EXPLICIT COUNT OF INTEGRAL IDEALS OF AN IMAGINARY QUADRATIC FIELD

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Abstract. We provide explicit bounds for the number of integral ideals of norms at most X is $\mathbb{Q}[\sqrt{d}]$ when $d<0$ is a fundamendal discriminant with an error term of size $\mathcal{O}(X^{1/3})$. In particular, we prove that, when χ is the non-principal character modulo 3 and $X\ge 1$, we have $\sum_{n\le X}(1\star\chi)(n)=\frac{\pi X}{3\sqrt{3}}+\mathcal{O}^*(1.94\,X^{1/3})$, and that , when χ is the non-principal character modulo 4 and $X\ge 1$, we have $\sum_{n\le X}(1\star\chi)(n)=\frac{\pi X}{4}+\mathcal{O}^*(1.4\,X^{1/3})$.

Résumé. Nous dénombrons de façon explicite avec un terme d'erreur $\mathcal{O}(X^{1/3})$ le nombre d'idéaux entiers de norme au plus X du corps $\mathbb{Q}[\sqrt{d}]$ lorsque $d<0$ est un discriminant fondamental. Nous montrons en particulier que, lorsque χ est le caractère non principal modulo 3 et $X\ge 1$, nous avons $\sum_{n\le X}(1\star\chi)(n)=\frac{\pi X}{3\sqrt{3}}+\mathcal{O}^*(1.94\,X^{1/3})$, et que , lorsque χ est le caractère non principal modulo 4 et $X\ge 1$, nous avons $\sum_{n\le X}(1\star\chi)(n)=\frac{\pi X}{4}+\mathcal{O}^*(1.4\,X^{1/3})$.

1. Introduction and results

General context. Let $\mathbb{K}$ be a number field, of degree $n_{\mathbb{K}}$, discriminant $\Delta(\mathbb{K})$. Its associated Dedekind zeta-function $\zeta_{\mathbb{K}}$ has a simple pole at $s=1$, with residue denoted by $\kappa_{\mathbb{K}}$. Counting the number of integral ideals of norm below some bound is a fundamental question that has been addressed by numerous authors. The explicit angle has been less popular and three pieces of works emerge: the paper [3] by K. Debaene, the PhD memoir [10] by J. Sunley and its upgraded version [9] by E.S. Lee. The first goes by lattice point counting, gets a dependence on the size of order $x^{1-1/n_{\mathbb{K}}}$ and treats the dependence in the field very finely. This approach is reused in [4] to enumerate integral ideals in ray classes. The second and third approaches follow the analytic treatment proposed by E. Landau: they get a better dependence on the size of order $x^{1-2/(n_{\mathbb{K}}+1)}$ but only rely on the discriminant of the field, an invariant which can be notoriously large. The constants obtained have the clear advantage of being explicit but they remain gigantic. For example, when $n_{\mathbb{K}}=2$, the error term in [9] reads

$$\mathcal{O}^*\left(8.81\cdot 10^{11}|\Delta_{\mathbb{K}}|^{\frac{1}{n_{\mathbb{K}}+1}}(\log|\Delta_{\mathbb{K}}|)^{n_{\mathbb{K}}-1}x^{1-\frac{2}{n_{\mathbb{K}}+1}}\right)$$

where $f=\mathcal{O}^*(g)$ means that $|f|\le g$. We aim here at being less demanding in generality but to gain in numerical precision.

Our results for imaginary quadratic number fields. Let d be a squarefree integer. We associate to this integer its fundamental discriminant defined by

$$\Delta_d=\begin{cases} d & \text{when } d\equiv 1[4],\\ 4d & \text{when } d\equiv 2,3[4].\end{cases} \tag{1}$$

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The associated character is given in terms of the Kronecker symbol by the formula $\chi(n) = (\frac{\Delta_d}{n})$. We refer to Chapter V, Section 4 of the book [?] by Z.I. Borevitch & I.R. Chafarevitch for the link between this character and the decomposition law of prime ideals in $\mathbb{Q}[\sqrt{-d}]$. Let us mention here that $\zeta_{\mathbb{Q}[\sqrt{d}]}(s) = \zeta(s)L(s,\chi)$.

Theorem 1.1. *When $X \ge \max(|\Delta_d|, 2c_0(d))$ and d is a negative squarefree integer, we have*

$$\sum_{n\le X}(1\star\chi)(n) = XL(1,\chi) + \frac{1}{2|\Delta_d|}\sum_{1\le r\le|\Delta_d|} r\chi(r) + \mathcal{O}^*\bigl(0.76\,L(1,\chi)c_0(d)X^{1/3}\bigr)$$

where

$$c_0(d) = \max(w(3/4), W(5/4))^{2/3}, \tag{2}$$

and, the function w and W are respectively defined, when $s\in(0,1)$ for w by

$$w(s) = \max_{M\ge1}\sum_{m\le M}\frac{(1\star\chi)(m)}{m^s M^{1-s}L(1,\chi)}, \tag{3}$$

and when $s>1$ for W by

$$W(s) = \max_{M\ge1}\frac{M^{s-1}}{L(1,\chi)}\sum_{m\ge M}\frac{(1\star\chi)(m)}{m^s}. \tag{4}$$

When $X \ge \max(130^2|\Delta_d|, 10c_0(d))$, the constant 0.76 may be replaced by 0.67.

