = 0.4282 + \mathcal{O}^*(10^{-4})$ *and*
$$b/a = \gamma + \sum_{p \geq 2} \frac{3p-2}{p^3 - 2p + 1} \log p = 2.046 + \mathcal{O}^*(10^{-4}).$$
Furthermore the 0.174 *can be reduced to* 0.0533 *when* $D \geq 10$ *and to* 0.0194 *when* $D \geq 48$.

Proof. We appeal to [42, Lemma 3.2]. First note that

$$\begin{aligned} D(s) &= \sum_{d\ge1}\frac{\mu^2(d)\varphi(d)}{d^{2+s}} = \prod_{p\ge2}\Bigl(1+\frac{p-1}{p^{2+s}}\Bigr) \\ &= \zeta(s+1)\prod_{p\ge2}\Bigl(1-\frac{1}{p^{2+s}}-\frac{1}{p^{2+2s}}+\frac{1}{p^{3+2s}}\Bigr) = \zeta(s+1)H(s) \end{aligned}$$

say. We thus get, for $D\ge1$:

$$\sum_{d\le D}\mu^2(d)\frac{\varphi(d)}{d^2} = H(0)\log D + H'(0) + \gamma H(0) + \mathcal{O}^*\bigl(c/D^{1/3}\bigr)$$

where the constants are given by

$$c = \prod_{p\ge2}\Bigl(1+\frac{1}{p^{5/3}}+\frac{1}{p^{4/3}}+\frac{1}{p^{7/3}}\Bigr) \le 6,$$

and

$$a = H(0) = \prod_{p\ge2}\frac{p^3-2p+1}{p^3} = 0.4282 + \mathcal{O}^*(10^{-4}).$$

Furthermore

$$\frac{H'(0)}{H(0)} = \sum_{p\ge2}\frac{3p-2}{p^3-2p+1}\log p = 1.4695 + \mathcal{O}^*(10^{-4})$$

We use the following Sage program, see [49], since it implements interval arithmetic from [21]:

```
# File lemma32.sage

R = RealIntervalField(64)

def g(n):
    res = 1
    l = factor(n)
    for p in l:
        if p[1] > 1:
            return R(0)
        else:
            res *= (p[0]-1)/p[0]^2
    return R(res)

P = 10000
aaa = R(1)
p = 2
while p <= P:
    aaa *= R(1-2/p^2+1/p^3)
    p = next_prime (p)
eps = 1/R(914)
x = 3*(1+eps)/R(P)/log(R(P))+3*eps/R(P)/log(R(P))+3/4/R(P)/log(R(P))^3
x = exp(-x)
aaa = aaa * x.union(R(1))
```

```
P = 100000
bbb = R(0)
p = 2
while p <= P:
    bbb += R((3*p-2)/(p^3-2*p+1))*log(R(p))
    p = next_prime (p)
x = (log(R(P))+1)/R(P)
bbb = bbb + x.union(R(0)) + R(euler_gamma)

ccc = R(6)

def model(z):
    return aaa * (log(R(z)) + bbb)

def getbounds (zmin, zmax):
    zmin = max (0, floor (zmin))
    zmax = ceil (zmax)
    res = R(0)
    for n in range (1, zmin + 1):
        res += g(n)
    maxi = abs(res - model (zmin)).upper()
    maxiall = maxi
    for n in xrange (zmin + 1, zmax + 1):
        m = model (n)
        maxi = max (maxi, abs(res - m).upper())
        res += g(n)
        maxi = max (maxi, abs(res - m).upper())
        if n % 100000 == 0:
            print "Upto ", n, " : ", maxi, cputime()
            maxiall = max (maxiall, maxi)
            maxi = R(-1000).upper()
    maxi = max (maxi, abs (res - model (zmax)).upper())
    maxiall = max (maxiall, maxi)
    print "La borne pour z >= ", zmax, " : "
    bound = ccc/R(zmax)^(1/3)
    print bound.upper()
    return [maxiall, maxi]
```

Assuming this file is called `lemma32.sage`, the command `load('lemma32.sage')` within Sage indeed loads the included functions. The command `getbounds(1000, 30000000)` brings the output

```
sage: getbounds(1000, 30000000)
...
La borne pour z >=  30000000  :
0.0193097876921125952
[0.00214646012080014072, 0.000202372251756890651]
```

showing that

$$\Big|\sum_{d\le D}\mu^2(d)\frac{\varphi(d)}{d^2} - a\log D - b\Big| \le 0.00215$$

when $1000 \le D \le 30\,000\,000$. We then check that we can in fact start at $x = 48$. The conclusion is easy. □

Lemma 3.4. *Let $N \ge 1$ be a real number. We have*

$$\frac{6}{\pi^2}\log N + 0.578 \le \sum_{n\le N}\mu^2(n)/n \le \frac{6}{\pi^2}\log N + 1.166.$$

When $N \ge 1000$, the couple $(0.578, 1.166)$ may be replaced by $(1.040, 1.048)$

A similar lemma occurs in [46], but with worse constants.

Proof. We proceed as above and get

$$\sum_{n\le N}\mu^2(n)/n = \frac{6}{\pi^2}\big(\log N + 2\sum_{p\ge 2}\frac{\log p}{p^2-1} + \gamma\big) + \mathcal{O}^*(3/N^{1/3}).$$

A similar script as in the previous lemma yields

$$\Big|\sum_{d\le D}\frac{\mu^2(d)}{d^2} - \frac{6}{\pi^2}\log D - b'\Big| \le 0.00340$$

when $1000 \le D \le 30\,000\,000$. We present here an easier GP script, see [38], to extend it. Though such a script is usually enough (by which we mean, its result can in most examples be certified by Sage as in the previous lemma), only the program using MPFR handles correctly the error term.

```
{g(n) =
   my(res = 1.0, dec = factor(n), P = dec[,1], E = dec[,2]);
   for(i = 1, #P,
      my(p = P[i]);
      if(E[i] != 1, return(0));
      res *= 1/p);
   return(res);}

aaa = 6/Pi^2;
bbb = 1.7171176851;
ccc = 3;

{model(z)=aaa*(log(z)+bbb)}

{getsidedbounds(zmin,zmax)=
   my(res = 0.0, m, maxiplus, maximinus, maxiplusall, maximinusall);
   zmin = max( 0, floor(zmin));
   zmax = ceil(zmax);
   for(n=1, zmin, res += g(n));
   m = model(zmin);
   maxiplus = res - m;
   maxiplusall = maxiplus;
   maximinus = res - m;
```

```
   maximinusall = maximinus;
   for(n = zmin+1, zmax,
      m = model(n);
      maxiplus = max(maxiplus, res-m);
      maximinus = min(maximinus, res-m);
      res += g(n);
      maxiplus = max(maxiplus, res-m);
      maximinus = min(maximinus, res-m);

      if(n%100000==0,
         print("Upto ",n," : ", maximinus, " / ", maxiplus);
         maxiplusall = max(maxiplusall, maxiplus);
         maximinusall = min(maximinusall, maximinus);
         maxiplus = -1000;
         maximinus = 1000));
   m = model(zmax);
   maxiplus = max(maxiplus, res - m);
   maxiplusall = max(maxiplusall, maxiplus);
   maximinus = min(maximinus, res - m);
   maximinusall = min(maximinusall, maximinus);
   print("La borne pour z >= ", zmax, " : ", ccc/zmax^(1/3));
   return( [maximinusall, maximinus, maxiplusall, maxiplus]);
}
```

The conclusion is easy. □

4. On the total weight

In this section we prove an upper bound for $\sum^{\star}_{\substack{q\le Q,\\ \chi \bmod^* q}} 1$.

Lemma 4.1. *The number $\varphi^*(q)$ of primitive characters modulo q is a multiplicative function given for any prime p by*

$$\varphi^*(p) = p-2, \quad \varphi^*(p^k) = p^k\Big(1-\frac{1}{p}\Big)^2 \quad (\forall k\ge 2).$$

This is [48, Theorem 8] with the notation φ^* of [29, (3.7)].

Proof. Indeed, there are $\varphi(q)$ characters modulo q, which we can split according to their conductor: for each $d|q$, there are $\varphi^*(d)$ characters modulo q of conductor d. Hence $\mathbb{1}\star\varphi^* = \varphi$ which is readily solved in $\varphi^* = \mu\star\varphi$. This expression proves the multiplicativity as well as the values we have given. □

By using a script similar to the one used for Lemma 3.4, we prove that

Lemma 4.2. *When $Q\in[10, 100\,000\,000]$, we have*

$$\sum_{q\le Q}\frac{q}{\varphi(q)}\varphi^*(q) \le 0.29\,Q^2$$

where the function φ^* is defined in Lemma 4.1.

In fact, computing up to $Q = 100$ would have been enough for our sole application in Lemma 4.4 below.

Here is a lemma that will lead to a proof similar to the one of [42, Lemma 3.2].

Lemma 4.3. *We have, for any real number $X \geq 0$ and any real number $c \in [1,2]$,*

$$\sum_{q\leq X} q = \tfrac{1}{2}X^2 + \mathcal{O}^*(\tfrac{1}{2}X^c).$$

Proof. When $X < 1$, we check that $\frac{1}{2}X^c \geq \frac{1}{2}X^2$ for any $c \in [1,2]$. This proves that

$$\sum_{q\leq X} q = \tfrac{1}{2}X^2 + \mathcal{O}^*(\tfrac{1}{2}X^c) \quad (\forall X \in [X^*, X^*+1)) \tag{4.1}$$

for any $c \in [1,2]$ and $X^* = 0$.

