AN EXPLICIT CROOT-ŁABA-SISASK LEMMA FREE OF PROBABILISTIC LANGUAGE

OLIVIER RAMARÉ

Abstract. We provide an explicit and probabilistic language-free proof of the famous Croot-Łaba-Sisask Lemma. In between, we do the same for the Khintchine and Marcinkiewicz-Zygmund inequalities and explicitate the implied constants for the upper bounds.

1. Introduction

After the fundatory papers of H. Rademacher [10] in 1922 and of A. Khintchine [8] in 1923, the usage of the so-called Rademacher system of functions, described thereafter, has known deep developments in L^p-space theory, then in Banach space theory, harmonic analysis and operator theory, for instance with the introduction of the Rademacher type and cotype. The n-th Rademacher function r_n is simply the function on $[0,1]$ that takes the value 1 at t when the integer part of $2^n t$ is even and -1 otherwise. They were introduced by Rademacher in [10, Part VI] in an L^2-setting and by Khintchine in [8, Section 1] in an L^p-setting. It turns out that this alternation of the ± 1 values is deeply connected with sums of Bernoulli variables and this introduces probability theory. We refer the reader to the book [1] by S. Astashkin. As the material stated in such a fashion may be difficult to grasp for a non probabilist, we propose here a fully elementary presentation of some part of it, where elementary means that any generic mathematical background should do.

Our main aim is to provide a proof of (a variant of) [3, Lemma 3.2] by E. Croot, I. Łaba & O. Sisask. We state this result in their notation and in particular, when z is a complex number, z° is defined to be $z/|z|$ when $z \neq 0$, and 0 when $z = 0$.

Theorem 1. *Let (X, μ) be a probability space, let $p \geq 2$ and a function f given in the form*

$$f = \sum_{k \leq K} \lambda_k g_k$$

where $(g_k)_k$ is a collection of measurable functions on X of $L^p(\mu)$-norm at most 1. Let finally $\varepsilon > 0$. There exists an L-tuple $(k_1, \cdots, k_L) \in \{1, \cdots, K\}^L$ of length $L \leq 20p/\varepsilon^2$ such that

$$\int_X \left| \frac{f(x)}{\|\lambda\|_1} - \frac{1}{L} \sum_{\ell \leq L} \lambda^\circ_{k_\ell} g_{k_\ell}(x) \right|^p d\mu \leq \varepsilon^p,$$

where $\|\lambda\|_1 = \sum_{k \leq K} |\lambda_k|$.

Thus L can be taken uniformly bounded, whatever rate of convergence (with respect to K) of the initial representation of f. This theorem has its origin in the paper [4] by E. Croot & O. Sisask. Since the Croot-Łaba-Sisask Lemma has

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important consequences, we thought it was worth presenting an elementary and self-contained proof of it.

We prove the upper Khintchine Inequality in Theorem 2 and the Marcinkiewicz-Zygmund Inequality in Theorem 3. We refer to the paper [9] by L. Pierce for more refined background on the Khintchine and Marcinkiewicz-Zygmund inequalities. Our treatment is far from being comprehensive and we should mention to the readers another important tool in this landscape: the Kahane-Salem-Zygmund Inequality, see for instance [5] by A. Defant & M. Mastyło and [11, Section 4] by A. Raposo, Jr. & D. Serrano-Rodríguez. The followings proofs borrow from several authors.

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2. An upper explicit Khintchine Inequality

Here is the main result of this section.

Theorem 2. *We have, when* $p \geq 1$,

$$(1/2^N) \sum_{(\varepsilon_n)\in\{\pm1\}^N} \Big| \sum_{n\leq N} c_n \varepsilon_n \Big|^p \leq p^{p/2} \Big(\sum_{n\leq N} |c_n|^2 \Big)^{p/2}.$$

This is only half of the Khintchine Inequality and in a special context, but it is explicit and will be enough for us. We followed [2, Chapter 10, Theorem 1, page 354] by Y.S. Show & H. Teicher. S. Astashkin in [1, Theorem 1.3] gives also a complete proof which is furthermore valid as soon as $p > 0$, up to a modification of the constant. To see the link between both results, let us mention that

$$(1/2^N) \sum_{(\varepsilon_n)\in\{\pm1\}^N} \Big| \sum_{n\leq N} c_n \varepsilon_n \Big|^p = \int_0^1 \Big| \sum_{n\leq N} c_n r_{n-1}(t) \Big|^p dt$$

where the (r_n) are the Rademacher functions defined in the introduction. This equality may be proved by considering the diadic expansion of $2^N t$, for each $t \in [0, 1]$.

