Convolution method

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Abstract

A step by step method to obtain a multiplicative function's average order (based on Dirichlet series identification).

Contents

1 Introduction

Many arithmetical functions have an erratic behavior which makes their arithmetic mean:

$$\frac{1}{X}\sum_{n\le X} f(n)$$

a much better way to "know" them.

Thus, the average order of an arithmetic function is some simpler or better-understood function which takes the same values "on average". We define an average order of an arithmetic function f to be any function g of a real variable such that

$$\sum_{n\le X} f(n) \sim \sum_{n\le X} g(n)$$

By $\sim$ (equivalent), we mean asymptotically equivalent such that if the limit $\lim_{x\to\infty} F(x)/G(x)$ exists it is equal to 1 (for example $\ln(x)+1 \sim \ln(x)$).

In general we try and find a function g which is commonly known (usually continuous and monotonic), and which asymptotic behavior is easy to determine. It is this asymptotic equivalent that we usually call the average order.

There does not always exist an simpler average order than f, but for many functions it does exist and it is a regular function. Through three examples, this article will explain the convolution method that helps determine the average order of some multiplicative functions.

We shall first consider the following function

$$f_0(n) = \prod_{p|n}(p+2), \tag{1}$$

in which p represents a prime number, so we are considering here the product for all prime numbers dividing n. We remind the reader than an empty product is to be assigned the value 1. We will use the same conventions in the rest of this document. Below are the values of $f_0(n)$ for n between 1 and 20:

$$1, 4, 5, 4, 7, 20, 9, 4, 5, 28, 13, 20, 15, 36, 45, 4, 19, 100, 21, 112.$$

This sequence does not give much information on the function's behavior, but we will prove the following theorem.

Theorem. *For all σ real in $]1/2, 1]$,*

$$\frac{1}{X}\sum_{n\le X} f_0(n) = \frac{1}{X}\sum_{n\le X}\prod_{p|n}(p+2) = C_0X + \mathcal{O}(X^\sigma) \tag{2}$$

where

$$C_0 = \frac{1}{2}\prod_{p\ge 2}\left(1+\frac{1}{p(p+1)}\right) = 0.6842...$$

We will then prove a second theorem that gives the average order of $\frac{\varphi(n)}{n}$, the function φ being Euler's totient function defined in part 2.1.

Theorem. *For all σ real in $]-1,0]$,*

$$\frac{1}{X}\sum_{n\leq X}\frac{\varphi(n)}{n}=\frac{6}{\pi^2}+\mathcal{O}(X^\sigma) \tag{3}$$

And at last, we will consider average orders of fuctions involving the Möbius function (defined in (2.1)) and prove the following theorem which shows that the method works with non positive multiplicative functions.

Indeed if we consider

$$f_3(n)=\frac{\mu(n)}{\varphi(n)} \qquad \text{and} \qquad g_3(n)=\frac{\mu(n)}{n} \tag{4}$$

we will find a function h_3 such that $f_3 = g_3 \star h_3$, as explained in the next paragraph, and obtain the following result.

Theorem. *For all $X > 1$,*

$$\left|\sum_{n\leq X}\frac{\mu(n)}{\varphi(n)}\right| \leq \frac{0.082}{\log X}+\frac{30}{X^{1/3}} \tag{5}$$

This theorem is proved for all $X > 100000$ and we check it holds for any $X > 1$ with GP calculator. We then easily use its proof to get a more general theorem below.

Theorem. *For all function f defined by $f = g_3 \star h$ where $g_3(n) = \frac{\mu(n)}{n}$ and h is such that $D(|h|, s)$ is convergent for all $s > -1$, we have for all $X > 100000$,*

$$\left|\sum_{n\leq X} f(n)\right| \leq \frac{C}{\log X}+\frac{C'}{X^{1/3}}$$

where $C = \frac{2}{69}D(|h|, 0)$ and $C' = D(|h|, -2/3) + D(|h|, 0)$.

The principle of the method is the following: we use the notion of abscissa of convergence to define a concept of *size* of an arithmetical function: a function f_1 is *greater* than a function f_2 if its abscissa of convergence is greater. The idea of the convolution method is to express a function f for which we would like to calculate the average order as a convolution $f = g\star h$ with the abscissa of convergence of g and f being the same and the abscissa of convergence of h being smaller. Hence we will consider h as being a *slight perturbation* and f being a slightly modified version of g. The function g being a known multiplicative function with a known average order.
The average order of f is then the average order of $g \star h$ which is usually dominated by the average order of g.

The key of the convolution method is the uniqueness Theorem (Theorem 3.7). If $D(f, s) = D(g, s)D(h, s)$ then $f = g \star h$. And f and g being multiplicative, we will express their Dirichlet series with Euler products (Theorem 3.6).

Before we establish those average orders with a step-by-step guide, we shall describe arithmetical functions and more precisely multiplicative functions, with a few of their useful properties. A complete course can be found in [1], [2] or [6] for example. We then shall define the Dirichlet series associated to an arithmetical function as they are one of the most useful tools in analytic number theory and specifically in the convolution method. These parts contain a few exercises so they might be used as a lesson.

The convolution method is highly effective, and though it doesn't work for some multiplicative functions , it does for most of the usual ones, so we can consider it as a great tool.

2 Arithmetical functions

An arithmetical function is a complex valued function defined on the positive integers. In other words, an arithmetical function can be seen as a sequence of complex values.

Multiplicative functions are a class of arithmetical functions that play an important role in number theory.

Definition 2.1. *An arithmetical function $f : \mathbb{N}^* \to \mathbb{C}$ is called multiplicative if*

$$f(1) = 1 \quad \textit{and} \quad f(nm) = f(n)f(m), \quad \textit{whenever} \quad \gcd(n, m) = 1.$$

The main interest of multiplicative functions is that the image of an integer is the product of the images of the prime powers of its primary decomposition. If f is a multiplicative function:

$$f(n) = \prod_{p^\alpha \| n} f(p^\alpha). \tag{6}$$

We remind the reader that $p^\alpha \| n$ means $p^\alpha | n$ and $p^{\alpha+1} \nmid n$. This implies that a multiplicative function is totally defined by its values on prime powers.

2.1 Inventory

In this section, and in the rest of this document, by divisor we mean positive divisor. Below is a list of the most common multiplicative functions:

- $\varphi(n)$: Euler's totient function. It gives the number of integers between 1 and n which are relatively prime to n. It has the following property for every prime number p and $\alpha \geq 1$: $\varphi(p^\alpha) = p^{\alpha-1}(p-1)$.

- $\tau(n)$: the number of divisors of n.
- $\sigma(n)$: the sum of the divisors of n.
- $\theta_\alpha(n)$: the function that associates the number n^α to n.
- $\mathbb{1}$ is the usual notation for θ_0.
- $\mu(n)$: the Moebius function. It is multiplicative and associates -1 to every prime number and 0 to all higher powers of prime numbers.
- $\mu^2(n)$: This function indicates square free numbers. It associates 1 to any square free number and 0 otherwise.
- $\lambda(n)$: the Liouville function. It associates the number $(-1)^k$ to all p^k.

