CORRIGENDUM TO "EXPLICIT ESTIMATES FOR THE SUMMATORY FUNCTION OF $\Lambda(n)/n$ FROM THE ONE OF $\Lambda(n)$"

OLIVIER RAMARÉ

Abstract. Paper [3] at the level of Lemma 4 and paper [2] at the level of Lemma 2.1 and 2.2 contain a sign errors. We correct them here. More seriously, the computer program used to check the result for values not more than 10000 had a typo. We correct this here also.

1. Correcting [2, Lemma 2.1]

This copy of [3, Lemma 4] contains a typo. Here is the corrected version.

Lemma 1.1 (Correction of [3, Lemma 4] and of [2, Lemma 2.1])**.** *Let g be a continuously differentiable function on $[a, b]$ with $2 \le a \le b < +\infty$. We have*

$$\int_a^b \psi(t)g(t)dt = \int_a^b tg(t)dt - \sum_\rho \int_a^b \frac{t^\rho}{\rho} g(t)dt - \int_a^b \bigl(\log 2\pi + \tfrac{1}{2}\log(1-t^{-2})\bigr)g(t)dt.$$

The error is at the level of the sign of $\log 2\pi$. On reading [1], chapter 17, formulae (9) and (10)), this constant is $-\zeta'(0)/\zeta(O)$ and we have

$$\zeta'(0) = -\tfrac{1}{2}\log 2\pi, \quad \zeta(0) = -\tfrac{1}{2}. \tag{1.1}$$

2. Correcting [2, Lemma 2.2]

As a consequence, and correcting another sign typo, [2, Lemma 2.2] should be as follows.

Lemma 2.1 (Correction of [2, Lemma 2.2])**.** *We have, for $x \ge 1$:*

$$\tilde{\psi}(x) = \log x - \gamma + \frac{\psi(x) - x}{x} - \sum_\rho \frac{x^{\rho-1}}{\rho(\rho-1)} + \frac{B(x)}{x}.$$

where the sum is over the zeroes ρ of the Riemann zeta function that lie in the critical strip $0 < \Im s < 1$ (the so-called non trivial zeroes) and $B(x)$ is the bounded function given by

$$B(x) = \log 2\pi + \frac{1}{2}\log(1-x^{-2}) + \frac{1}{2}\biggl(x \log\frac{x+1}{x-1} - 2\biggr).$$

2010 *Mathematics Subject Classification.* Primary: 11N05, 11M36, secondary: 11N37.
Key words and phrases. Explicit estimates, von Mangoldt function.

We have $0 \le B(x) \le \log 2\pi$.

[2, Theorem 1.1] does not need to be modified. The sign in front of the $\log 2\pi$ comes from the typo above, the sign in front of the function of x in $B(x)$ comes from the third line of the proof of this lemma: the integral from x to ∞ should have a minus sign in front. We reach in this manner

$$\tilde{\psi}(x) = \log x - \gamma + \frac{\psi(x)-x}{x} - \sum_\rho \frac{x^{\rho-1}}{\rho(\rho-1)} + \int_x^\infty \left(\log 2\pi + \tfrac{1}{2}\log(1-t^{-2})\right)\frac{dt}{t^2}.$$

The claimed inequality on $B(x)$ is obvious from the integral expression one infers from this expression. In order to reach the exact expression, it is enough to mention that:

$$\begin{aligned}\frac{d}{dx}\left(\frac{1}{x}\log(1-x^{-2}) + \log\frac{x+1}{x-1} - \frac{2}{x}\right) \\ &= \frac{-1}{x^2}\log(1-x^{-2}) + \frac{2}{x^4(1-x^{-2})} - \frac{2}{x^2-1} + \frac{2}{x^2} \\ &= \frac{-1}{x^2}\log(1-x^{-2})\end{aligned}$$

as required.

3. Correcting [2, Corollary]

As [2, Theorem 1.1] does not need to be modified, the spread of this error stops here, but another and more annoying error comes from the Pari/GP code used to check the finite part of [2, Corollary].