Proof. Let us start with $p = 2k \geq 2$, so that we may open the inner sum and get

$$\begin{aligned} 2^N S(2k) &= \sum_{(\varepsilon_n)\in\{\pm1\}^N} \Big| \sum_{n\leq N} c_n \varepsilon_n \Big|^p \\ &= \sum_{\substack{s_1+s_2+\cdots+s_N=2k,\\ s_n\geq 0}} \binom{2k}{s_1, s_2, \cdots, s_N} \prod_{1\leq n\leq N} c_n^{s_n} \sum_{(\varepsilon_n)\in\{\pm1\}^N} \prod_{n\leq N} \varepsilon_n^{s_n} \end{aligned}$$

by the multinomial theorem. The inner summand vanishes as soon as some s_n is odd, whence, by letting $2t_n = s_n$, we get

$$\begin{aligned} 2^N S(2k) &= \sum_{\substack{t_1+t_2+\cdots+t_N=k,\\ t_n\geq 0}} \binom{2k}{2t_1, 2t_2, \cdots, 2t_N} \prod_{n\leq N} (c_n^2)^{t_n} \\ &\leq C \sum_{\substack{t_1+t_2+\cdots+t_N=k,\\ t_n\geq 0}} \binom{k}{t_1, t_2, \cdots, t_N} \prod_{n\leq N} (c_n^2)^{t_n} = C \Big(\sum_{n\leq N} c_n^2 \Big)^k \end{aligned}$$

where

$$C = \max \binom{2k}{2t_1, 2t_2, \cdots, 2t_N} \binom{k}{t_1, t_2, \cdots, t_N}^{-1}$$
$$\leq \max \frac{2k(2k-1)\cdots(k+1)}{\prod_j 2t_j(2t_j-1)\cdots(t_j+1)} \leq \max \frac{k^k}{2^{t_1+\cdots+t_k}} = (k/2)^k.$$

As $S(p)$ is increasing, we simply choose $k = \lceil p/2 \rceil$ (the upper integer part of $p/2$). This gives us $2k \geq p+2$ and thus

$$(k/2)^k \leq (p+2)^{1+p/2} \leq (30\,p)^{p/2}.$$

This concludes the main part of the proof, except for the constant 30. We will not continue the proof but simply refer to the paper [7] by U. Haagerup who shows that best constant is (be careful: the abstract of this paper misses a closing parenthesis for the value of B_p, but the value of B_p displayed in the middle of page 232 misses a squareroot-sign around the π, as an inspection of the proof at the end the paper rapidly reveals)

$$\begin{cases} 1 & \text{when } 0 < p \leq 2, \\ \sqrt{2}\left(\frac{\Gamma((p+1)/2)}{\sqrt{\pi}}\right)^{1/p} & \text{when } 2 < p. \end{cases}$$

We readily check that this implies that the constant 1 rather than 30 is admissible. □

3. An upper explicit Marcinkiewicz-Zygmund Inequality

Here is the main result of this section.

Theorem 3. *Let (X,μ) be a probability space. When $p \geq 1$, let $(f_n)_{n\leq N}$ be a system of functions such that $\int_X f_n(x)d\mu = 0$. We have*

$$\int_{(x_n)\in X^N} \Big| \sum_{1\leq n\leq N} f_n(x_n) \Big|^p d(x_n)$$
$$\leq (4p)^{p/2} \int_{(x_n)\in X^N} \Big(\sum_{1\leq n\leq N} |f_n(x_n)|^2 \Big)^{p/2} d(x_n).$$

The power of this inequality is that the implied constant does not depend on N, the effect of some orthogonality. Again, this is only half of the Marcinkiewicz-Zygmund Inequality and in a special context, but the constants are explicit. This will be enough for us. We followed [2, Chapter 10, Theorem 2, page 356] by Y.S. Show & H. Teicher. The relevant constant is the subject of [12] by Y.-F. Ren & H.-Y. Liang (their value is slightly worse than ours) and [6] by D. Ferger, where the best constant is determined provided the f_n's are "symmetric".