The functions μ and λ can also be defined without using their multiplicativity property with the omega functions: if $\omega(n)$ gives the number of distinct prime factors of n and $\Omega(n)$ gives the number of prime factors of n counted by multiplicities, then $\mu(n) = (-1)^{\omega(n)}$ if $\omega(n) = \Omega(n)$ and 0 otherwise, and $\lambda(n) = (-1)^{\Omega(n)}$.

2.2 Dirichlet convolution

The Dirichlet convolution of two arithmetical functions f and g is defined by:

$$(f \star g)(n) = \sum_{d|n} f(n/d)g(d). \tag{7}$$

The function $\delta_{n=1}$ also named δ_1, defined by $\delta_1(1) = 1$ and $\delta(n) = 0$ for $n \neq 1$, is the identity element for this multiplication, as for all arithmetical function g, we have

$$(\delta_1 \star g)(n) = \sum_{\ell m = n} \delta_1(\ell)g(m) = g(n).$$

The reader can easily verify the two following properties:

Property 2.2. *Dirichlet convolution is commutative and associative.*

Property 2.3. *Dirichlet convolution distributes over addition.*

Thus, these two operations give the set of arithmetical functions the structure of an algebra over a commutative ring.

A very important theorem for multiplicative functions is:

Theorem 2.4. *If f and g are multiplicative, so is their Dirichlet product $f \star g$.*

To prove the theorem above, we use the following lemma:

Lemma 2.5. *Let m and n be two relatively prime integers. For all function F, we have*

$$\sum_{d|mn} F(d) = \sum_{d_1|m} \sum_{d_2|n} F(d_1 d_2).$$

Proof. Let n be an integer, and let $\mathcal{D}(n)$ be the set of its positive divisors, for example $\mathcal{D}(12) = \{1, 2, 3, 4, 6, 12\}$. We consider the two following functions:

$$\begin{aligned} \Phi : \mathcal{D}(m) \times \mathcal{D}(n) &\to \mathcal{D}(mn), & \Psi : \mathcal{D}(mn) &\to \mathcal{D}(m) \times \mathcal{D}(n), \\ (d_1, d_2) &\mapsto d_1 d_2 & d &\mapsto (\gcd(d, m), \gcd(d, n)). \end{aligned}$$

We now prove that $\Psi \circ \Phi = \mathrm{Id}$ and $\Phi \circ \Psi = \mathrm{Id}$, hence the lemma is a direct consequence of the equivalence $d/mn \Leftrightarrow d = d_1 d_2$ with d_1/m and d_2/n.

First, let (d_1, d_2) from $D(m) \times \mathcal{D}(n)$.
We have $\gcd(d_1 d_2, m) = d_1$ and $\gcd(d_1 d_2, n) = d_2$, hence $\Psi \circ \Phi = \mathrm{Id}$.
Conversely, if d divides mn, we have
$\gcd(d, mn) = \gcd(d, m) \gcd(d, n)$ therefore $\Phi \circ \Psi = \mathrm{Id}$. □

Using this lemma, we can now prove Theorem 2.4 :

Proof : Clearly $(f \star g)(1) = f(1)g(1) = 1$. Then, given m and n two relatively prime numbers,

$$(f \star g)(mn) = \sum_{d|mn} f\left(\frac{mn}{d}\right) g(d).$$

Lemma 2.5 tells us that

$$\begin{aligned} (f \star g)(mn) &= \sum_{d_1|m} \sum_{d_2|n} f\left(\frac{mn}{d_1 d_2}\right) g(d_1 d_2) \\ &= \sum_{d_1|m} \sum_{d_2|n} f(m/d_2) f(n/d_2) g(d_1) g(d_2) = (f \star g)(m)(f \star g)(n) \end{aligned}$$

□

Note that τ the function that gives the number of divisors is multiplicative, as $\tau(n) = (\mathbb{1} \star \mathbb{1})(n)$.

Exercise 1. *Prove that $\mathbb{1} \star \lambda(n) = 1$ if n is a square and 0 otherwise.*

Exercise 2. *Let f and g so that $f(n) = \tau(n)^2$ et $g(n) = \tau(n^2)$. Prove that $f = \mathbb{1} \star g$.*

Exercise 3. *Prove that $\theta_1 = \mathbb{1} \star \phi$.*

3 Dirichlet series

Definition 3.1. *Let f be an arithmetical function. The Dirichlet series associated with f is defined for all real number s for which the series is convergent by:*

$$D(f, s) = \sum_{n \geq 1} f(n) n^{-s}. \tag{8}$$

Note: There might not exist values of s for which the Dirichlet series is defined, for example if $f(n) = e^n$.
Also, Dirichlet series are usually defined on $\mathbb{C}$ but in this article we only need s to be a real number. More information about Dirichlet series can be found in [4].

3.1 Abscissa of absolute convergence

Definition 3.2. *The* abscissa of convergence *for a function f, is the smallest real σ_c such as the Dirichlet series $D(f,s)$ is convergent for all $s > \sigma_c$. If $D(f,s)$ is convergent for all s, we say that the abscissa of convergence is $-\infty$.*

Example 1. *The abscissa of convergence is $-\infty$ for $f(n) = e^{-n}$.*

Definition 3.3. *The* abscissa of absolute convergence *for a function f, is the smallest real σ_a such as the Dirichlet series $D(|f|,s)$ is convergent for all $s > \sigma_a$.*

Note: The series might not be convergent for $s = \sigma_c$ (or σ_a). Landau's theorem stipulates that if f is positive and σ_c is finite, the Dirichlet series associated with f diverges in σ_c (see [1] for example, Thm 11.13).

Property 3.4. *Let f be an arithmetical function such as the associated Dirichlet series is absolutely convergent for some s. Thus, for all $r \geq s$, the series $D(f,r)$ is absolutely convergent.*

Proof : We have

$$D(f,r) = \sum_{n\geq 1} \frac{f(n)}{n^s}\frac{n^s}{n^r}.$$

$r \geq s$ so $n^s/n^r \leq 1$, hence $D(|f|,r) \leq D(|f|,s)$, i.e. the Dirichlet series associated with f is absolutely convergent for all $r \geq s$. □

3.2 Euler product

Infinite sums of multiplicative functions can be expanded into infinite products over prime numbers, with the same abscissa of absolute convergence. This is a consequence of a theorem of Godement (see [3]):

Theorem 3.5. *Considering a sequence (u_n) such that $\sum_{n\geq 1} |u_n| = M < \infty$, then*

1. $P_n = \prod_{1\leq j\leq n} (1 + u_j) \to P \in \mathbb{C}$.

2. *If for all j, $1 + u_j \neq 0$ then $P \neq 0$.*

Proof : 1. We have $P_n - P_{n-1} = u_n P_{n-1}$ so

$$| P_n - P_{n-1} | \leq | u_n || P_{n-1} | \leq | u_n | \prod_{1 \leq j \leq n-1} e^{|u_j|} \leq | u_n | e^M.$$

2. We find some product Q such that $PQ = 1$.

Indeed, for all j we define

$$v_j = \frac{1}{1+u_j} - 1 = -\frac{u_j}{1+u_j}$$

(v_j) is well defined as for all j, $1 + u_j \neq 0$. And $\sum_{n \geq 1} |u_j| < \infty$ as $|v_j| \sim |u_j|$. So if $Q = \prod_{n \geq 1}(1+v_n)$ then, as we have for all j, $(1+u_j)(1+v_j) = 1$, we easily deduce $PQ = 1$. □

We will say that a product is absolutely convergent in Godement's criterion if it can be writen $\prod(1 + u_j)$ with $\sum |u_j|$ bounded. And this can be applied to Dirichlet series of multiplicative functions: if $D(f, s)$ can be expanded into a Euler product, it means that the product is zero if and only if it contains at least one zero factor.