Let Q be a positive integer and $N = \sum_{q\leq Q} q = Q(Q+1)/2$. Note that $Q \leq \sqrt{2N} < Q+1$. Then

- When $Q \leq X < \sqrt{2N}$, we have $|N - \frac{1}{2}X^2| \leq \frac{1}{2}X^c$ for every $c \in [1,2]$. Indeed this is equivalent to $N \leq \frac{1}{2}X^2 + \frac{1}{2}X^c$ which is implied by $N \leq \frac{1}{2}Q^2 + \frac{1}{2}Q \leq \frac{1}{2}X^2 + \frac{1}{2}X^c$.
- When $\sqrt{2N} \leq X < Q+1$, the inequality $|N - \frac{1}{2}X^2| \leq \frac{1}{2}X^c$ is equivalent to $X^c - X^2 + 2N \geq 0$. The derivative of the involved function is $cX^{c-1} - 2X$ which is non-positive ($c < 2$). So we have to check that $(Q+1)^c - (Q+1)^2 + 2N \geq 0$ i.e. $2N \geq Q^2 + Q$ which is true.

This proves that (4.1) holds for $X^* = Q$, for all Q. □

Lemma 4.4. *When $Q \geq 1$, we have*

$$\sum_{Q_0<q\leq Q} \frac{q}{\varphi(q)}\varphi^*(q) \leq 0.240\, Q^2 + 2\, Q^{3/2}$$

where the function φ^* is defined in Lemma 4.1. Hence the sum in question is $\leq 0.29\, Q^2$ when $Q \geq 10$. And, when $Q \geq Q_0 = 400\,000$, we have

$$\sum_{Q_0<q\leq Q} \frac{q}{\varphi(q)}\varphi^*(q) \leq 0.240\left(Q^2 - Q_0^2\right) + 2\left(Q^{3/2} + Q_0^{3/2}\right).$$

Proof. We denote by S the sum to be studied. We introduce the multiplicative function

$$g(q) = \frac{q}{\varphi(q)}\varphi^*(q).$$

We have $g(q)/q = (\mathbb{1}\star h)(q)$ where h is the multiplicative function defined on powers of each prime p by

$$h(p) = \frac{-1}{p-1}, \quad h(p^2) = \frac{1}{p(p-1)}, \quad h(p^k) = 0 \quad (\forall k \geq 3).$$

This enables us to write

$$g(q) = q \sum_{\substack{ab^2|q,\\ (a,b)=1}} \frac{\mu(a)\mu^2(b)}{\varphi(a)b\varphi(b)}. \tag{4.2}$$

This expression together with Lemma 4.3 with some parameter $c \in (1,2)$ yields

$$\begin{aligned} S &= \sum_{\substack{a,b\ge 1,\\ (a,b)=1}} \frac{\mu(a)\mu^2(b)}{\varphi(a)b\varphi(b)} \sum_{ab^2|q\le Q} q \\ &= \sum_{\substack{a,b\ge 1,\\ (a,b)=1}} \frac{\mu(a)\mu^2(b)ab}{\varphi(a)\varphi(b)} \left(\frac{Q^2}{2a^2b^4} + \mathcal{O}^*\left(\frac{Q^c}{2a^cb^{2c}}\right)\right) \\ &= \frac{Q^2}{2}\prod_{p\ge 2}\left(1 - \frac{1}{p(p-1)} + \frac{1}{p^3(p-1)}\right) + \mathcal{O}^*\left(\frac{Q^c}{2}\prod_{p\ge 2}\left(1 + \frac{1}{p^{c-1}(p-1)}\left(1+\frac{1}{p^c}\right)\right)\right). \end{aligned}$$

We choose $c = 3/2$ and compute

$$S \le 0.240\, Q^2 + 2Q^{3/2}.$$

On appealing to Lemma 4.2, the second part of our lemma follows readily. The third part is straightforward. □

5. Estimates concerning the Moebius function

Here is a handy lemma taken from [43, Theorem 1.1], generalizing [25, Lemma 10.2].

Lemma 5.1. *We have uniformly for any real numbers $N \ge 1$ and $\varepsilon > 0$, and any integer d*

$$\Bigl|\sum_{\substack{n\le N,\\ (n,d)=1}} \mu(n)/n^{1+\varepsilon}\Bigr| \le 1 + \varepsilon.$$

Lemma 5.2. *When $\sigma \ge 1$ and $q \ge 2$, we have $q^{-\sigma} - q^{-2\sigma} \le q^{-1} - q^{-2}$.*

Proof. We consider the auxiliary function $f(\sigma) = e^{-\sigma y} - e^{-2\sigma y}$, whose derivative is $f'(\sigma) = -ye^{-\sigma y}(1 - 2e^{-\sigma y})$. When $y \ge \log 2$ and $\sigma \ge 1$, we have $1 - 2e^{-\sigma y} \ge 0$. The proof is complete. □

Lemma 5.3. *For s real satisfying $|s-1| \le 1/2$, we have*

$$\zeta(s) = \frac{1}{s-1} + \gamma - \gamma_1(s-1) + \mathcal{O}^*(20|s-1|^2)$$

where $\gamma = 0.57721\cdots$ and $\gamma_1 = 0.07281\cdots$ are the Laurent-Stieltjes constants.

See [19] for the latest bounds on the Laurent-Stieltjes constants.

Proof. We first note the inequality $|e^{-z} - 1 + z| \le |z|^2 e^{|z|}$. We then proceed as follows:

$$\begin{aligned} \zeta(1+z) &= \frac{1+z}{z} - (1+z)\int_1^\infty \{t\}\frac{dt}{t^{2+z}} \\ &= \frac{1+z}{z} - (1+z)\int_1^\infty \{t\}(1 - z\log t)\frac{dt}{t^2} + \mathcal{O}^*\Bigl(|z^2+z^3|\int_1^\infty \frac{(\log t)^2 dt}{t^{2-|z|}}\Bigr) \\ &= \frac{1}{z} + \gamma - \gamma_1 z + \mathcal{O}^*\Bigl(|z^2+z^3|\int_1^\infty (\log t + (\log t)^2)\frac{dt}{t^{2-|z|}}\Bigr). \end{aligned}$$

Note that these lines show that the constant γ and γ_1 exist. Since they are unique, we can identify them with the usual ones. An integration by parts takes care of the remainder term. □

Lemma 5.4. *When $s > 1$ is real, we have $\zeta(s) \le e^{\gamma(s-1)}/(s-1)$.*

In [5], the authors prove among other things this inequality with $\log 2$ instead of the optimal γ.

Proof. Since $x \mapsto e^{\gamma x}/x$ is increasing when $\gamma x \ge 1$, while $x \mapsto \zeta(1+x)$ is decreasing, and $\zeta(1+\gamma^{-1}) \le e\gamma$, the inequality is proved for $x \ge 1/\gamma$.

By splitting the interval $[0,2]$ in $K+1 = 10001$ subintervals $[k/K,(k+1)/K]$ and checking numerically that $\zeta(1+2k/K) \le K\, e^{2\gamma(k+1)/K}/(2(k+1))$ we obtain that it fails when $k \le 632$ and succeeds otherwise: we have proved the inequality for $s > 1 + 2 \times 633/10000 = 1.1266$. We reiterate this process, but replacing the interval $[0,2]$ by the interval $[0, 2\times 633/10000]$ and prove the inequality for $s > 1 + 2\times 633/10^4 \times 4025/10^4$, and in particular for $s > 1.06$.

We should now prove the inequality in the vicinity of $s = 1$, for which we use Lemma 5.3. We find that

$$\zeta(1+\varepsilon) \le \frac{1}{\varepsilon} + \gamma - \gamma_1\varepsilon + 20\varepsilon^2$$

and we check that this is not more than $e^{\gamma\varepsilon}/\varepsilon$ when $\varepsilon \in [0, 1/10]$. The lemma follows readily. □

Lemma 5.5. *For $\sigma > 1$ and $X \ge 10^9$, we have*

$$\sum_{n\ge 1} \frac{\Bigl(\sum_{\substack{d|n,\\ d\le X}} \mu(d)\Bigr)^2}{\sigma n^\sigma} \le 0.529 \frac{e^{\gamma(\sigma-1)}\sigma}{\sigma-1}\log X.$$

Proof. Let $G(\sigma)$ be our sum. On expanding the square, we find that

$$G(\sigma) = \frac{\zeta(\sigma)}{\sigma} \sum_{d_1,d_2\le X} \frac{\mu(d_1)\mu(d_2)}{[d_1,d_2]^\sigma}.$$

We define for any σ the auxiliary function

$$\varphi_\sigma(d) = d^\sigma \prod_{p|d}(1-p^{-\sigma}) \tag{5.1}$$

which verifies $d^\sigma = (\mathbb{1} \star \varphi_\sigma)(d)$; here $\star$ denotes the usual convolution product. On using Lemma 5.1 and Selberg diagonalisation method, we get

$$G(\sigma)/\zeta(\sigma) = \frac{1}{\sigma}\sum_{\delta\le X} \varphi_{\sigma-1}(\delta)\Biggl(\sum_{\delta|d\le X}\frac{\mu(d)}{d^\sigma}\Biggr)^2 \le \sigma \sum_{\delta\le X}\frac{\mu^2(\delta)\varphi_\sigma(\delta)}{\delta^{2\sigma}}.$$

We readily check that $\varphi_\sigma(\delta)/\delta^{2\sigma} \le \varphi(\delta)/\delta^2$, since Lemma 5.2 establishes this fact on prime powers. We are now in a position to appeal to Lemma 3.3 and reach

$$G(\sigma) \le \sigma\zeta(\sigma)\bigl(0.4283\log X + 2.047 + 0.0194\bigr).$$

We conclude by appealing to Lemma 5.4. □

Lemma 5.6. *When $X \geq 10^9$, we have*

$$\sum_{X<n<5X} \frac{\Bigl(\sum_{\substack{d|n,\\ d\leq X}} \mu(d)\Bigr)^2}{n^2} \leq \frac{0.605}{X}$$