Lemmas 3.3 and 3.4 propose upper bounds for $w(3/4)$ and $W(5/4)$. It is in particular proved that $\min(w(3/4), W(5/4)) \ge 4$. Therefore imposing larger bounds on X would reduce only marginally the final constant. It may still be of interest for very small values of d, where the range in X can be completed by direct computations.

Initial numerical data. We start by some numerical verification done with a GP-Pari script.

Theorem 1.2. *Let χ be the non principal character modulo 4. For $X\in[1,10^8]$, we have*

$$\sum_{n\le X}(1\star\chi)(n) = \frac{\pi X}{4} + \mathcal{O}^*\bigl(2.08\,X^{1/4}\bigr).$$

The constant in the big-O seems to increase slightly when X increases.

Theorem 1.3. *Let χ be the non principal character modulo 3. For $X\in[1,10^8]$, we have*

$$\sum_{n\le X}(1\star\chi)(n) = \frac{\pi X}{3\sqrt{3}} + \mathcal{O}^*\bigl(1.63\,X^{1/4}\bigr).$$

Special cases. When we specialise Theorem 1.1 to $-19\le d\le -1$ and incorporate Theorem 1.3 and 1.2, we get the next results.

Corollary 1.4. *Let χ be the non principal character modulo 4. For $X\ge1$, we have*

$$\sum_{n\le X}(1\star\chi)(n) = \frac{\pi X}{4} + \mathcal{O}^*\bigl(1.4\,X^{1/3}\bigr).$$

Corollary 1.5. *Let χ be the non principal character modulo 3. For $X\ge1$, we have*

$$\sum_{n\le X}(1\star\chi)(n) = \frac{\pi X}{3\sqrt{3}} + \mathcal{O}^*\bigl(1.94\,X^{1/3}\bigr).$$

Corollary 1.6. *Let χ be the quadratic character of $\mathbb{Q}[\sqrt{d}]$, where $-19 \le d \le -1$. For $X \ge 68$, we have*

$$\sum_{n \le X} (1 \star \chi)(n) = X L(1,\chi) + \frac{1}{2|\Delta|} \sum_{1 \le r \le |\Delta_d|} r\chi(r) + \mathcal{O}^*\big(3.4\, X^{1/3}\big).$$

Methodology. E. Landau's approach in [7] (See also [8, Satz 210]) relies on several ingredients, but the first and main one is the functional equation of the associated Dedekind zeta function. E. Landau sends the line of integration to $\Re s = -1/2$, then uses the functional equation to study the last integral. This Landau's approach is described in modern language in [9]. The process used has become known as the Voronoï Summation Formula(s), based on [11, 12], though this latter is more commonly used for the divisor function. In fact, though the papers of Voronoï largely predates the ones of Landau, and it cannot be assumed that Landau did not know of them, Landau does not mention the Voronoï approach, a surprising fact as this author has most of the time been very prompt in explaining the genesis of ideas. This absence may be due to the combination of two facts: Landau worked in great generality, with intricate Gamma-factors, and a general view of the Voronoï process was missing at the time. This process is now well understood and is for instance well-documented in Chapter 10 of the book [2] by H. Cohen. The addition of Voronoï is to recognize the involved Mellin transform as a Bessel function and to consider a functional transform of the initial weight function, see Lemma 4.1 below. We follow this approach here.

Two more ingredients are being used: a non-negative smoothing device and an apriori trivial upper bound for the number of integral ideals below some bound to avoid divisor functions of x in the remainder term, see Lemma 3.3 and 3.4 below.

We do not examine what happens for the small values of the size with respect to the discriminant.

Generalization. The present method may be used for other fields, and this is currently being investigated. We aim at exploring here with precision this simplest case. Moreover, this case is the only one where we have $\zeta_{\mathbb{K}}(0) \neq 0$, introducing an additional term in Lemma 4.1.

Notation. We use $f = \mathcal{O}^*(g)$ to mean that $|f| \le g$ and, for a real number x, we write $x = [x] + \{x\}$ where $[x]$ is the integer part of x while $\{x\}$ denotes its fractional part.

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2. On the Bessel functions

We need in this work the J-Bessel functions of the first kind. They are the subject of several treatises, of which we only cite the reference book [1] by M. Abramowitz & I.A. Stegun. We shall use explicit estimates due to Krasikov in [6] and only for J_0, J_1 and J_2.