Property 3.6. *Let f be a multiplicative function and assume that the Dirichlet series associated with f is absolutely convergent for some s. Hence, $D(f, s)$ is expandable into the Euler product in Godement's criterion:*

$$D(f, s) = \prod_{p \geq 2} \sum_{k \geq 0} \frac{f(p^k)}{p^{ks}}.$$

Proof : The idea is to express n as its product of prime numbers. Then, f being multiplicative, this proves the theorem. A complete proof is in [6]. □

The Riemann zeta function is the Dirichlet series associates with the constant function 1 which we named θ_0 and $\mathbb{1}$ in the inventory. Thus it is the simplest (and most famous) Dirichlet series. It is defined for all $s > 1$ by

$$\zeta(s) = \sum_{n \geq 1} n^{-s}.$$

Note: Riemann considered complex values of s and connected the distribution of primes to analytic properties of the function ζ.

As the function $\mathbb{1}$ is multiplicative, the function ζ is expandable to the Euler product :

$$\zeta(s) = \prod_{p \geq 2} (\sum_{k \geq 0} \frac{1}{p^{ks}}) = \prod_{p \geq 2} (1 - p^{-s})^{-1}, \tag{9}$$

which is absolutely convergent for $s > 1$.

3.3 Uniqueness theorem

For every arithmetical function we can associate a Dirichlet series, as long as the last is convergent for at least some s. We see below that a function f is uniquely determined by its Dirichlet series.

Property 3.7. *Given two functions f and g such that their Dirichlet series are both absolutely convergent for some s and that $D(f,r) = D(g,r)$ for all $r > s$, then $f = g$.*

Proof : Let $h_1 = f - g$, we have $D(h_1, r) = 0$ for all $r > s$. As the series is convergent for $r = s + 1$, $h_2(n) = h_1(n)/n^{s+1}$ is bounded (in absolute value) and $D(h_2, r) = 0$ for all $r > -1$. We will prove that $h_2 = 0$.
Let's assume that $h_2 \neq 0$. Let n_0 be the smallest integer for which $h_2(n) \neq 0$. We have, for $r > 1$:

$$n_0^r D(h_2, r) - h_2(n_0) = \sum_{n \geq n_0} h_2(n) \frac{n_0^r}{n^r} - h_2(n_0) = \sum_{n \geq n_0+1} h_2(n) \frac{n_0^r}{n^r}$$

Hence, if comparing the sum to an integral we have:

$$\begin{aligned} |n_0^r D(h_2, r) - h_2(n_0)| &\leq \max_n |h_2(n)| \sum_{n \geq n_0+1} \frac{n_0^r}{n^r} \\ &\leq \max_n |h_2(n)| n_0^r \int_{n_0}^{\infty} \frac{dt}{t^r} \leq \frac{n_0 \max_n |h_2(n)|}{(r-1)}, \end{aligned}$$

$n_0 \max_n |h_2(n)|/(r-1) \to 0$ when $r \to +\infty$. And as $D(h_2, r) = 0$ (and so $n_0^r D(h_2, r) = 0$), we should have $h_2(n_0) = 0$, hence the contradiction.

□

3.4 Dirichlet series and convolution

The two operations defined in chapter 2.2 on the set of arithmetical functions are precisely the ones involved when adding or multiplying two Dirichlet series.

- Addition (+): given two functions f and g which Dirichlet series are absolutely convergent for some s, we have

$$D(f + g; s) = D(f; s) + D(g; s).$$

- Multiplication ($\star$): given two functions f and g which Dirichlet series are absolutely convergent for some s, we have

$$D(f \star g; s) = D(f; s) D(g; s).$$

The last identity, which is easy to establish, means the Dirichlet series of a convolution is the product of the Dirichlet series the same way the Fourier transform of a convolution is the product of the Fourier transforms for real values.

Note: the abscissa of convergence of the Dirichlet series associated with $f \star g$ is clearly less than or equal to the maximum of the abscissas of convergence of f and g. There is usually equality when those abscissas of convergence are distinct and none of the factors is nul.

Exercise 4. *Show that*

1. $D(\varphi, s) = \zeta(s-1)/\zeta(s)$,
2. *if* $f(n) = \tau^2(n)$, *then* $D(f, s) = \zeta^4(s)/\zeta(2s)$,
3. *if* $f(n) = \tau(n^2)$, *then* $D(f, s) = \zeta^3(s)/\zeta(2s)$,
4. $D(\lambda, s) = \zeta(2s)/\zeta(s)$,
5. $D(\mu^2, s) = \zeta(s)/\zeta(2s)$.

Exercise 5. *Show that the Dirichlet series associated to the Moebius function* μ *is* $1/\zeta(s)$ *and exhibit an example that will prove that the abscissa of absolute convergence of a convolution can be strictly smaller than the maximum of the abscissas of convergence of the two functions involved. (consider* $\mathbb{1} \star \mu$*).*

4 Results

As stated in the introduction, we now explain the convolution method in a step-by-step guide with three examples. We remind the reader that the results are:

Theorem 4.1. *Let* X *be a positive real number. For all* σ *real in* $]1/2, 1]$, *we have*

$$\frac{1}{X}\sum_{n\leq X} f_0(n) = C_0 X + \mathcal{O}(X^\sigma)$$

where the $\mathcal{O}$ *contains a constant number depending on* σ *and where*

$$C_0 = \tfrac{1}{2}\prod_{p\geq 2}\left(1+\frac{1}{p(p+1)}\right) = 1.3684...$$

Then if

$$f_1(n) = \frac{\varphi(n)}{n}, \tag{10}$$

we have

Theorem 4.2. *Let X be a positive real number. For all σ real in $]-1,0]$, we have*

$$\frac{1}{X}\sum_{n\leq X} f_1(n) = \frac{6}{\pi^2} + \mathcal{O}(X^\sigma)$$

where the $\mathcal{O}$ contains a constant number depending on σ.

At last if we consider

$$f_3(n) = \frac{\mu(n)}{\varphi(n)} \qquad \text{and} \qquad g_3(n) = \frac{\mu(n)}{n} \tag{11}$$

we will find a function h_3 such that $f_3 = g_3 \star h_3$ and obtain the following result.

Theorem 4.3. *For all $X > 100000$,*

$$\left|\sum_{n\leq X}\frac{\mu(n)}{\varphi(n)}\right| \leq \frac{0.082}{\log X} + \frac{30}{X^{1/3}}$$

We let the reader quickly check that f_0, f_1 and f_3 are indeed multiplicative functions, f_0 from its definition, f_1 and f_3 as quotients of two multiplicative functions.