Proof. We first notice that, since $\int_0^1 f_n(x)dx = 0$, we may introduce a symmetrization through

$$\sum_{n\leq N} f_n(x_{2n-1}) = -\int_{(x_{2n})\in X^N} \sum_{n\leq 2N} (-1)^n f_{\lceil n/2 \rceil}(x_n) d(x_{2n}).$$

Jensen's inequality gives us that

$$\int_{(x_{2n-1})\in X^N}\left|\int_{(x_{2n})\in X^N}\sum_{n\le 2N}(-1)^n f_{\lceil n/2\rceil}(x_n)d(x_{2n})\right|^p d(x_{2n-1})$$
$$\le \int_{(x_{2n-1})\in X^N}\int_{(x_{2n})\in X^N}\left|\sum_{n\le 2N}(-1)^n f_{\lceil n/2\rceil}(x_n)\right|^p d(x_{2n})d(x_{2n-1})$$

from which we deduce that the L^p-norm of the symmetrization controls the one of the initial sum:

$$\int_{(x_{2n-1})\in X^N}\left|\sum_{n\le N} f_n(x_{2n-1})\right|^p d(x_{2n-1}) \le \int_{(x_n)\in X^{2N}}\left|\sum_{n\le 2N}(-1)^n f_{\lceil n/2\rceil}(x_n)\right|^p d(x_n).$$

We next notice that, for any $(\varepsilon_n)\in\{\pm1\}^N$, we have

$$\int_{(x_n)\in X^{2N}}\left|\sum_{n\le 2N}\varepsilon_{\lceil n/2\rceil}(-1)^n f_{\lceil n/2\rceil}(x_n)\right|^p d(x_n)$$
$$= \int_{(x_n)\in X^{2N}}\left|\sum_{n\le 2N}(-1)^n f_{\lceil n/2\rceil}(x_n)\right|^p d(x_n).$$

Indeed, consider the indices $n\in\{2k-1,2k\}$. When $\varepsilon_k=1$, we do not do anything while, when $\varepsilon_k=-1$, we exchange $n=2k-1$ and $n'=2k$. This enables us to introduce the Rademacher system:

$$(1/2^N)\sum_{(\varepsilon_n)\in\{\pm1\}^N}\int_{(x_n)\in X^{2N}}\left|\sum_{n\le 2N}\varepsilon_{\lceil n/2\rceil}(-1)^n f_{\lceil n/2\rceil}(x_n)\right|^p d(x_n)$$
$$= \int_{(x_n)\in X^{2N}}\left|\sum_{n\le 2N}(-1)^n f_{\lceil n/2\rceil}(x_n)\right|^p d(x_n).$$

We may now remove the symmetrization since:

$$\int_{(x_n)\in X^{2N}}\left|\sum_{n\le 2N}\varepsilon_{\lceil n/2\rceil}(-1)^n f_{\lceil n/2\rceil}(x_n)\right|^p d(x_n)$$
$$\le \int_{(x_n)\in X^{2N}} 2^{p-1}\left(\left|\sum_{n\le N}\varepsilon_n f_n(x_{2n})\right|^p + \left|\sum_{n\le N}\varepsilon_n f_n(x_{2n-1})\right|^p\right)d(x_n)$$
$$\le 2^p\int_{(x_{2n})\in X^N}\left|\sum_{n\le N}\varepsilon_n f_n(x_{2n})\right|^p d(x_{2n}).$$

The Khintchine Inequality from Theorem 2 finally gives us that

$$(1/2^N)\sum_{(\varepsilon_n)\in\{\pm1\}^N}\left|\sum_{n\le N}\varepsilon_n f_n(x_{2n})\right|^p \le p^{p/2}\left(\sum_{n\le N}|f_n(x_{2n})|^2\right)^{p/2}.$$

The proof is then complete. Concerning the constant 4 in the Theorem, the paper [12] by Y.-F. Ren & H.-Y. Liang gives the upper bound 9/2, which is worse than the above one. □