Proof. We compute separately the contributions arising from $n \in (X, 2X]$ and from $n \in (2X, 5X)$. When $n \in (X, 2X]$, the coefficient $\sum_{\substack{d|n,\\ d\leq X}} \mu(d)$ equals $-\mu(n)$, which means we have to bound above

$$S_1 = \sum_{X<n\leq 2X} \frac{\mu^2(n)}{n^2}.$$

We proceed by integration by parts, relying on Lemma 3.1:

$$\begin{aligned} S_1 &= \frac{\sum_{X<n\leq 2X} \mu^2(n)}{(2X)^2} + 2\int_X^{2X} \sum_{X<n\leq t} \mu^2(n) \frac{dt}{t^3} \\ &\leq \frac{\frac{6}{\pi^2}X + 0.1333(1+\sqrt{2})\sqrt{X}}{(2X)^2} + 2\int_X^{2X} \Bigl(\frac{6}{\pi^2}(t-X) + 0.1333(\sqrt{X}+\sqrt{t})\Bigr)\frac{dt}{t^3} \\ &\leq \frac{3}{\pi^2 X} + \frac{0.1333}{X^{3/2}}\biggl(1+\sqrt{2}+1-\frac{1}{4}+\frac{2}{3/2}\Bigl(1-\frac{1}{2^{3/2}}\Bigr)\biggr) \leq 0.304/X. \end{aligned}$$

When $n \in (2X, 5X)$, we readily see by inspecting all the possible cases that the coefficient $\sum_{\substack{d|n,\\ d\leq X}} \mu(d)$ takes values in $\{-1, 0, 1\}$. The only non-trivial cases is when n is divisible by 2 and 3, where the coefficient has value $-\mu(n) - \mu(n/2) - \mu(n/3)$. When $\mu(n) \neq 0$, the conclusion is straighforward, but otherwise we are left with $-\mu(n/2) - \mu(n/3)$. However, if both $\mu(n/2)$ and $\mu(n/3)$ do not vanish, then so does $\mu(n)$. It is thus enough to bound $S_2 = \sum_{2X<n<5X} 1/n^2$. We write simply

$$S_2 \leq \frac{1}{(2X)^2} + \int_{2X}^{5X} \frac{dt}{t^2} = \Bigl(\frac{1}{2} - \frac{1}{5} + \frac{1}{4X}\Bigr)\frac{1}{X} \leq 0.301/X.$$

□

6. Large sieve estimates and the like

Here is the classical large sieve inequality for primitive characters (see [36], [22, Lemma 1]) stated with notation (1.5):

Lemma 6.1. *We have*

$$\sum_{\substack{q\leq Q,\\ \chi \bmod^* q}}^{\star} \Bigl|\sum_{1\leq n\leq N} b_n \chi(n)\Bigr|^2 \leq (N - 1 + Q^2)\sum_n |b_n|^2.$$

Theorem 6.2. *Let $(u_n)_{n\geq 1}$ be a sequence of complex numbers. Let $k \geq 1$ be an integer parameter, and $c^* \geq 2/k$ and $T^* > 0$ be two real parameters. For any real* *numbers* $T \geq 0$ *we have*

$$\sum_{\substack{q \leq Q,\\ \chi \bmod^* q}}^{\star} \int_{-T}^{T} \Bigl|\sum_n u_n \chi(n) n^{it}\Bigr|^2 dt$$
$$\leq \beta_k^2 \biggl(\frac{\pi/c^*}{\sin\frac{2\pi}{c^* k}}\biggr)^{2k} \sum_{n \leq N} |u_n|^2 \Bigl(4\pi \frac{\sinh\frac{4\pi}{c^* T^*}}{4\pi/(c^* T^*)} n + c^* Q^2 \max(T^*, T)\Bigr)$$

where the positive constants β_k *are defined in* (6.5)*.*

When $c^* \geq 5.03$ and T^* is large enough, it is best to select $k = 1$. The choice $c^* = 12.5876$ leads to the following corollary:

Corollary 6.3. *We have, for* $T \geq 0$*:*

$$\sum_{\substack{q \leq Q,\\ \chi \bmod^* q}}^{\star} \int_{-T}^{T} \Bigl|\sum_n a_n \chi(n) n^{it}\Bigr|^2 dt \leq 7 \sum_n |a_n|^2 (n + Q^2 \max(T, 3)).$$

It is possible to diminish the constant in front of the $Q^2 T$-term at the cost of a higher one in front of the n-term. For instance, on selecting $c^* = 1.21$ and $k = 18$, we get

Corollary 6.4. *We have, for* $T \geq 0$*:*

$$\sum_{\substack{q \leq Q,\\ \chi \bmod^* q}}^{\star} \int_{-T}^{T} \Bigl|\sum_n a_n \chi(n) n^{it}\Bigr|^2 dt \leq \sum_n |a_n|^2 (43n + \tfrac{33}{8} Q^2 \max(T, 70)).$$

We follow the idea of [36, Corollary 3] but rely on [40] to get that

Lemma 6.5. *We have*

$$\int_0^T \Bigl|\sum_n u_n n^{it}\Bigr|^2 dt \leq \sum_{n \leq N} |u_n|^2 (2\pi c_0 \,(n+1) + T),$$

where $c_0 = \sqrt{1 + \frac{2}{3}\sqrt{\frac{6}{5}}}$*. Moreover, when* u_n *is real-valued, the constant* $2\pi c_0$ *may be reduced to* πc_0*.*

Proof. As noted in [36, last paragraph], we have $|\log(m/n)| \geq 1/(n+1)$ when m and n are distinct positive integers and it is thus a triviality to give explicit constants in [36, Corollary 3]. When the sequence (u_n) is real-valued, we write

$$\int_0^T \Bigl|\sum_n u_n n^{it}\Bigr|^2 dt = T \sum_n |u_n|^2 + \sum_{m \neq n} \frac{u_m m^{iT} \overline{u_n} n^{-iT}}{\log(m/n)} - \sum_{m \neq n} \frac{u_m \overline{u_n}}{\log(m/n)}.$$

This third summand vanishes identically when (u_n) is real-valued as shown by combining the pairs (m, n) and (n, m). The conclusion is straightforward. □

6.1. **Proof of Theorem 6.2, I: a generic proof.** We follow closely the proof of a lemma due to Gallagher (this is [22, Lemma 1], as well as [9, Theorem 9]). We first present a "generic" proof and choose the parameters later. Let F be a function to be chosen later. We assume that $F(t)=0$ as soon as $|t|\ge\eta$ for some parameter $\eta>0$. Let $\delta>0$ be a parameter that we shall also chose later. We define

$$F_\delta(x)=F(x/\delta).$$

Let us start from an arbitrary sequence of complex numbers (v_n) such that $\sum_n|v_n|<\infty$. We readily get

$$\sum_n v_n e^{2i\pi t(\log n)/(2\pi)}\hat F_\delta(t)=\sum_n v_n \widehat{F_\delta\Bigl(t-\frac{\log n}{2\pi}\Bigr)}.$$

Parseval identity yields

$$\begin{aligned}\int_{-\infty}^{\infty}\Bigl|\sum_n v_n e^{2i\pi t(\log n)/(2\pi)}\hat F_\delta(t)\Bigr|^2dt &= \int_{-\infty}^{\infty}\Bigl|\sum_n v_nF_\delta\Bigl(x-\frac{\log n}{2\pi}\Bigr)\Bigr|^2dx\\ &=\frac{1}{2\pi}\int_0^\infty\Bigl|\sum_n v_nF_\delta\Bigl(\frac{\log(y/n)}{2\pi}\Bigr)\Bigr|^2dy/y.\end{aligned}$$

Our hypothesis on the support of F implies that the y's in the relevant range verify $e^{-\eta\pi\delta}\le y/n\le e^{\eta\pi\delta}$. We apply the above to $v_n=u_n\chi(n)$, apply the large sieve inequality recalled in Lemma 6.1 and find that

$$\begin{aligned}&\sideset{}{^\star}\sum_{\substack{q\le Q,\\ \chi \bmod^* q}}\int_{-\infty}^{\infty}\Bigl|\sum_n u_n n^{it}\chi(n)\hat F_\delta(t)\Bigr|^2dt\\ &\le\frac{1}{2\pi}\int_0^\infty\sum_n|u_n|^2\Bigl|F_\delta\Bigl(\frac{\log(y/n)}{2\pi}\Bigr)\Bigr|^2\bigl(y(e^{2\eta\pi\delta}-e^{-2\eta\pi\delta})+Q^2\bigr)dy/y\\ &\le\frac{1}{2\pi}\sum_n|u_n|^2\int_0^\infty\Bigl|F_\delta\Bigl(\frac{\log u}{2\pi}\Bigr)\Bigr|^2\bigl(n(e^{2\eta\pi\delta}-e^{-2\eta\pi\delta})+Q^2u^{-1}\bigr)du.\end{aligned}$$

We change variable by setting $u=\exp(2\pi\delta w)$ and recall that the kernel function F is assumed to be even. The right-hand side is thus bounded above by

$$\begin{aligned}&\delta\sum_n|u_n|^2\int_{-\infty}^{\infty}|F(w)|^2\Bigl(n(e^{2\eta\pi\delta}-e^{-2\eta\pi\delta})e^{2\pi\delta w}+Q^2\Bigr)dw\\ &\le\delta\sum_n|u_n|^2\bigl(n(e^{2\eta\pi\delta}-e^{-2\eta\pi\delta})(e^{2\eta\pi\delta}+e^{-2\eta\pi\delta})+2Q^2\bigr)\int_0^\infty|F(w)|^2dw.\end{aligned}$$

Since $\hat F_\delta(t)=\delta\hat F(\delta t)$, we have finally reached

$$\begin{aligned}\sideset{}{^\star}\sum_{\substack{q\le Q,\\ \chi \bmod^* q}}\int_{-\infty}^{\infty}\Bigl|\sum_n u_n n^{it}\chi(n)\Bigr|^2|\hat F(\delta t)|^2dt&\\ \le\sum_n|u_n|^2\Bigl(n\frac{\sinh(4\eta\pi\delta)}{\delta}+Q^2\delta^{-1}\Bigr)\int_{-\infty}^{\infty}|F(w)|^2dw.&\end{aligned}\tag{6.1}$$