5 The step-by-step guide to convolution method and proof of Theorem 4.1

Step one: finding Dirichlet series such that $D(f_0,s) = D(g,s)D(h,s)$

Let us express the Dirichlet series associated with f_0 as an Euler product, we have:

$$D(f_0,s) = \sum_{n\geq 1}\frac{\prod_{p'|n}(p'+2)}{n^s} = \prod_{p\geq 2}\left(1+\sum_{k\geq 1}\frac{\prod_{p'|p^k}(p'+2)}{p^{ks}}\right).$$

But p' and p being prime numbers, the only p' that divides p^k is $p'=p$. So what remains from the sum inside the brackets is:

$$\sum_{k\geq 1}\frac{(p+2)}{p^{ks}} = (p+2)\sum_{k\geq 1}\frac{1}{p^{ks}} = \frac{p+2}{p^s-1}.$$

The Dirichlet series associated with f_0 is:

$$D(f_0,s) = \prod_{p\geq 2}\left(1+\frac{p+2}{p^s-1}\right). \tag{12}$$

We now notice that this product is similar to

$$\prod_{p\geq 2}\left(1+\frac{1}{p^{s-1}-1}\right)$$

which is equal to $\zeta(s-1)$.

We remind the reader that our goal is to find a function g and a function h such that $f_0 = g \star h$. According to the property 3.7 and the fact that the Dirichlet series of a convolution is the product of the Dirichlet series, we are looking for a Dirichlet series $H(s) = D(h,s)$ such that $D(f_0,s) = \zeta(s-1)H(s)$ So our function g here would be the function θ_1 that associates n to n and which Dirichlet series is $\zeta(s-1)$ we now need to find the function h, but first to find H(s).

Let us "take" $\zeta(s-1)$ "out" of our product:

$$\begin{aligned}
D(f_0,s) = \prod_{p\geq 2}\left(1+\frac{p+2}{p^s-1}\right) &= \zeta(s-1)\prod_{p\geq 2}\left(1+\frac{p+2}{p^s-1}\right)\prod_{p\geq 2}\left(1-\frac{1}{p^{s-1}}\right) \\
&= \zeta(s-1)\prod_{p\geq 2}\left(1-\frac{1}{p^{s-1}}+\frac{p+2}{p^s-1}-\frac{p+2}{p^{s-1}(p^s-1)}\right) \\
&= \zeta(s-1)\prod_{p\geq 2}\left(1+\frac{2}{p^s-1}-\frac{1}{p^{s-2}(p^s-1)}-\frac{1}{p^{s-1}(p^s-1)}\right) \\
&= \zeta(s-1)H(s).
\end{aligned}$$

Step two: identifying the function h

We must now express H(s) as a Dirichlet series and at the same time consider the values of s for which $H(s)$ is absolutely convergent. We remind the reader that we are looking for a function h such that:

$$H(s) = D(h,s) = \sum_{n\geq 1} h(n)/n^s \tag{13}$$

And because we only consider multiplicative functions, the series can be expanded into the Euler product

$$D(h,s) = \prod_{p\geq 2}\left(1+\sum_{k\geq 1}\frac{h(p^k)}{p^{ks}}\right).$$

Hence we have

$$1+\sum_{k\geq 1}\frac{h(p^k)}{p^{ks}} = 1+\frac{2}{p^s-1}-\frac{1}{p^{s-2}(p^s-1)}-\frac{1}{p^{s-1}(p^s-1)}. \tag{14}$$

Because h is multiplicative, the series $H(s)$ will be absolutely convergent if in the right member of this Euler product we have $\sum \frac{2}{n^s-1}$, $\sum \frac{1}{n^{s-2}(n^s-1)}$ and $\sum \frac{1}{n^{s-1}(n^s-1)}$ that are convergent. Which means we must have at the same time $s > 1$, $2s-2 > 1$ hence $s > 3/2$ and $2s-1 > 1$ hence $s > 1$ again. Thus, the

abscissa of absolute convergence of h is at most $3/2$.
This result is coherent with our goal which was to find a function h "smaller" than f and g as defined in the introduction of this section. Indeed, the abscissa of convergence of $g = \theta_1$ (associated with $\zeta(s-1)$) is 2, and so will be f_0's as we can guess from the way we defined g and h and as we can verify when f_0 is determined.

Let us now explicit $H(s)$:

$$1 + \sum_{k \geq 1} \frac{h(p^k)}{p^{ks}} = 1 + \frac{2}{p^s - 1} - \frac{1}{p^{s-2}(p^s - 1)} - \frac{1}{p^{s-1}(p^s - 1)}.$$

Let us modify the right hand member so that it can be identified with the left one. We use the simple fact that

$$\frac{1}{1 - \frac{1}{p^s}} = \sum_{k \geq 0} \frac{1}{p^{ks}} \tag{15}$$

Hence

$$\begin{aligned}
1 + \frac{2}{p^s - 1} - \frac{1}{p^{s-2}(p^s - 1)} - \frac{1}{p^{s-1}(p^s - 1)} &= 1 + \frac{2}{p^s}\left(\frac{1}{1 - \frac{1}{p^s}}\right) - \frac{1}{p^{2s-2}}\left(\frac{1}{1 - \frac{1}{p^s}}\right) - \frac{1}{p^{2s-1}}\left(\frac{1}{1 - \frac{1}{p^s}}\right) \\
&= 1 + \frac{2}{p^s} \sum_{k \geq 0} \frac{1}{p^{ks}} - \frac{1}{p^{2s-2}} \sum_{k \geq 0} \frac{1}{p^{ks}} - \frac{1}{p^{2s-1}} \sum_{k \geq 0} \frac{1}{p^{ks}} \\
&= 1 + 2 \sum_{k \geq 0} \frac{1}{p^{(k+1)s}} - p^2 \sum_{k \geq 0} \frac{1}{p^{(k+2)s}} - p \sum_{k \geq 0} \frac{1}{p^{(k+2)s}} \\
&= 1 + \frac{2}{p^s} + \sum_{k \geq 2} \frac{-p^2 - p + 2}{p^{ks}}
\end{aligned}$$

Identifying each term of these series, we define the multiplicative function h by:

$$\begin{cases} h(p) = 2, \\ h(p^k) = -p^2 - p + 2 \quad \text{for } k \geq 2, \end{cases} \tag{16}$$

which meets our requirements.

Step three: determination of the average order of f_0

We will use the following lemma:

Lemma 5.1. *For all $\alpha \in [1, 2]$ and all $M \geq 0$.*

$$\sum_{m \leq M} m = \frac{1}{2} M^2 + \mathcal{O}(M^\alpha) \tag{17}$$

Proof : for all integer N we have:

$$\sum_{n \leq N} n = N(N+1)/2,$$

thus, for $M \geq 1$, M being real we have:

$$\sum_{m \leq M} m = \tfrac{1}{2}M(M+1) + \mathcal{O}(M) = \tfrac{1}{2}M^2 + \mathcal{O}(M). \tag{18}$$

This estimation is true for any M greater than or equal to 0, and obviously for all $\alpha \in [1,2]$,

$$\sum_{m \leq M} m = \tfrac{1}{2}M^2 + \mathcal{O}(M^\alpha)$$

□

As $D(f_0, s) = \zeta(s-1)H(s)$, we already explained in the first step why $f_0 = \theta_1 \star h$.