Corollary. *We have for* $x > 1$,

$$\tilde{\psi}(x) = \log x - \gamma + \mathcal{O}^*\left(1.833/\log^2 x\right).$$

Furthermore, for $1 \le x \le 10^{10}$, *we have* $\tilde{\psi}(x) = \log x - \gamma + \mathcal{O}^*(1.31/\sqrt{x})$.
For $x \ge 1.52 \cdot 10^6$, *we have* $\tilde{\psi}(x) = \log x - \gamma + \mathcal{O}^*(0.0067/\log x)$.
For $x \ge 468\,000$, *we have* $\tilde{\psi}(x) = \log x - \gamma + \mathcal{O}^*(0.01/\log x)$.
For $x \ge 115$, *we have* $\tilde{\psi}(x) = \log x - \gamma + \mathcal{O}^*(1/(4\log x))$.

The code used previously has been lost, so here is the code we presently use, in order to make checking easier:

```
{Lambda(d)=my(dec = factor(d),P = dec[,1]);
   if(#P!=1, return(0), return(log(P[1])));}

{check(borneinf, bornesup, model = (t->log(t))) =
   /* We assume that between d and d+1 the function to be tested is
      concave. It is true when model = (t->log(t)).
      The values obtained are also valid for x < bornesup+1 */
   my(mymin = 1000, mymax = -1000, psitilde = 0, val wheremin, wheremax);
   for(d = 1, borneinf -1, psitilde += Lambda(d)/d);
   for(d = borneinf, bornesup,
      psitilde += Lambda(d)/d;
      val = (psitilde - log(d) + Euler)*model(d);
```

```
    if(val < mymin, mymin = val; wheremin = d);
    if(val > mymax, mymax = val; wheremax = d);
    val = (psitilde - log(d+1) + Euler)*model(d+1);
    if(val < mymin, mymin = val; wheremin = d + 0.9999);
    if(val > mymax, mymax = val; wheremax = d + 0.9999);
 );
 print("The minimum of (psitilde - log(d) + Euler)*model(d)");
 print("On [", borneinf, ", ", bornesup, "] is reached at d = ", wheremin);
 print("   with value = ", mymin);
 print("");
 print("The maximum of (psitilde - log(d) + Euler)*model(d)");
 print("On [", borneinf, ", ", bornesup, "] is reached at d = ", wheremax);
 print("   with value = ", mymax);

 return([mymin, mymax]);
 }
```

Section 6 of [2] does not need to be modified except that the value $1.68 \cdot 10^{-12}$ should throughout be replaced by $1.75 \cdot 10^{-12}$ with no consequence. Furthermore, in this same Section 6, the inequality When $8\,950 \le x \le 10^{10}$, we have

$$\left|\tilde{\psi}(x) - \log x + \gamma\right| \log x \le \frac{1.3 \log x}{\sqrt{x}} + 1.68 \cdot 10^{-12} \log x \overset{?}{\le} 0.0003$$

is another typo. It should be replaced by When $9.75 \cdot 10^6 \le x \le 10^{10}$, we have

$$\left|\tilde{\psi}(x) - \log x + \gamma\right| \log x \le \frac{1.3 \log x}{\sqrt{x}} + 1.68 \cdot 10^{-12} \log x \le 0.0067.$$

By direct inspection, we show this bound is valid in the range $x \ge 1.52 \cdot 10^6$.

References

1. H. Davenport, *Multiplicative Number Theory*, third edition ed., Graduate texts in Mathematics, Springer-Verlag, New-York, 2000.
2. O. Ramaré, *Explicit estimates for the summatory function of* $\Lambda(n)/n$ *from the one of* $\Lambda(n)$, Acta Arith. **159** (2013), no. 2, 113–122.
3. O. Ramaré and Y. Saouter, *Short effective intervals containing primes*, J. Number Theory **98** (2003), 10–33.

CNRS / Laboratoire Paul Painlevé, Université Lille 1, 59 655 Villeneuce d'Ascq

Email address: ramare@math.univ-lille1.fr