6.2. **Proof of Theorem 6.2, II: searching for a good kernel.** Now that we have this generic proof at our disposal, we seek to optimise the choice of the function F. We want $|\hat{F}(\delta t)| > 0$ when $|t| \le T$ as well as $F(t) = 0$ when $|t| \ge \eta$. The only regularity conditions are that F is even and belongs to $\mathrm{L}^2[-\eta, \eta]$. We obviously have $\hat{F}(y) = \int_{-\eta}^{\eta} F(t)e(yt)dt$ and we need to maximise

$$m(F, c, \eta) = \min_{|y| \le 1/c} |\hat{F}(y)|$$

where $c = 1/(\delta T)$. We assume further that $\int_{-\eta}^{\eta} |F(x)|^2 dx = 1$ since $\int_{-\infty}^{\infty} |\hat{F}(y)|^2 dy = \int_{-\eta}^{\eta} |F(t)|^2 dt$. This would imply, via (6.1) and provided that $m(F, c, \eta) > 0$, that

$$\sum_{\substack{q \le Q, \\ \chi \bmod^* q}}^{\star} \int_{-T}^{T} \Big| \sum_n u_n n^{it} \chi(n) \Big|^2 dt \le \frac{\sum_n |u_n|^2}{m(F, c, \eta)^2} \big(ncT \sinh \frac{4\eta\pi}{cT} + cQ^2 T \big). \tag{6.2}$$

We define $G(x) = \sqrt{\eta} F(\eta x) \in \mathrm{L}^2[-1, 1]$, which verifies $\int_{-1}^{1} |G(x)|^2 dx = 1$. Furthermore $\hat{G}(y) = \hat{F}(y/\eta)/\sqrt{\eta}$, and thus the right-hand side of (6.2) becomes

$$\frac{1}{m(G, c/\eta, 1)^2 \eta} \sum_n |u_n|^2 \big(ncT \sinh \frac{4\eta\pi}{cT} + cQ^2 T \big).$$

On setting $c^* = c/\eta$ we get

$$\sum_{\substack{q \le Q, \\ \chi \bmod^* q}}^{\star} \int_{-T}^{T} \Big| \sum_n u_n n^{it} \chi(n) \Big|^2 dt \le \frac{\sum_n |u_n|^2}{m(G, c^*, 1)^2} \big(nc^* T \sinh \frac{4\pi}{c^* T} + c^* Q^2 T \big). \tag{6.3}$$

By using Cauchy's inequality, we see that the condition $\int_{-1}^{1} |G(x)|^2 dx = 1$ implies that $m(G, c^*, 1)^2 \le 2$. When c^* tends to ∞, reducing $m(G, c^*, 1)$ to $\hat{G}(0)$, the bound $m(G, \infty, 1)^2 = 2$ is reached with the choice $G(x) = \mathbb{1}_{|x| \le 1}$. Let us now select a positive integer k and consider $g_k(x) = \mathbb{1}_{|x| \le 1/k}$. Its k-th convolution power $G_k = g_k^{(\star k)}/\beta_k$ verifies the condition support, is indeed even and we have

$$\hat{G}_k(y) = \left(\frac{\sin \frac{2\pi y}{k}}{\pi y} \right)^k / \beta_k.$$

The constant β_k is $\|g_k\|_2$. We use [24, (3.836), part 2] with $m = 0$ to get

$$\int_0^{\infty} \left(\frac{\sin x}{x} \right)^n dx = \frac{n\pi}{2^n} \sum_{0 \le \ell \le n/2} \frac{(-1)^\ell (n - 2\ell)^{n-1}}{\ell!(n - \ell)!}. \tag{6.4}$$

We infer from this formula that

$$\|g_k\|_2^2 = \int_{-\infty}^{\infty} \left(\frac{\sin \frac{2\pi y}{k}}{\pi y} \right)^{2k} dy = \left(\frac{2}{k} \right)^{2k} \frac{k}{2\pi} \frac{4k\pi}{2^{2k}} \sum_{0 \le \ell \le k} \frac{(-1)^\ell (2k - 2\ell)^{2k-1}}{\ell!(2k - \ell)!}$$

from which we infer that

$$\beta_k^2 = \|g_k\|_2^2 = \frac{2^{2k}}{k^{2k-2}} \sum_{0 \le \ell \le k-1} \frac{(-1)^\ell (k - \ell)^{2k-1}}{\ell!(2k - \ell)!}. \tag{6.5}$$

This gives

$$\beta_1^2 = \|g_1\|_2^2 = 2, \quad \beta_2^2 = \|g_2\|_2^2 = \frac{2}{3}, \quad \beta_3^2 = \|g_3\|_2^2 = \frac{88}{1215}.$$

Numerically, we find that $\beta_k^2 = \|g_k\|_2^2 \le (2/k)^{2k}\frac{\sqrt{k}}{2}$ and in fact may be asymptotic to this expression. If $c^* \ge 2/k$, which implies that $2\pi/(kc^*) \le \pi$, we have

$$m(G_k, c^*, 1)^{-2} = \beta_k^2 \biggl(\frac{\frac{\pi}{c^*}}{\sin\frac{2\pi}{c^* k}}\biggr)^{2k}. \tag{6.6}$$

This together with (6.3) ends the proof of Theorem 6.2: indeed, we first check that it is enough to prove the stated inequality for $T \ge T^*$ and the result then follows.

6.3. **Proof of Corollary 6.3.** Let us start by a remark. Numerically, we see that, given c^*, the sequence $(m(G_k, c^*, 1)^{-2})_k$ decreases and then increases. It has thus most probably a limit and the guess $\beta_k^2 \sim (2/k)^{2k}\frac{\sqrt{k}}{2}$ implies that the limit is infinity. As a consequence, letting k go to infinity is ruled out and we can only look for the best value of k. It is then straighforward to compute numerical values.

Concerning Lemma 6.3, we first took $c = 4\pi$ and rounding the constant in front of the Q^2T-term led to the value 7 with $k = 1$. Once this value 7 was set, we deacreased the value of c so as to maintain a constant not more than 7 in front of the Q^2T-term but decrease the constant in front of the n-term. We reached $c = 12.93$ is this manner. This process has been carried with $T^* = 1000$, and we finally check that it can be reduced to $T^* = 3$.

Here is the GP-script we have used:

```
{norme(k) = my(res = 0, sgn = 1);
   for(l = 0, k-1,res += sgn*(k-l)^(2*k-1)/l!/(2*k-l)!;
                  sgn = -sgn);
  return(res*2^(2*k)/(k^(2*k-2)));}

{ct(k,c) = my(u = Pi/c);
   return(norme(k)*(u/sin(2*u/k))^(2*k));}

{show(c, borneinf = 10, bornesup = 100, Tstar = 1000) =
   my(ct0, corr = sinh(4*Pi/c/Tstar)/4/Pi*c*Tstar);
   for(k = max(borneinf,ceil(2/c+0.00000000001)), bornesup,
      ct0 = ct(k,c);
      print(" k = ", k,
            " --> ( ", ct0*4*Pi*corr, ", ", ct0*c, ")"));}
```

The value $\frac{33}{8}$ is the smallest value with a denominator ≤ 10 we have been able to get in front of the Q^2T-term, with $T^\star = 1000$. The best actual value we have been able to reach is $4.121\cdots$ by taking $k = 18$ and $c = 1.21$. We then check it is possible to take $T^\star = 70$ (and even $T^\star = 62$ would do).

7. Usage of Theorem 6.2

Lemma 7.1. *We have, for $X \ge 10^9$, $X \ge 2\,000\,Q^2$, and $Q \ge 10$, $T \ge 0$,*

$$\sum_{\substack{q \le Q,\\ \chi \bmod^* q}}^{\star} \int_0^T |M_X(\tfrac12 + it, \chi)|^2 dt \le (1.36\,Q^2T + 0.692\,X)\log X.$$

We also have

$$\int_0^T |M_X(\tfrac12 + it, \mathbb{1})|^2 dt \le (0.77T + 0.126\,X)\log X.$$

Proof. From Corollary 6.4, Lemma 3.4 and 3.1, we readily get the first quantity to be not more than (note that the integration is between 0 and T and *not* between $-T$ and T)

$$\sum_{n\le X}\frac{\mu^2(n)}{n}(21.5n+\tfrac{33}{16}Q^2\max(T,70))$$
$$\le \tfrac{33}{16}Q^2\max(T,70)\Big(\frac{6}{\pi^2}\log X+1.048\Big)+21.5\times 0.62X.$$

This ensures, on taking into account the bound $X\ge 10^9$, that we have

$$\sideset{}{^\star}\sum_{\substack{q\le Q,\\ \chi \bmod^* q}}\int_0^T|M_X(\tfrac12+it,\chi)|^2dt\le (1.36\,Q^2\max(T,70)+0.644\,X)\log X.$$

Thus, when $T\le 70$, we have

$$\sideset{}{^\star}\sum_{\substack{q\le Q,\\ \chi \bmod^* q}}\int_0^T|M_X(\tfrac12+it,\chi)|^2dt\le (1.36\,Q^2\times 70+0.644\,X)\log X$$
$$\le\Big(\frac{1.36\times 70}{2000}+0.644\Big)X\log X.$$

Hence the lemma.

When considering only the principal character modulo 1, we can rely on Lemma 6.5, which gives us the bound

$$\sum_{n\le X}\frac{\mu^2(n)}{n}(4.2\,n+T+4.2)\le 0.77(T+4.2)\log X+4.2\times 0.62X.$$

□

Lemma 7.2. *We have, for $X\ge 10^9$, $X\ge 2\,000\,Q^2$, $Q\ge 10$ and $T\ge 0$,*

$$\tfrac12 F_Q(1/2,T)\le 1.40\,Q^{1/2}(Q^2T+0.51\,X)(2T^{1/4}\log(QT)+3\log Q)^2\log X.$$

When $Q=1$, we get:

$$\tfrac12 F_1(1/2,T)\le 4.45(T+0.164X)(T^{1/6}\log(T)+2.83)^2\log X.$$

Note that it is important that this lemma should hold for small T's as well. The method developped here is of course very elementary since we want to be able to compute all the involved constants, and has nothing in common with the technology developped for instance in [14].