Let us express f_0 as the convolution of θ_1 by h.

$$f_0(n) = \sum_{\ell m = n} h(\ell)\theta_1(m) = \sum_{\ell m = n} h(\ell)m.$$

We are now ready to calculate the average order of f_0. We have

$$\sum_{n \leq X} f_0(n) = \sum_{\ell m \leq X} h(\ell)m = \sum_{\ell \leq X} h(\ell) \sum_{m \leq X/\ell} m. \tag{19}$$

The lemma 5.1 works as we actually need the error term to be less good because of the abscissa of absolute convergence of h which is $3/2$, and this allows us to easily finish our calculation.

$$\sum_{n \leq X} f_0(n) = \sum_{\ell \leq X} h(\ell)\Big(\tfrac{1}{2}\frac{X^2}{\ell^2} + \mathcal{O}(\frac{X^\alpha}{\ell^\alpha})\Big)$$

We conveniently notice that the condition $\ell \leq X$ can be replaced with $\ell \geq 1$ as the identity (and estimation) remains true. And so:

$$\sum_{n \leq X} f_0(n) = \frac{X^2}{2}\sum_{\ell \geq 1}\frac{h(\ell)}{\ell^2} + \mathcal{O}\Big(X^\alpha \sum_{\ell \geq 1}\frac{|h(\ell)|}{\ell^\alpha}\Big).$$

We now understand better why we needed the error term power in (5.1) to be greater than 1. H(s) is absolutely convergent for $s > 3/2$ so $\sum_{\ell \leq 1} |h(\ell)|/\ell^\alpha$ is finite for all $\alpha > 3/2$. And the error term only makes sense for $\alpha \leq 2$.

In conclusion, if we let $\sigma = \alpha - 1$ we find that

$$\frac{1}{X}\sum_{n \leq X} f_0(n) = \frac{X}{2}\sum_{\ell \geq 1}\frac{h(\ell)}{\ell^2} + \mathcal{O}\Big(X^\sigma \sum_{\ell \geq 1}\frac{|h(\ell)|}{\ell^\sigma}\Big).$$

for all $\sigma \in]1/2, 1[$ which concludes the demonstration of Theorem 4.1.

Expression of the constant C_0

In Theorem 4.1 we stated that

$$\frac{1}{X}\sum_{n\le X} f_0(n) = C_0 X + \mathcal{O}(X^\sigma) \quad \text{so } C_0 = \frac{1}{2}\sum_{\ell\ge 1}\frac{h(\ell)}{\ell^2} \tag{20}$$

with

$$\begin{aligned}\sum_{\ell\ge 1}\frac{h(\ell)}{\ell^2} &= \prod_{p\ge 2}\left(1+\sum_{k\ge 1}\frac{h(p^k)}{p^{2k}}\right)\\ &= \prod_{p\ge 2}\left(1+\frac{2}{p^2}+(-p^2-p+2)\sum_{k\ge 2}\frac{1}{p^{2k}}\right)\\ &= \prod_{p\ge 2}\left(1+\frac{2}{p^2}+\frac{(-p^2-p+2)}{p^2(p^2-1)}\right)\\ &= \prod_{p\ge 2}\left(1+\frac{(p^2-p)}{p^2(p^2-1)}\right) = \prod_{p\ge 2}\left(1+\frac{1}{p(p+1)}\right)\end{aligned}$$

Hence $C_0 = \frac{1}{2}\prod_{p\ge 2}\left(1+\frac{1}{p(p+1)}\right)$.

6 Second example of the convolution method, proof of Theorem 4.2

In this part, we are considering the function below

$$f_1(n) = \frac{\varphi(n)}{n}$$

for which we cannot use directly use the convolution method to determine its average order for divergence reasons. So we determine the average order of

$$f_2(n) = \frac{\varphi(n)}{n^2} \tag{21}$$

and then use a summation by parts explained in the next section.

6.1 Determination of the average order of f_2

Step one

Let us express the Dirichlet series of f_2 as the product of two Dirichlet series. f_2 being multiplicative, we have

$$D(f_2, s) = \sum_{n\ge 1}\frac{\varphi(n)}{n^{s+2}} = \prod_{p\ge 2}\left(1+\sum_{k\ge 1}\frac{\varphi(p^k)}{p^{(s+2)k}}\right) \tag{22}$$

but for all $k \geq 1$ we have $\varphi(p^k) = p^{k-1}(p-1)$ so

$$D(f_2, s) = \prod_{p\geq 2}\left(1 + \sum_{k\geq 1} \frac{p^{k-1}(p-1)}{p^{(s+2)k}}\right) = \prod_{p\geq 2}\left(1 + \sum_{k\geq 1}\frac{1}{p^{(s+1)k}} - \sum_{k\geq 1}\frac{1}{p}\frac{1}{p^{(s+1)k}}\right)$$
$$= \prod_{p\geq 2}\left(1 + \frac{1}{p^{s+1}-1} - \frac{1}{p}\frac{1}{p^{s+1}-1}\right)$$

We notice that this product is similar to

$$\prod_{p\geq 2}\left(1 + \frac{1}{p^{s+1}-1}\right) = \zeta(s+1). \tag{23}$$

So we now write

$$D(f_2, s) = \zeta(s+1)\prod_{p\geq 2}\left(1 + \frac{1}{p^{s+1}-1} - \frac{1}{p(p^{s+1}-1)}\right)\left(1 - \frac{1}{p^{s+1}}\right)$$
$$= \zeta(s+1)\prod_{p\geq 2}\left(1 + \frac{1}{p^{s+1}-1} - \frac{1}{p(p^{s+1}-1)} - \frac{1}{p^{s+1}} - \frac{1}{p^{s+1}(p^{s+1}-1)} + \frac{1}{p^{s+2}(p^{s+1}-1)}\right).$$

Fortunately, a few simplifications occur here as

$$\frac{1}{p^{s+1}-1} - \frac{1}{p^{s+1}} = \frac{1}{p^{s+1}(p^{s+1}-1)}$$

so finally

$$D(f_2, s) = \zeta(s+1)\prod_{p\geq 2}\left(1 - \frac{1}{p(p^{s+1}-1)} + \frac{1}{p^{s+2}(p^{s+1}-1)}\right). \tag{24}$$

Step two

Our goal is again to find a function g and a function h such that $f_2 = g \star h$ ans so that $D(f_2, s) = D(g, s)D(h, s)$. So our function g here would be the function that associates $1/n$ to n and which Dirichlet series is $\zeta(s+1)$ we now need to find the function h such that

$$D(h, s) = \prod_{p\geq 2}\left(1 - \frac{1}{p(p^{s+1}-1)} + \frac{1}{p^{s+2}(p^{s+1}-1)}\right) = \prod_{p\geq 2}\left(1 + \sum_{k\geq 1}\frac{h(p^k)}{p^{ks}}\right).$$

Note that the absolute convergence of this series (expressed as a Euler product here) will be secured by having $s+2 > 1$ and $2s+3 > 1$ hence $s > -1$. We now need to identify the terms inside the Euler products

$$\sum_{k\geq 1}\frac{h(p^k)}{p^{ks}} = -\frac{1}{p(p^{s+1}-1)} + \frac{1}{p^{s+2}(p^{s+1}-1)} = -\frac{1}{p^{s+2}(1-\frac{1}{p^{s+1}})} + \frac{1}{p^{2s+3}(1-\frac{1}{p^{s+1}})}$$