4. Proof of Theorem 1

Proof. We define $\Omega=\{1,\cdots,K\}$ which we equip with the probability measure defined by $\nu(\{k\})=|\lambda_k|/\|\lambda\|_1$. It induces a product measure on Ω^L, and, when $\mathbf{u}=(u_h)_{h\le L}\in\Omega^L$, we shall simply write $d\mathbf{u}$ when integrating with respect to this measure. Given a positive integer L, we consider the family of functions φ_ℓ, for $\ell \le L$ given by

$$\begin{array}{rcl} \varphi_\ell : \Omega^L \times X & \to & \mathbb{C} \\ (\mathbf{u}, x) & \mapsto & \lambda^\circ_{u_\ell} g_{u_\ell}(x) \end{array}$$

so that

$$\int_{\Omega^L} \varphi_\ell(\mathbf{u}, x) d\mathbf{u} = \sum_{k \in \Omega} \frac{|\lambda_k|}{\|\lambda\|_1} \lambda^\circ_k g_k(x) = \frac{f(x)}{\|\lambda\|_1} = f_0(x)$$

say. We aim at showing that $(1/L)\sum_{\ell \le L} \varphi_\ell(\mathbf{u}, x)$ closely approximates f_0 for most values of $\mathbf{u} = (u_h)_{h \le L}$. Selecting one such value gives qualitatively our result. To do so, we write

$$\int_{\Omega^L} \int_X \left| \frac{1}{L} \sum_{\ell \le L} \varphi_\ell(\mathbf{u}, x) - f_0(x) \right|^p d\mathbf{u} dx = \frac{1}{L^p} \int_{\Omega^L} \int_X \left| \sum_{\ell \le L} \big(\varphi_\ell(\mathbf{u}, x) - f_0(x)\big) \right|^p d\mathbf{u} dx.$$

We apply the Marcinkiewicz-Zygmung Inequality, i.e. Theorem 3, to this latter expression, getting for fixed $\mathbf{u}$,

$$\begin{aligned} \int_X \left| \frac{1}{L} \sum_{\ell \le L} \varphi_\ell(\mathbf{u}, x) - f_0(x) \right|^p dx &\le \frac{(4p)^{p/2}}{L^{p/2}} \int_X \left| \frac{1}{L} \sum_{\ell \le L} \big|\varphi_\ell(\mathbf{u}, x) - f_0(x)\big|^2 \right|^{p/2} dx \\ &\le \frac{(4p)^{p/2}}{L^{p/2}} \int_X \frac{1}{L} \sum_{\ell \le L} \big|\varphi_\ell(\mathbf{u}, x) - f_0(x)\big|^p dx, \end{aligned}$$

the second step having been obtained through the Hölder inequality. We next integrate over $\mathbf{u}$ and notice that $\int_{\Omega^L} |\varphi_\ell(\mathbf{u}, x) - f_0(x)|^p d\mathbf{u}$ is independent of ℓ to infer that

$$\int_{\Omega^L} \int_X \left| \frac{1}{L} \sum_{\ell \le L} \varphi_\ell(\mathbf{u}, x) - f_0(x) \right|^p d\mathbf{u} dx \le \frac{(4p)^{p/2}}{L^{p/2}} \int_X \int_{\Omega^L} \big|\varphi_1(\mathbf{u}, x) - f_0(x)\big|^p d\mathbf{u} dx.$$

Concerning the relevant p-norms, we make the following observations:

$$\int_X \int_{\Omega^L} \big|\varphi_1(\mathbf{u}, x)\big|^p d\mathbf{u} dx = \int_{\Omega^L} \left(\int_X \big|\varphi_1(\mathbf{u}, x)\big|^p dx \right) d\mathbf{u} \le 1,$$

on the one side while on the other side, by the triangle inequality, we have

$$\|f_0\|_p \le \sum_{k \le K} \frac{|\lambda_k|}{\|\lambda\|_1} \|g_k\|_p \le 1.$$

Therefore

$$\left(\int_{\Omega^L} \int_X \left| \frac{1}{L} \sum_{\ell \le L} \varphi_\ell(\mathbf{u}, x) - f_0(x) \right|^p d\mathbf{u} dx \right)^{1/p} \le \frac{(4p)^{1/2}}{L^{1/2}} (1 + 1) = \sqrt{16p/L}.$$

We deduce from this inequality that the set of $\mathbf{u}$ for which

$$\int_X \left| \frac{1}{L} \sum_{\ell \le L} \varphi_\ell(\mathbf{u}, x) - f_0(x) \right|^p dx > \varepsilon^p$$

has measure at most $\sqrt{16p/(\varepsilon^2 L)}$ which is strictly less than 1 by our assumption on L. The theorem follows readily. □

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`Orcid number of the author: 0000-0002-8765-0465`

(O. Ramaré) CNRS/ Institut de Mathématiques de Marseille, Aix Marseille Université, U.M.R. 7373, Site Sud, Campus de Luminy, Case 907, 13288 Marseille Cedex 9, France.

Email address: `olivier.ramare@univ-amu.fr`