Proof. On using (1.7), the Minkowski inequality together with Lemma 4.4, we readily see that

$$\sqrt{\tfrac12 F_Q(1/2,T)}\le\sqrt{A}+\sqrt{0.3\,TQ^2}$$

where

$$A=\sideset{}{^\star}\sum_{\substack{q\le Q,\\ \chi \bmod^* q}}\int_0^T|M_X(\tfrac12+it,\chi)|^2dt\max_{\substack{q\le Q,\\ \Re s=\frac12,|\Im s|\le T}}|L(s,\chi)|^2.$$

On appealing to Lemma 2.4 and 7.1, we reach the upper bound

$$A\le A_0=Q^{1/2}(1.36\,Q^2T+0.692\,X)(2T^{1/4}\log(QT)+3\log Q)^2\log X.$$

We notice that $2T^{1/4}\log(QT)+3\log Q = 8Q^{-1/4}x\log x+3\log Q$ with $x=(QT)^{1/4}$. The minimum of this quantity is obtained with $x=1/e$ and then $Q=10$ and has value ≥ 5.25. We check that $A_0 \ge 1.36Q^{5/2}T\cdot 5.25\log X \ge 918\cdot 0.3\,Q^2T$ and, as a consequence

$$\tfrac12 F_Q(1/2,T) \le (1+\sqrt{1/918})^2 1.36\,Q^{1/2}\times \\ (Q^2T+0.51\,X)(2T^{1/4}\log(QT)+3\log Q)^2\log X.$$

When $Q=1$, we start from $\sqrt{\tfrac12 F_1(1/2,T)} \le \sqrt{A}+1$ and use the second part of Lemma 2.4 and 7.1 to get the bound:

$$A \le A_0 = (2.4T^{1/6}\log(T)+6.8)^2(0.77T+0.126X)\log X.$$

We check that $T\mapsto 2.4T^{1/6}\log(T)+6.8$ stays greater than 1.5025, and then that $A_0 \ge 1.5025^2\cdot 0.126\cdot 10^9\cdot\log(10^9)\ge 10^9$ and thus

$$\tfrac12 F_1(1/2,T) \le (1+\sqrt{1/10^9})^2 2.403^2(T^{1/6}\log T+\tfrac{6.8}{2.403})^2(0.77T+0.126X)\log X.$$

We pulled out 2.403 and not 2.4 so that we get in front the nicely rounded coefficient 4.45 and inside the equally nicely rounded coefficient 2.83. □

Lemma 7.3. *We have, when $\delta = 1/\log X$, $X\ge 2000Q^2$, $X\ge 10^9$ and $T\ge 0$,*

$$\tfrac12 F_Q(1+\delta,T) \le \Big(1.18+0.155\frac{Q^2T}{X}\Big)(\log X)^2.$$

When $Q=1$, we get:

$$\tfrac12 F_1(1+\delta,T) \le \Big(1.40+0.0442\frac{T}{X}\Big)(\log X)^2.$$

Proof. We readily get from (1.7) and Lemma 6.3 the upper bound

$$\frac72\sum_{X<n}\Big(\sum_{\substack{d|n,\\ d\le X}}\mu(d)\Big)^2 n^{-2-2\delta}(n+Q^2\max(T,3)).$$

We note that, by Lemma 5.5, we have

$$\sum_{X<n}\Big(\sum_{\substack{d|n,\\ d\le X}}\mu(d)\Big)^2 n^{-1-2\delta} \le \frac{(1+2\delta)^2e^{2\gamma\delta}}{2\delta}0.529\log X \\ \le 0.337(\log X)^2$$

Concerning the second sum, we appeal to Lemma 5.6 together with a simple version of Rankin's trick[1] to bound above this quantity by

$$Q^2\max(T,3)\bigg(\frac{0.605}{X^{1+2\delta}}+\sum_{n\ge1}\Big(\sum_{\substack{d|n,\\ d\le X}}\mu(d)\Big)^2 n^{-2-2\delta}\Big(\frac{n}{5X}\Big)^{1+\delta}\bigg).$$

---

[1]The reason we use the exponent $1+\delta$ is the following: on using $1+k\delta$ say, we get Q^2T/X divided by $(2-k)X^{k\delta}$. With the choice $\delta=1/\log X$, we check that the maximum of this denominator is attained close to the point $k=1$. Hence our choice.

By Lemma 5.5, this is not more than

$$Q^2\max(T,3)\biggl(\frac{0.605}{X^{1+2\delta}}+\frac{(1+\delta)^2e^{\gamma\delta}}{5X^{1+\delta}\delta}0.529\log X\biggr)\\ \le 0.0442\frac{Q^2\max(T,3)}{X}(\log X)^2.$$

All of that gives us

$$\frac{7}{2}\biggl(0.337+0.0442\frac{Q^2\max(T,3)}{X}\biggr)(\log X)^2$$

When $T\le 3$, we find

$$\tfrac12 F_Q(1+\delta,T)\le\frac{7}{2}\Bigl(0.337+\frac{0.0442\times 3}{2000}\Bigr)(\log X)^2\le 1.18\,(\log X)^2$$

We include this contribution to our estimate by replacing $\frac72\times 0.337$ by 1.18. The first part of the lemma follows readily.

When considering only the principal character modulo 1, we can rely on Lemma 6.5 and get the upper bound

$$\sum_{X<n}\biggl(\sum_{\substack{d|n,\\ d\le X}}\mu(d)\biggr)^2 n^{-2-2\delta}(4.14\,n+T+4.14).$$

We proceed as above via Rankin's trick, after some steps similar to what has been done, we reach the bound

$$4.14\frac{(1+2\delta)^2e^{2\gamma\delta}}{2\delta}0.529\log X+\frac{T+4.14}{X}\biggl(\frac{0.605}{X^{2\delta}}+\frac{(1+\delta)^2e^{\gamma\delta}}{5X^{\delta}\delta}0.529\log X\biggr)$$

amounting to

$$\Bigl(1.393+0.0442\frac{T+4.14}{X}\Bigr)(\log X)^2.$$

The lemma follows readily. □

8. Computing some values of Γ and its derivatives

We shall require values of Γ and Γ' at special points. The values we require are tabulated in [1, Section 6, pages 253–277] and the values of the Γ-function may also be asked to GP/Pari, but some explanations are called for. We get to Γ' via $\Gamma'(s)=\psi(s)\Gamma(s)$, where ψ is the Digamma function which is well known. In particular it verifies $\psi(x+1)=\psi(x)+(1/x)$. There are ways to compute explicitely the values of the ψ-function at rational arguments (see Gauss's Formula), but we will simply use the `psi` function of Gp/Pari.

We proceed in a similar fashion for the trigamma function $\psi_1(x)=\psi'(x)$. It verifies $\psi_1(x+1)=\psi_1(x)-(1/x^2)$. Again some values are missing and we recall the following simplistic representation of ψ_1 that we used to compute $\psi_1(4/3)$:

$$\psi_1(x)=\sum_{n\ge 0}\frac{1}{(x+n)^2}. \tag{8.1}$$

This series converges rather slowly but we can use the `sumpos` function of Gp/Pari via `psi1(x) = sumpos( X = 0,` $1/(X+x)^2$ `)` to get excellent results instantly.

Here are the values we will need ($\Gamma'' = (\psi^2 + \psi_1)\Gamma$):

$$
\begin{array}{llll}
 & \Gamma(1) = 1 & & \\
 & \Gamma(7/6) = 0.927\cdots & \Gamma'(7/6) = -0.308\cdots & \\
 & \Gamma(4/3) = 0.892\cdots & \Gamma'(4/3) = -0.117\cdots & \Gamma''(4/3) = 0.993\cdots \\
 & \Gamma(3/2) = 0.886\cdots & \Gamma'(3/2) = 0.0323\cdots & \Gamma''(3/2) = 0.829\cdots \\
(8.2) & \Gamma(2) = 1 & & \\
 & \Gamma(13/6) = 1.08\cdots & \Gamma'(13/6) = 0.568\cdots & \\
 & \Gamma(9/4) = 1.14\cdots & \Gamma'(9/4) = 1.20\cdots & \\
 & \Gamma(7/3) = 1.19\cdots & \Gamma'(7/3) = 0.735\cdots & \Gamma''(7/3) = 1.08\cdots \\
 & \Gamma(5/2) = 1.32\cdots & \Gamma'(5/2) = 0.934\cdots & \Gamma''(5/2) = 1.30\cdots
\end{array}
$$

9. On the total number of zeroes

Here is a lemma we took from [35].

Lemma 9.1. *If χ is a Dirichlet character of conductor k, if $T \geq 1$ is a real number, and if $N(T,\chi)$ denotes the number of zeros $\beta + i\gamma$ of $L(s,\chi)$ in the rectangle $0 < \beta < 1$, $|\gamma| \leq T$, then*

$$\left|N(T,\chi) - \frac{T}{\pi}\log\left(\frac{qT}{2\pi e}\right)\right| \leq C_2\log(qT) \ + C_3$$

with $C_2 = 0.9185$ and $C_3 = 5.512$.

We recall that D. Platt in his thesis [39] has shown that no Dirichlet L-series with conductor $\leq Q_0 = 400\,000$ has no zeros of height $10^8/Q_0$ off the critical line and (his result is somewhat stronger). In particular $N(\sigma, 6, \chi) = 0$ whenever $\sigma > 1/2$.