As we have

$$\frac{1}{1-\frac{1}{p^{s+1}}} = \sum_{k\geq 0} \frac{1}{p^{k(s+1)}}, \tag{25}$$

we can deduce that

$$\begin{aligned}\sum_{k\geq 1} \frac{h(p^k)}{p^{ks}} &= -\frac{1}{p^{s+2}} \sum_{k\geq 0} \frac{1}{p^{k(s+1)}} + \frac{1}{p^{2s+3}} \sum_{k\geq 0} \frac{1}{p^{k(s+1)}} \\ &= -\frac{1}{p} \sum_{k\geq 0} \frac{1}{p^{(k+1)s+k+1}} + \frac{1}{p} \sum_{k\geq 0} \frac{1}{p^{(k+2)s+k+2}} \\ &= -\frac{1}{p}\frac{1}{p^{s+1}} = -\frac{1}{p^2}\frac{1}{p^s}.\end{aligned}$$

And finally, if we define the multiplicative function h by

$$\begin{cases} h(p) = -\frac{1}{p^2}, \\ h(p^k) = 0 \quad \text{for } k \geq 2, \end{cases} \tag{26}$$

it meets our requirements. We note that we have $h(n) = \frac{\mu(n)}{n^2}$!

Step three

Let us calculate the average order of f_2 using the convolution $g \star h$.

$$f_2(n) = \sum_{\ell m = n} h(\ell) \frac{1}{m}.$$

We have

$$\sum_{n\leq X} f_2(n) = \sum_{\ell m \leq X} h(\ell) \frac{1}{m} = \sum_{\ell \leq X} h(\ell) \sum_{m \leq X/\ell} \frac{1}{m}, \tag{27}$$

but for all real number $M \geq 1$ we have

$$\sum_{m\leq M} \frac{1}{n} = \ln M + \gamma + \mathcal{O}(1/M). \tag{28}$$

Please note that we also have:

$$\sum_{m\leq M} \frac{1}{n} = \ln M + \gamma + \mathcal{O}(M^{\sigma}) \tag{29}$$

for all $\sigma \in [-1, 0]$ and all $M \geq 1$ and where γ is the Euler constant.

Here again we need the error term to be less good because of the abscissa of absolute convergence of h which is -1.

$$\sum_{n\leq X} f_2(n) = \sum_{\ell \leq X} h(\ell) \Big(\ln \frac{X}{\ell} + \gamma + \mathcal{O}(\frac{X^{\sigma}}{\ell^{\sigma}}) \Big)$$

And here again, the condition $\ell \leq X$ can be replaced with $\ell \geq 1$ as the identity (and estimation) remains true. And so:

$$\sum_{n \leq X} f_2(n) = \ln X \sum_{\ell \geq 1} h(\ell) - \sum_{\ell \geq 1} h(\ell) \ln \ell + \gamma \sum_{\ell \geq 1} h(\ell) + \mathcal{O}\Big(X^\sigma \sum_{\ell \geq 1} \frac{|h(\ell)|}{\ell^\sigma}\Big).$$

H(s) is absolutely convergent for $s > -1$ so $\sum_{\ell \leq 1} |h(\ell)|/\ell^\sigma$ is finite for all $\sigma > -1$. And the error term makes sense for $\sigma \leq 0$ and so for all σ in $]-1, 0]$,

$$\sum_{n \leq X} f_2(n) = \ln X \sum_{\ell \geq 1} h(\ell) - \sum_{\ell \geq 1} h(\ell) \ln \ell + \gamma \sum_{\ell \geq 1} h(\ell) + \mathcal{O}(X^\sigma).$$

Determination of the constants

First we have

$$\sum_{\ell \geq 1} h(\ell) = \prod_{p \geq 2} \left(1 - \frac{1}{p^2}\right) = \frac{1}{\zeta(2)} = \frac{6}{\pi^2}.$$

Then the series

$$\sum_{\ell \geq 1} h(\ell) \ln \ell = \sum_{\ell \geq 1} \frac{\mu(\ell) \ln \ell}{\ell^2} \tag{30}$$

is absolutely convergent, as $\sum \frac{\ln \ell}{\ell^2}$ is convergent.
Note: if we define $H(s) = D(h, s)$, the series (30) is actually the opposite of $H'(0)$ where H' is the derivative of H.

6.2 Summation by parts

Lemma 6.1. *If for some σ in $]-1, 0]$,*

$$\sum_{n \leq X} \frac{\varphi(n)}{n^2} = a \ln X + b + \mathcal{O}(X^\sigma) \tag{31}$$

where a and b are real numbers, then we have

$$\sum_{n \leq X} \frac{\varphi(n)}{n} = aX + \mathcal{O}(X^{\sigma+1}).$$

Proof: We consider that we have $\sum_{n \leq X} \frac{\varphi(n)}{n^2} = a \ln X + b + \mathcal{O}(X^\sigma)$ and deduce $\sum_{n \leq X} \frac{\varphi(n)}{n}$. There are two important things that we will use in this technique. The first one being the simple fact that

$$n = \int_0^n dt$$

and the second one being that for all function f and for all positive real number X we have

$$\sum_{n\leq X} f(n)\int_0^n dt = \int_0^X \sum_{t\leq n\leq X} f(n)dt \tag{32}$$

Let us name S the sum $S(X) = \sum_{n\leq X}\frac{\varphi(n)}{n^2} = a\ln X + b + \mathcal{O}(X^\sigma)$, we then have

$$\begin{aligned}
\sum_{n\leq X}\frac{\varphi(n)}{n} &= \sum_{n\leq X}\frac{\varphi(n)}{n^2}\int_0^n dt = \int_0^X \sum_{t\leq n\leq X}\frac{\varphi(n)}{n^2}dt \\
&= \int_0^X (S(X)-S(t))dt = XS(X) - \int_0^X S(t)dt \\
&= aX\ln X + bX + \mathcal{O}(X^{\sigma+1}) - \big[a(t\ln t - t)\big]_0^X - bX + \mathcal{O}(X^{\sigma+1}) \\
&= aX + \mathcal{O}(X^{\sigma+1}).
\end{aligned}$$

□

6.3 Conclusion

So if we identify the constants used in the summation by parts in (31), we have $a = \frac{1}{\zeta(2)}$ and $b = \frac{\gamma}{\zeta(2)} + H'(0)$.

From the above and the summation by parts, we deduce that for all σ in $]-1,0]$ we have

$$\frac{1}{X}\sum_{n\leq X}\frac{\varphi(n)}{n} = \frac{6}{\pi^2} + \mathcal{O}(X^\sigma).$$

which concludes Theorem 4.2.

7 Third and last example, proof of Theorem 4.3

In this last example, $f_3(n) = \frac{\mu(n)}{\varphi(n)}$ and $g_3(n) = \frac{\mu(n)}{n}$, we just want to find h_3 so that $f_3 = g_3 \star h_3$, expressing their Dirichlet series with Euler products:

$$\begin{aligned}
D(f_3,s) = \sum_{n\geq 1}\frac{\mu(n)}{\varphi(n)n^s} &= \prod_{p\geq 2}\left(1 + \sum_{k\geq 1}\frac{\mu(p^k)}{\varphi(p^k)p^{ks}}\right) \\
&= \prod_{p\geq 2}\left(1 - \frac{1}{(p-1)p^s}\right)
\end{aligned}$$

As fo all $k \geq 2$, $\mu(p^k) = 0$.

And on the other hand,

$$D(g_3,s) = \sum_{n\geq 1}\frac{\mu(n)}{n^{s+1}} = \prod_{p\geq 2}\left(1 - \frac{1}{p^{s+1}}\right) = \frac{1}{\zeta(s+1)}$$

so

$$D(f_3,s) = D(g_3,s)\prod_{p\geq 2}\left(1-\frac{1}{(p-1)p^s}\right)\left(\frac{1}{1-\frac{1}{p^{s+1}}}\right).$$

So we need to identify:

$$\prod_{p\geq 2}\left(1-\frac{1}{(p-1)p^s}\right)\left(\frac{1}{1-\frac{1}{p^{s+1}}}\right) = D(h_3,s) = \prod_{p\geq 2}\left(1+\sum_{k\geq 1}\frac{h_3(p^k)}{p^{ks}}\right). \quad (33)$$

We have

$$\begin{aligned}\prod_{p\geq 2}\left(1-\frac{1}{(p-1)p^s}\right)\left(\frac{1}{1-\frac{1}{p^{s+1}}}\right) &= \prod_{p\geq 2}\left(1-\frac{1}{(1-\frac{1}{p})p^{s+1}}\right)\left(1+\sum_{k\geq 1}\frac{1}{p^{(s+1)k}}\right)\\ &= \prod_{p\geq 2}\left(1-\frac{1}{p^{s+1}}\sum_{k\geq 0}\frac{1}{p^k}\right)\left(1+\sum_{k\geq 1}\frac{1}{p^{(s+1)k}}\right)\\ &= \prod_{p\geq 2}\left(1-\sum_{k\geq 1}\frac{1}{p^{k+s}}\right)\left(1+\sum_{k\geq 1}\frac{1}{p^{(s+1)k}}\right)\end{aligned}$$

We expand the product and find that it is equal to

$$= \prod_{p\geq 2}\left(1-\sum_{k\geq 1}\frac{1}{p^{k+s}}+\sum_{k\geq 1}\frac{1}{p^{(s+1)k}}-\sum_{\substack{k\geq 1\\ k'\geq 1}}\frac{1}{p^{k+s+(s+1)k'}}\right)$$

$$= \prod_{p\geq 2}\left(1+\frac{1}{p^s}\left(-\sum_{k\geq 1}\frac{1}{p^k}+\frac{1}{p}\right)+\frac{1}{p^{2s}}\left(\frac{1}{p^2}-\sum_{k\geq 1}\frac{1}{p^{k+1}}\right)+\frac{1}{p^{3s}}\left(\frac{1}{p^3}-\sum_{k\geq 1}\frac{1}{p^{k+2}}\right)+...\right)$$

We can now define h_3 with its value for prime powers as h_3 is multiplicative:

$$\begin{cases} h_3(1) = 1 \\ h_3(p) = -\sum_{k\geq 1}\frac{1}{p^k}+\frac{1}{p} = -\frac{1}{p(p-1)} \\ h_3(p^\alpha) = \frac{1}{p^\alpha}-\sum_{k\geq 1}\frac{1}{p^{\alpha+k-1}} = \frac{1}{p^\alpha}-\frac{1}{p^{\alpha-1}}\frac{1}{p-1} \qquad \text{for } \alpha\geq 2\end{cases} \quad (34)$$

So $h_3(1) = 1$ and for all prime p and for all integer $\alpha \geq 1$, $h_3(p^\alpha) = -\frac{1}{p^\alpha(p-1)} = -\frac{1}{p\varphi(p^\alpha)}$. If we define h_3 for all integer n, we have

$$h_3(n) = \frac{(-1)^{\omega(n)}}{\varphi(n)F(n)}$$

where

$$F(n) = \prod_{p^\alpha | n} p$$

F is the function that removes the multiciplity order to each prime factor of n. And $\omega(n) = \sum_{p|n} 1$.

Note: we expect to find the abscissa of absolute convergence of $D(h_3, s)$ strictly less than the one of $D(f_3, s)$ (which is the same as $D(g_3, s)$). Indeed, the abscissa of absolute convergence of $D(g_3, s)$ is 1 when the abscissa of absolute convergence of $D(h_3, s)$ is -1. As for any real number s,

$$\sum_{\alpha \geq 1} \frac{|h_0(p^\alpha)|}{p^{\alpha s}} = \mathcal{O}\left(1 + \sum_{\alpha \geq 1} \frac{1}{p^{\alpha s + s + 1}}\right) \tag{35}$$

And

$$1 + \sum_{\alpha \geq 1} \frac{1}{p^{\alpha s + s + 1}} = 1 + \frac{1}{p^{(s+1)+1}} \sum_{\beta \geq 0} \frac{1}{p^{\beta(s+1)}} = \text{Cst} \times \left(1 + \frac{1}{p^{s+2}}\right)$$

So $D(|h_3|, s)$ is convergent for all $s + 2 > 1$ hence $s > -1$.

We now find that

Lemma 7.1. *h_3 being the function defined above,*

$$\sum_{\ell \geq 1} h_3(\ell) = 0 \qquad \textit{and} \qquad \sum_{\ell \geq 1} |h_3(\ell)| = \prod_{p \geq 2} \left(1 + \frac{1}{(p-1)^2}\right) = 2.8264...$$

Proof : We use a Euler product:

$$\sum_{\ell \geq 1} h_3(\ell) = \prod_{p \geq 2} \left(1 + \sum_{k \geq 1} h_3(p^k)\right) = \prod_{p \geq 2} \left(1 - \sum_{k \geq 1} \frac{1}{p^k(p-1)}\right)$$
$$= \prod_{p \geq 2} \left(1 - \frac{1}{p-1} \sum_{k \geq 1} \frac{1}{p^k}\right)$$

hence

$$\sum_{\ell \geq 1} h_3(\ell) = \prod_{p \geq 2} \left(1 - \frac{1}{(p-1)^2}\right) = 0.$$

And we deduce easily the second sum as it is equal to $\prod_{p \geq 2} \left(1 + \sum_{k \geq 1} \frac{1}{p^k(p-1)}\right)$.
□

We can now prove Theorem 4.3:

Theorem. *For all $X > 100000$,*

$$\left| \sum_{n \leq X} \frac{\mu(n)}{\varphi(n)} \right| \leq \frac{0.082}{\log X} + \frac{30}{X^{1/3}} \tag{36}$$

Proof : We know from [5], Teorem 1.2 that for $X > 100000$

$$\left|\sum_{n\leq X}\frac{\mu(n)}{n}\right| < \frac{1}{69\log X} \tag{37}$$

Using our identity $f_3 = g_3 \star h_3$ where $g_3(n) = \frac{\mu(n)}{n}$, we have

$$\begin{aligned}\sum_{n\leq X} f_3(n) &= \sum_{\ell\leq X} h_3(\ell)\sum_{m\leq \frac{X}{\ell}} g_3(m)\\ &= \sum_{\ell\leq\sqrt{X}} h_3(\ell)\sum_{m\leq\frac{X}{\ell}} g_3(m) + \sum_{\sqrt{X}\leq\ell\leq X} h_3(\ell)\sum_{m\leq\frac{X}{\ell}} g_3(m)\end{aligned}$$