Platt's result together with Lemma 9.1 imply that, on using the third part of Lemma 4.4 and provided when $Q \geq 10$ and $\sigma > 1/2$, we have

$$\sum_{\substack{q\leq Q,\\ \chi \bmod^* q}}^{\star} N(\sigma,6,\chi) \leq \Big(0.240\left(Q^2 - Q_0^2\right) + 2\left(Q^{3/2} + Q_0^{3/2}\right)\Big) \tag{9.1}$$

$$\times\left(\frac{6}{2\pi}\log\frac{6Q}{2\pi e} + \tfrac{1}{2}(0.92\log(6Q) + 5.6)\right)$$

$$\leq 0.273\, Q^2 \log Q. \tag{9.2}$$

The maximal value being about $0.2729167881609\cdots$ reached next to $Q = 4\,833\,287$.

10. Bounding $F_Q(T_2,\sigma) - F_Q(T_1,\sigma)$

This part contains the heart of the argument. Here are the results we prove in this section.

Lemma 10.1. *Let $T \geq T_1 \geq 2$ and Q be positive real parameters that verify $X \geq 10^9$, $X \geq 2\,000\, Q^2$ and $Q \geq 10$. When $X = Q^2T$, we have, for any $\sigma \in [1/2, 1]$:*

$$F_Q(T,\sigma) - F_Q(T_1,\sigma) \leq \frac{2.7}{0.367}(61Q^5T^3)^{1-\sigma}\log^{4-2\sigma}(Q^2T).$$

And here its counterpart concerning solely the principal character:

Lemma 10.2. *Let $T \geq 3\cdot 10^{10}$ and $\sigma \in [1/2, 1]$. We have*

$$F_1(T,\sigma) - F_1(6,\sigma) \leq \frac{2.91}{0.367}(2.77\,T)^{8(1-\sigma)/3}\log^{4-2\sigma}(T).$$

Both proofs are rather easy in principle: we majorize $F_Q(T,\sigma) - F_Q(T_1,\sigma)$ by a smoother quantity (replacing the cutoff at T by essentially an exponential smoothing). This is done at subsection 10.5. We evaluate this smoother version by a convexity argument which we develop at Subsection 10.1. In order to apply the resulting bound, we need to bound the smoothed version on $\sigma = 1/2$ (this is subsection 10.2) and on $\sigma = 1+\delta$ for some small δ (this is subsection 10.3). This last part is where the fact that the coefficients of the Dirichlet series f_X vanish at the beginning will be used.

10.1. **A convexity argument.** To evaluate $\int_{T_1}^{T_2} \sum^\star_{\substack{q\le Q,\\ \chi \bmod^* q}} |f_X(\sigma_0+it,\chi)|^2 dt$, we use a slight extension convexity argument due to [26]. We first are to evaluate this integral in $\frac12$ and in $1+\delta$. We set

$$\Phi(s) = \frac{s-1}{s(\cos s)^{1/(2\tau)}} \qquad \Re s \in [\tfrac12, 1+\delta] \tag{10.1}$$

for some parameter $\tau \ge 2\,000$ that we will at the end take to be T_2. Here $\delta = 1/\log(Q^2T_2)$. The function $s \mapsto \cos s$ does not vanish in the strip we consider since $|\cos(\sigma+it)|^2 = (\cos\sigma)^2 + (\sinh t)^2$. The factor $s-1$ is to take care of the pole of ζ at $s=1$, and its growth is compensated by the $1/s$. The $(\cos s)^{1/2\tau}$ is here so that $\Phi(s) f_X(s,\chi) = o(1)$ uniformly in $\Re s$ and as $|\Im s|$ goes to infinity while giving enough weight to the s with $|\Im s|$ between 0 and T. Let us set

$$a = \frac{1+\delta-\sigma}{1+\delta-\frac12} \quad , \quad b = \frac{\sigma-\frac12}{1+\delta-\frac12}. \tag{10.2}$$

A slight extension of the Hardy-Ingham-Pólya inequality which we prove thereafter reads

$$\mathfrak{M}_Q(\sigma) \le \mathfrak{M}_Q(1/2)^a \mathfrak{M}_Q(1+\delta)^b \tag{10.3}$$

with

$$\mathfrak{M}_Q(\sigma) = \int_{-\infty}^{\infty} \sum^\star_{\substack{q\le Q,\\ \chi \bmod^* q}} |\Phi(\sigma+it) f_X(\sigma+it,\chi)|^2 dt. \tag{10.4}$$

The extension comes from the fact that we have added a summation over characters instead of considering a single function.

Proof of (10.3). Indeed we follow closely [51, section 7.8] and set

$$\phi(z,\chi) = \frac{1}{2i\pi}\int_{\sigma-i\infty}^{\sigma+i\infty} \Phi(s) f_X(s,\chi) z^{-s} ds \qquad (\sigma \ge 1/2, |\arg z| < \pi/2). \tag{10.5}$$

Setting $z = ixe^{-i\delta}$ with $0<\delta<\pi/2$, we readily see that

$$\Phi(\sigma+it) f_X(\sigma+it,\chi) e^{-i(\sigma+it)(\frac12\pi-\delta)} \quad \text{and} \quad \phi(ixe^{-i\delta},\chi)$$

are a Mellin pair. Using Parseval's formula and Hölder inequality, we obtain:

$$\begin{aligned}\mathfrak{M}_Q(\sigma) &= 2\pi\int_0^\infty \sideset{}{^\star}\sum_{\substack{q\le Q,\\ \chi \bmod^* q}} |\phi(ixe^{-i\delta},\chi)|^2 x^{2\sigma-1}dx \\ &\le 2\pi\Big(\int_0^\infty \sideset{}{^\star}\sum_{\substack{q\le Q,\\ \chi \bmod^* q}} |\phi(ixe^{-i\delta},\chi)|^2 dx\Big)^a\Big(\int_0^\infty \sideset{}{^\star}\sum_{\substack{q\le Q,\\ \chi \bmod^* q}} |\phi(ixe^{-i\delta},\chi)|^2 x^{1+2\delta}dx\Big)^b \\ &\le \mathfrak{M}_Q(1/2)^a\mathfrak{M}_Q(1+\delta)^b.\end{aligned}$$

□

Lemma 10.3. *We have* $|\cos(\sigma+it)|\ge|\cos\sigma\sin\sigma|\,e^{|t|}$.

Proof. We have $G=e^{-2|t|}|\cos(\sigma+it)|^2=(\cos\sigma)^2e^{-2|t|}+(e^{-|t|}\sinh t)^2$. We develop $(e^{-|t|}\sinh|t|)^2$ in $(1-2x+x^2)/4$ with the notation $x=e^{-2|t|}$. As a result we find that $4G=4(\cos\sigma)^2x+1-2x+x^2$: this is a convex quadratic polynomial in x. It thus takes its minimum at the unique zero $x=1-2\cos^2\sigma$ of its derivative; this minimum is $4\cos^2\sigma-4\cos^4\sigma$, i.e. $4|\cos\sigma\sin\sigma|^2$. Our inequality is proved. □

We now exploit inequality (10.3), still following [51, section 7.8], on appealing to Lemma 10.3 as well as $|(s-1)/s|\le 1$ when $1/2\le\Re s\le 1+\delta\le 3/2$. We bound above the RHS of (10.4) via

$$\mathfrak{M}_Q(\sigma)\le\Big(\frac{1}{\cos\sigma\sin\sigma}\Big)^{1/\tau}\int_{-\infty}^{\infty}e^{-|t|/\tau}\sideset{}{^\star}\sum_{\substack{q\le Q,\\ \chi \bmod^* q}}|f_X(\sigma+it,\chi)|^2dt.$$

On recalling (1.7), we see that an integration by parts give us

$$\begin{aligned}\mathfrak{M}_Q(\sigma) &\le \Big(\frac{1}{\cos\sigma\sin\sigma}\Big)^{1/\tau}\int_0^\infty e^{-T/\tau}F_Q(\sigma,T)dT/\tau \\ &\le \Big(\frac{2}{\sin 2\sigma}\Big)^{1/\tau}\int_0^\infty e^{-t}F_Q(\sigma,t\tau)dt\end{aligned}$$

10.2. **An upper bound for $\mathfrak{M}_Q(1/2)$.**

Lemma 10.4. *Let T and Q be positive real parameters that verify $Q^2T\ge 10^9$, $X\ge 2\,000\,Q^2$ and $Q\ge 10$. With the choice $X=Q^2T$ and $\tau=T$, we have*

$$\mathfrak{M}_Q(1/2)\le 21(Q^5T^3)^{1/2}\log^3(Q^2T).$$

Proof. We appeal to Lemma 7.2 to infer that $\mathfrak{M}_Q(1/2)/(2\times 1.40\sqrt{Q}\log X)$ is bounded above by

$$\int_0^\infty \mathfrak{N}_Q(t)e^{-t}dt$$

where $\mathfrak{N}_Q(t)$ is $(Q^2\tau t+0.51X)(2\tau^{1/4}t^{1/4}\log(Q\tau)+2\tau^{1/4}t^{1/4}\log t+3\log Q)^2$ i.e.

$$\begin{aligned}&4((\log t)^2+2(\log Q\tau)\log t)+(\log Q\tau)^2)Q^2\tau^{3/2}t^{3/2}\\ &+12(\log Q)((\log t)+(\log Q\tau))Q^2\tau^{5/4}t^{5/4} \qquad +9(\log Q)^2Q^2\tau t\\ &+2\times 0.51((\log t)^2+2(\log Q\tau)(\log t)+(\log Q\tau)^2)X\tau^{1/2}t^{1/2} \qquad +9\times 0.51(\log Q)^2X\end{aligned}$$

Note that in this proof, we keep X and τ independant of T and Q until the integration has been done. On using values of Γ, Γ' and Γ'' (see section 8), we get the bound

$$\begin{array}{ll} 4(1.31+2(\log Q\tau)0.935+1.33(\log Q\tau)^2)Q^2\tau^{3/2} & \\ +12(\log Q)(1.21+1.15(\log Q\tau))Q^2\tau^{5/4} & +9(\log Q)^2Q^2\tau \\ +1.02(0.830+2\times 0.0324(\log Q\tau)+(\log Q\tau)^2 0.887)X\tau^{1/2} & +4.49(\log Q)^2X \end{array}$$

We now take $X = Q^2T$ and $\tau = T$. We use the inequalities $\log Q \le \log T$, $\log Q \le (1/3)\log(Q^2T)$ and $\log(Q\tau) \le \log(Q^2T)$. Here is the upper bound we get for $\mathfrak{M}_Q(1/2)/(2.8\,Q^{5/2}T^{3/2}\log^3(Q^2T))$

$$\begin{array}{ll} 4(\frac{1.31}{\log^2(Q^2T)}+2\frac{0.935}{\log Q^2T}+1.33) & \\ +\frac{12}{3}(\frac{1.21}{\log Q^2T}+1.15)T^{-1/4} & +T^{-1/2} \\ +0.54(\frac{0.830}{\log^2(Q^2T)}+2\frac{0.0324}{\log Q^2T}+0.887) & +\frac{2.43}{9}T^{-1/2} \end{array}$$

which simplifies into the claimed quantity since $\log(Q^2T) \ge 9\log(10)$. □

And here is the counterpart corresponding to the case $Q = 1$.