In the first term, we use (37) as

$$\left|\sum_{m\leq\frac{X}{\ell}} g_3(m)\right| < \frac{1}{69\log\frac{X}{\ell}} \leq \frac{1}{69\log\sqrt{X}}$$

And so

$$\begin{aligned}\left|\sum_{\ell\leq\sqrt{X}} h_3(\ell)\sum_{m\leq\frac{X}{\ell}} g_3(m)\right| &\leq \frac{1}{69\log\sqrt{X}}\sum_{\ell\leq\sqrt{X}}|h_3(\ell)|\\ &\leq \frac{2}{69\log X}\sum_{\ell\geq 1}|h_3(\ell)| = \frac{C_3}{\log X}\end{aligned}$$

As $\sum |h_3(n)|$ is convergent. And from Lemma 7.1 we decuce $C_3 = 0.0819....$

For the second term, we use the Rankin method: for all $a > 0$ we have

$$\begin{aligned}\sum_{\sqrt{X}\leq\ell\leq X}|h_3(\ell)| \leq \sum_{\sqrt{X}\leq\ell}|h_3(\ell)| &\leq \sum_{\sqrt{X}\leq\ell}|h_3(\ell)|\left(\frac{\ell}{\sqrt{X}}\right)^a\\ &\leq \sum_{1\leq\ell}|h_3(\ell)|\left(\frac{\ell}{\sqrt{X}}\right)^a\\ &\leq \sum_{1\leq\ell}\frac{|h_3(\ell)|}{\ell^{-a}}\left(\frac{1}{\sqrt{X}}\right)^a\end{aligned}$$

But $\sum_{1\leq\ell}\frac{|h_3(\ell)|}{\ell^{-a}} = D(|h_3|, -a)$ and we know that the Dirichlet series $D(|h_3|, s)$ is convergent for all $s > -1$, so here for all a such that $-a > -1$. We choose $a = \frac{2}{3}$ and so we find that

$$\sum_{\sqrt{X}\leq\ell\leq X}|h_3(\ell)| = \mathcal{O}\left(\frac{1}{X^{1/3}}\right). \tag{38}$$

More precisely, we expand $D(|h_3|, -2/3)$ into a Euler product to calculate it, the same way as we did in Lemma 7.1 and we obtain:

$$D(|h_3|, -2/3) = \sum_{1 \le n} |h_3(n)| n^{2/3} = \prod_{p \ge 2} \left(1 + \frac{1}{(p-1)(p^{1/3}-1)}\right) \le 27 \qquad (39)$$

Now because $\sum g_3(n)$ is also convergent, $\left|\sum_{m \le \frac{X}{\ell}} g_3(m)\right|$ is bounded by 1 (from (37)). Hence

$$\left|\sum_{\sqrt{X} \le \ell \le X} h_3(\ell) \sum_{m \le \frac{X}{\ell}} g_3(m)\right| = \mathcal{O}\left(\frac{1}{X^{1/3}}\right).$$

More precisely, $\sum_{1 \le n} |h_3(n)|$ as estimated in Lemma 7.1,

$$\begin{aligned}\left|\sum_{\sqrt{X} \le \ell \le X} h_3(\ell) \sum_{m \le \frac{X}{\ell}} g_3(m)\right| &\le \left|\sum_{m \le \frac{X}{\ell}} g_3(m)\right| \sum_{1 \le n} |h_3(n)| \frac{1}{X^{1/3}} \\ &\le \sum_{1 \le n} |h_3(n)| \frac{1}{X^{1/3}} \le \frac{3}{X^{1/3}}\end{aligned}$$

□

The theorem holds for any $X > 100000$ but we can check it holds also for $2 \le X \le 100000$ with GP PARI and the following script:

```
somme=1.0;
for(k = 2, 100000, somme += moebius(k)/eulerphi(k);
if(abs(somme)>0.082/log(k)+30/$k^{1/3}$,print("Problem at ", k)))
```

We can generalize this last result to any function $f = g_3 \star h$ where h is a *slight* perturbation.

Theorem 7.2. *For all function f defined by $f = g_3 \star h$ where $g_3(n) = \frac{\mu(n)}{n}$ and h is such that $D(|h|, s)$ is convergent for all $s > -1$, we have*

$$\left|\sum_{n \le X} f(n)\right| \le \frac{C}{\log X} + \frac{C'}{X^{1/3}}$$

where $C = \frac{2}{69} D(|h|, 0)$ and $C' = D(|h|, -2/3) + D(|h|, 0)$.

Proof : We follow exactly the method above, using the same bound (37), we use the fact that $D(g_3, s) = \frac{1}{\zeta(s+1)}$ and we can find h from its Dirichlet series which will be convergent for all $s > -1$. □

Divertimento

The summation by part is such an interesting tool that we wish to use it again to show its strength. Let us suppose we are interested in finding the average order of f_0/n, it will now be very simple!

This time the idea is to use the property:

$$\sum_{n\leq t} 1 = t + \mathcal{O}(1).$$

And to notice that:

$$\frac{1}{n} = \frac{1}{X} + \int_n^X \frac{dt}{t^2}. \tag{40}$$

And so:

$$\sum_{n\leq X} \frac{1}{n} = \frac{\sum_{n\leq X} 1}{X} + \int_1^X \sum_{n\leq t} 1 \frac{dt}{t^2} = \ln X + \mathcal{O}(1).$$

We use these properties to show that $\sum_{n\leq X} \frac{f_0(n)}{n} = 2C_0X + \mathcal{O}(X^\sigma)$ for all σ real in $]1/2, 1]$.

Indeed, as in (40) we have:

$$\begin{aligned}\sum_{n\leq X} \frac{f_0(n)}{n} &= \sum_{n\leq X} f_0(n)\Big(\frac{1}{X} + \int_n^X \frac{dt}{t^2}\Big) = \frac{\sum_{n\leq X} f_0(n)}{X} + \int_1^X \sum_{n\leq t} f_0(n)\frac{dt}{t^2} \\ &= C_0X + \mathcal{O}(X^\sigma) + C_0 \int_1^X dt + \mathcal{O}\Big(\int_1^X t^{\sigma-1} dt\Big) \\ &= 2C_0X + \mathcal{O}(X^\sigma).\end{aligned}$$

Et voilà !

References

[1] Tom M. Apostol. *Introduction toAnalytic Number Theory.* Undergraduate Texts in Mathematics. Springer, 1976.

[2] P.T. Bateman and H.G. Diamond. *Analytic number theory: An introductory course.* Monographs in Number Theory. World Scientific Publishing Co, 2004.

[3] Roger Godement. *Analysis I, Convergence, Elementary functions.* Universitext. Springer, Dec. 2012.

[4] G.H. Hardy and Marcel Riesz. *The general theory of Dirichlet's series.* Cambridge Univeristy Press, 1915.

[5] Olivier Ramaré. Explicit estimates on several summatory functions involving the moebius function. *Mathematics of Computation*, 84(293):1359–1387, May 2015.

[6] Gérald Tenenbaum. *Introduction to analytic and probabilistic number theory.* Cambridge studies in advanced mathematics, 1995.