Lemma 10.5. *Let $T \ge 3\cdot 10^{10}$. On selecting $X = \tau = T$, we have*

$$\mathfrak{M}_1(1/2) \le 11.3\,T^{4/3}\log^3(T).$$

Proof. We appeal to Lemma 7.2 to infer that $\mathfrak{M}_1(1/2)/(2\times 4.45\log X)$ is bounded above by

$$\int_0^\infty \mathfrak{N}_1(t)e^{-t}dt$$

where $\mathfrak{N}_1(t)$ is $(t\tau + 0.164X)((t\tau)^{1/6}\log\tau + (\tau t)^{1/6}\log t + 2.83)^2$, i.e.

$$\begin{array}{ll} ((\log\tau)^2+2(\log t)(\log\tau)+(\log t)^2)\tau^{4/3}t^{4/3} & \\ +5.66(\log\tau+\log t)\tau^{7/6}t^{7/6} & +8.0089\tau t \\ +0.164(X(\log\tau)^2+2X(\log t)(\log\tau)+X(\log t)^2)\tau^{1/3}t^{1/3} & \\ +0.92824(X\log\tau+X\log t)\tau^{1/6}t^{1/6} & +1.3134596X \end{array}$$

Note again that in this proof, we keep X and τ independant of T until the integration has been done. On using values of Γ, Γ' and Γ'' (see section 8), and substituting $X = \tau = T$. we get the bound

$$\begin{array}{ll} (1.20(\log T)^2+1.472\log T+1.09)T^{4/3} & \\ +5.66(0.569\log T+1.09)T^{7/6} & +8.0089T \\ +0.164(0.893(\log T)^2-0.117\log T+0.0994)T^{1/3} & \\ +0.92824(0.927\log T-0.308)T^{1/6} & +1.3134596T \end{array}$$

Here is the upper bound we get when $T \ge 3\cdot 10^{10}$

$$\mathfrak{M}_1(1/2) \le 11.3\,T^{4/3}\log^3(T).$$

□

10.3. **An upper bound for $\mathfrak{M}_Q(1+\delta)$.** We appeal to Lemma 7.3 (and recall that $\delta = 1/\log X$) to infer that

$$\begin{aligned}\mathfrak{M}_Q(1+\delta) &\leq 2\int_0^\infty \left(1.18 + 0.155Q^2 t\tau X^{-1}\right) e^{-t} dt (\log X)^2 \\ &\leq 2\left(1.18 + 0.155Q^2 TX^{-1}\right)(\log X)^2 \leq 2.67\log^2(Q^2T).\end{aligned}$$

In case $Q = 1$, we get

$$\begin{aligned}\mathfrak{M}_1(1+\delta) &\leq \int_0^\infty 2\left(1.41 + 0.0443\frac{t\tau}{eX}\right) e^{-t} dt (\log X)^2 \\ &\leq 2.909(\log T)^2.\end{aligned}$$

10.4. **An upper bound for $\mathfrak{M}_Q(\sigma)$.** We thus conclude via (10.3) that (note that $b = 1 - a$)

$$\begin{aligned}\mathfrak{M}_Q(\sigma) &\leq \left(21(Q^5T^3)^{1/2}\log^3(Q^2T)\right)^a \left(2.7\log^2(Q^2T)\right)^b \\ &\leq 2.7\,(21/2.7)^a\,(Q^5T^3)^{a/2}\log^{2+a}(Q^2T).\end{aligned}$$

We note the inequality

$$(21/2.7)\frac{\sqrt{Q^5T^3}}{\log(Q^2T)} = (21/2.7)\sqrt{QT}\frac{Q^2T}{\log(Q^2T)} \geq 1.$$

This enables us to use monotonicity in the exponent:

$$\left(\frac{21Q^{5/2}T^{3/2}\log^2(Q^2T)}{2.7}\right)^{1-\frac{\sigma-\frac12}{1+\delta-\frac12}} \leq \left(\frac{21Q^{5/2}T^{3/2}\log^2(Q^2T)}{2.7}\right)^{1-\frac{\sigma-\frac12}{1-\frac12}}.$$

Hence the bound:

$$\boxed{\mathfrak{M}_Q(\sigma) \leq 2.7\big(61Q^5T^3\big)^{1-\sigma}\log^{4-2\sigma}(Q^2T).}$$

Let us now prove the counterpart of this bound when $Q = 1$.

$$\begin{aligned}\mathfrak{M}_Q(\sigma) &\leq \left(11.3\,T^{4/3}\log^3(T)\right)^a \left(2.909\log^2(T)\right)^b \\ &\leq 2.91\,(11.3/2.91)^a\,T^{4a/3}\log^{2+a}(T).\end{aligned}$$

We proceed as above to simplify this bound into

$$\boxed{\mathfrak{M}_Q(\sigma) \leq 2.91\,(2.77\,T)^{8(1-\sigma)/3}\log^{4-2\sigma}(T)} \tag{10.6}$$

10.5. **End of the proof of Lemma 10.1 and 10.2.** We first notice that

$$\left|\frac{s-1}{s(\cos s)^2}\right| = \left|1 - \frac{1}{s}\right|\frac{1}{\cos^2\sigma + \sinh^2 t} \geq \left(1 - \frac{1}{|t|}\right)\frac{1}{1+\sinh^2|t|}.$$

The derivative of the right-hand-side is $(1 + \sinh^2 t - 2(t^2 - t)\sinh t\cosh t)/(t(1 + \sinh^2 t))^2$ which is negative when $t \geq 2$. The assumption $T_1 \geq 2$ comes here into play. We deduce from the above inequality that

$$\min_{T_1\leq|t|\leq T_2}\frac{|s-1|^2}{|s|^2|\cos s|^{1/\tau}} \geq \frac{(1-\frac{1}{T_2})^2}{(1+\sinh^2 T_2)^4}\frac{1}{\sqrt{1+\sinh^2 T_2}^{\,\tau^{-1}-4}}.$$

We continue by (with $T_2 = \tau = T \geq 2000$)

$$\left(1 - \frac{1}{T}\right)^2 \frac{1}{(1+\sinh^2 T)^{2/T}} \geq \left(1 - \frac{1}{T}\right)^2 \frac{2.00001^{1/T}}{e} \geq 0.367.$$

This leads to

$$\mathfrak{M}_Q(\sigma) \geq 0.367 \int_{T_1 \leq |t| \leq T_2} \sum_{\substack{q \leq Q, \\ \chi \bmod^* q}}^{\star} |f_X(\sigma + it, \chi)|^2 dt. \tag{10.7}$$

11. The Zero Detection Lemma and proof of Theorem 1.1

11.1. **$N_1(\sigma_1, 6, T, \chi)$: from a pointwise version to an averaged one.** We use $\sigma_0 = \sigma_1 - 1/(3\log(Q^2T))$ and write

$$N_1(\sigma_1, 6, T, \chi) \leq \int_{\sigma_0}^{\sigma_1} N_1(\sigma, 6, T, \chi) d\sigma/(\sigma_1 - \sigma_0).$$

We have to note here that the condition $\sigma \geq 0.65$ of Theorem 1.1 ensures that $\sigma_0 > 1/2$. Indeed the parameter σ from this theorem is σ_1.

11.2. **From the averaged version to $F_Q(T,\sigma) - F_Q(6,\sigma)$: character per character.** In 1914, H. Bohr and E. Landau proved in [8] for the first time that the number of zeroes off the critial line but within the critical strip is negligible when compared to the total number of zeroes. Their argument was qualitative and H.-E. Littlewood made it quantitative in [32]. We follow this approch as reproduced in [51, section 9.9]. For $\sigma_0 \in [\frac{1}{2}, 1]$, we have

(11.1)

$$\begin{aligned} 2\pi \int_{\sigma_0}^{2} N_1(\sigma, T_1, T_2, \chi) d\sigma = &\int_{T_1}^{T_2} \big(\log|g_X(\sigma_0 + it, \chi)| - \log|g_X(2+it,\chi)|\big) dt \\ &+ \int_{\sigma_0}^{2} \big(\arg g_X(\sigma + iT_2, \chi) - \arg g_X(\sigma + iT_1, \chi)\big) d\sigma \end{aligned}$$

where $\arg g_X(s,\chi)$ is taken to be 0 on the line $\Re s = 2$.

There are two ways of studying the first integral. They both start by noticing

$$\log|h_X(s,\chi)| \leq \log(1 + |f_X(s,\chi)|^2). \tag{11.2}$$

The usual fashion is to continue by the inequality $\log(1+|f_X(s,\chi)|^2) \leq |f_X(s,\chi)|^2$. We can however also appeal to the Jensen inequality and use instead:

$$\begin{aligned} \int_{T_1}^{T_2} \log|g_X(\sigma_0 + it, \chi)| dt &\leq \int_{T_1 \leq |t| \leq T_2} \sum_{\substack{q \leq Q, \\ \chi \bmod^* q}}^{\star} \log(1 + |f_X(s,\chi)|^2) dt \\ &\leq W \log\left(1 + \frac{1}{W} \int_{T_1 \leq |t| \leq T_2} \sum_{\substack{q \leq Q, \\ \chi \bmod^* q}}^{\star} |f_X(s,\chi)|^2 dt\right) \end{aligned}$$

with $W = 2(T_2 - T_1) \sum_{\substack{q \leq Q, \\ \chi \bmod^* q}}^{\star} 1$. This inequality is increasing in W and we can take for W and upper bound for the stated value. And in fact, when W tends to infinity, we reach the former inequality. We will use this variation when $Q = 1$, with $W = 2T_2$.

Concerning the other summand in (11.1), we note that

$$-\log|h_X(2+it,\chi)| \leq -\log(1-|f_X(2+it,\chi)|^2) \leq 2|f_X(2+it,\chi)|^2 \tag{11.3}$$

provided $|f_X(2+it,\chi)|^2 \leq 1/2$ which we prove now:

$$\begin{aligned} |f_X(2+it,\chi)| &\leq \sum_{n\geq X}\frac{|\sum_{d|n}\mu(d)|}{n^2} \leq \sum_{n\geq X}\frac{2^{\omega(n)}}{n^2} \\ &\leq \sqrt{8/3}\sum_{n\geq X}\frac{1}{n^{3/2}} \leq \frac{2\sqrt{8/3}}{(X-1)^{1/2}} \leq 1/\sqrt{2} \end{aligned}$$

since $X \geq 2000$ and $2^{\omega(n)} \leq \sqrt{8/3}\sqrt{n}$ (use multiplicativity).

Bounding the argument. Getting an upper bound for the argument is more tricky and relies on the following lemma from [51, section 9.4]:

Lemma 11.1. *Let $0 \leq \alpha < \beta \leq 2$ and F be an analytical function, real for real s, holomorphic for $\sigma \geq \alpha$ except maybe at $s = 1$. Let us assume that $|\Re F(2+it)| \geq m > 0$ and that $|F(\sigma'+it')| \leq M$ for $\sigma' \geq \sigma$ and $T \geq t' \geq T_0 - 2$. Then, if $T - 2 \geq T_0$ is not the ordinate of a zero of $F(s)$, we have*

$$|\arg F(\sigma+iT)| \leq \frac{\pi}{\log\frac{2-\alpha}{2-\beta}}\log\bigl(M/m\bigr) + \frac{3\pi}{2}$$

valid for $\sigma \geq \beta$.

The condition concerning the *ordinate* comes from the way we define the logarithm, and hence the argument. It is usually harmless since one can otherwise argue by continuity at the level of the resulting bound.

We use this lemma with $\alpha = 0$, $\beta = 1/2$ and $F = g_X(s,\chi)$ which is indeed real on the real axis. We already showed that

$$|\Re g_X(2+it,\chi)| \geq (1-|f_X(2+it,\chi)^2|)(1-|f_X(2+it,\overline{\chi})^2|) \geq (1-0.214^2)^2 \geq 0.91.$$

Hence, for $j = 1, 2$, on using Lemma 2.3 together with Lemma 3.1 to bound $|M_X(s)|$ by $0.62\,X$, we find that

$$|\arg g_X(\sigma+iT_j,\chi)| \leq 11\times 2\log\bigl(1+(0.62(qT_j)^{5/8}X)^2\bigr) + 17.$$

The use of this lemma asks for $T_1 = 4 + 2$ (the smallest value available). Since we fix this value, we can dispense with the index in T_2 and denote it by T. We continue as follows, since $X = Q^2T$:

$$\begin{aligned} &|\arg g_X(\sigma+iT,\chi)| + |\arg g_X(\sigma+6i,\chi)| \\ &\qquad \leq 22\log\bigl(1+(0.62(QT)^{5/8}Q^2T)^2\bigr) + 22\log\bigl(1+(0.62(6Q)^{5/8}Q^2T)^2\bigr) + 34. \end{aligned}$$

We have to compare this quantity with $\log(Q^2T)$, knowing that $Q^2T \geq 10^9$ and $T \geq 2000$. We use

$$\begin{aligned} \log\bigl(1+(0.62(QT)^{5/8}Q^2T)^2\bigr) &\leq \frac{13}{4}\log(Q^2T) + \log\bigl(0.62^2 + 10^{-9\times 13/4}\bigr) \\ &\leq \frac{13}{4}\log(Q^2T) - 0.956 \end{aligned}$$

and

$$\begin{aligned}\log\bigl(1+(0.62(6Q)^{5/8}Q^2T)^2\bigr) &\le \log\bigl(1+(0.62(\tfrac{6}{2000}TQ)^{5/8}Q^2T)^2\bigr)\\ &\le \log\bigl(1+(0.62(\tfrac{6}{2000}TQ^2)^{5/8}Q^2T)^2\bigr)\\ &\le \frac{13}{4}\log(Q^2T)+\log\bigl(10^{-9\times 13/4}+(0.62(\tfrac{6}{2000})^{5/8})^2\bigr)\\ &\le \frac{13}{4}\log(Q^2T)-8.21.\end{aligned}$$

This finally amounts to

$$|\arg g_X(\sigma+iT,\chi)|+|\arg g_X(\sigma+6i,\chi)|\le 143\log(Q^2T)-167.$$

We will multiply this bound by $3/2$ to take care of the integration over σ in $[\sigma_0,2]$ in (11.1).

Partial conclusion. Since $|f_X(2+it)|\le 1/(X-1)$, we get for $\sigma_0\ge 1/2$

$$2\pi\int_{\sigma_0}^{2}N_1(\sigma,6,T,\chi)d\sigma\le\int_6^T\bigl(|f_X(\sigma_0+it,\chi)|^2+|f_X(\sigma_0+it,\overline{\chi})|^2\bigr)dt + \frac{4(T-6)}{X-1}+215\log(Q^2T)-250. \tag{11.4}$$

We have been careful not to use the bound $Q\ge 10$ up to now to cover the two cases $Q=1$ but $T\ge 3\cdot 10^{10}$, and $Q\ge 10$, $T\ge 2000$ and $Q^2T\ge 10^9$. We now have to distinguish both cases as the estimate from Lemma 4.4 requires a bound on Q.

11.3. **From the averaged version to $F_Q(T,\sigma)-F_Q(6,\sigma)$: summing over characters.** We sum (11.4) over q, use Lemma 4.4, and join the two previous steps. We get

$$\sum_{\substack{q\le Q,\\ \chi \bmod^* q}}^{\star} N_1(\sigma_1,6,T,Q)\le\int_6^T\sum_{\substack{q\le Q,\\ \chi \bmod^* q}}^{\star}|f_X(\sigma_0+it,\chi)|^2dt\,\frac{3\log(Q^2T)}{\pi} + 31\,Q^2\log(Q^2T).$$

But one should be careful: the variable t ranges positive values only while it ranges a symmetric interval in $F_Q(T,\sigma)-F_Q(6,\sigma)$.

11.4. **Using Lemma 10.1.** We finally use $X=Q^2T$, $Q\ge 10$ and $T\ge 2\,000$ and Lemma 10.1 to get

$$N_1(\sigma_1,6,T,Q)\le\frac{2.7\times 3}{0.367\times 2\pi}\bigl(61Q^5T^3\bigr)^{1-\sigma_0}\log^{5-2\sigma_0}(Q^2T)+31\,Q^2\log(Q^2T).$$

We have to replace σ_0 by σ_1. We define $\delta_1=1/(3\log(Q^2T))$ and note that, with $x=\log(Q^2T)$, we have

$$\bigl(61\,Q^5T^3\bigr)^{\delta_1}\log^{2\delta_1}(Q^2T)\le 61^{\delta_1}\exp\left(3x\frac{1}{3x}+(\log x)\frac{2}{3x}\right)\le 5.44.$$

All of that amounts to

$$\begin{aligned}N_1(\sigma_1,6,T,Q)&\le 19.2\bigl(61\,Q^5T^3\bigr)^{1-\sigma_1}\log^{5-2\sigma_1}(Q^2T)+31\,Q^2\log^2(Q^2T).\\ &\le 20\bigl(56\,Q^5T^3\bigr)^{1-\sigma_1}\log^{5-2\sigma_1}(Q^2T)+31\,Q^2\log^2(Q^2T).\end{aligned}$$

We simplify and use (9.2) to get the first part of Theorem 1.1.

11.5. **Using Lemma 10.2.** Here is the counterpart when $Q = 1$. We combine (11.4) together with Lemma 10.2 to get

$$N_1(\sigma_1, 6, T, 1) \le \frac{2.91 \times 3}{0.367 \times 2\pi}(2.77\,T)^{8(1-\sigma_0)/3}\log^{5-2\sigma_0}(T) + 103\,(\log T)^2.$$

Hence

$$N_1(\sigma_1, 6, T, 1) \le 9.72(3T)^{8(1-\sigma_1)/3}\log^{5-2\sigma_1}(T) + 103(\log T)^2.$$

If we are to use the variation implying the Jensen inequality, we reach

$$N_1(\sigma_1, 6, T, 1) \le 2T\log\left(1 + \frac{9.72}{2T}(3T)^{\frac{8}{3}(1-\sigma_1)}\log^{5-2\sigma_1}(T)\right) + 103(\log T)^2.$$

The main theorem follows readily.

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Laboratoire Paul Painlevé / CNRS, Université Lille 1, 59655 Villeneuve d'Ascq, France

E-mail address: ramare@math.univ-lille1.fr