Lemma 2.1. *When $a > 0$, we have*

$$\int_0^X J_0(a\sqrt{t})dt = \frac{2\sqrt{X}}{a} J_1(a\sqrt{X})$$

and

$$\int_0^X t J_0(a\sqrt{t})dt = \frac{4X}{a^2} J_2(a\sqrt{X}) + \frac{2X^{3/2}}{a} J_1(a\sqrt{X}).$$

Proof. Indeed, when $\nu \geq 1$, we have $J_\nu(t)' = J_{\nu-1}(t) - J_\nu(t)\nu/t$. Hence

$$(\sqrt{t}J_1(a\sqrt{t}))' = \frac{1}{2\sqrt{t}}J_1(a\sqrt{t}) + \frac{a}{2}\biggl(J_0(a\sqrt{t}) - \frac{1}{a\sqrt{t}}J_1(a\sqrt{t})\biggr) = \frac{a}{2}J_0(a\sqrt{t})$$

which proves the first formula. For the second one, we notice similarly that

$$\begin{aligned}(t^{3/2}J_1(a\sqrt{t}))' &= \frac{3\sqrt{t}}{2}J_1(a\sqrt{t}) + \frac{at}{2}\biggl(J_0(a\sqrt{t}) - \frac{1}{a\sqrt{t}}J_1(a\sqrt{t})\biggr)\\ &= \frac{a}{2}tJ_0(a\sqrt{t}) - \sqrt{t}J_1(a\sqrt{t})\end{aligned}$$

and

$$(tJ_2(a\sqrt{t}))' = J_2(a\sqrt{t}) + \frac{a\sqrt{t}}{2}\biggl(J_1(a\sqrt{t}) - \frac{2}{a\sqrt{t}}J_2(a\sqrt{t})\biggr) = \frac{a\sqrt{t}}{2}J_1(a\sqrt{t})$$

and therefore

$$\frac{d}{dx}\biggl(\frac{at^{3/2}}{2}J_1(a\sqrt{t}) + tJ_2(a\sqrt{t})\biggr) = \frac{a^2}{4}tJ_0(a\sqrt{t})$$

□

Lemma 2.2. *When $\nu > 0$ and $x \geq 0$, we have*

$$\left|J_\nu(x) - \sqrt{\frac{2}{\pi x}}\cos(x - (2\nu+1)\tfrac{\pi}{4})\right| \leq \frac{4|\nu^2 - 1/4|}{5x^{3/2}}.$$

When $\nu > 1/2$ and $x \geq 0$, we have $|x^2 - \nu^2 + \frac{1}{4}|^{1/4}|J_\nu(x)| \leq \sqrt{2/\pi}$.

Proof. The first inequality is given in a consequence of [6, Theorem 4] by Krasikov while the second one comes from [6, Theorem 3]. □

Lemma 2.3. *With*

$$T(z;a) = \frac{(1+z)J_2(a\sqrt{1+z}) - J_2(a)}{z} \tag{5}$$

we have, when $a \geq 4\pi$ and $z \in (0, 1/3]$:

$$|T(z,a)| \leq \min\biggl(0.53\sqrt{a}, \frac{7}{3z\sqrt{a}}\biggr).$$

When $a \geq 130 \cdot 4\pi$ and $z \in (0, 1/10)$:

$$|T(z,a)| \leq \min\biggl(0.4\sqrt{a}, \frac{2.1}{z\sqrt{a}}\biggr).$$

Proof. By the mean value theorem and when $z > 0$, we find that

$$\begin{aligned}|T(z;a)| &\leq \frac{a(\sqrt{1+z}-1)}{z}\max_{a\leq t\leq a\sqrt{1+z}}|J_2'(t)| + |J_2(a\sqrt{1+z})|\\ &\leq \frac{a}{2}\max_{a\leq t\leq a\sqrt{1+z}}\biggl(|J_1(t)| + \frac{2}{t}|J_2(t)|\biggr) + |J_2(a\sqrt{1+z})|.\end{aligned}$$

The map $x \mapsto |x^2 - 3/4|^{1/4}$ is non-decreasing when $x \geq \sqrt{3}/2$ and the map $x \mapsto |x^2 - 7/4|^{1/4}$ is also non-decreasing when $x \geq \sqrt{7}/2$. On assuming that $a \geq \sqrt{7}/2$, Lemma 2.2 thus gives us that

$$\sqrt{\pi/2}|T(z;a)| \leq \frac{a}{2|a^2 - 3/4|^{1/4}} + \frac{2}{|a^2 - 7/4|^{1/4}}.$$

When $a \geq 4\pi$, a rapid plot shows that $\sqrt{\pi/2}|T(z;a)| \leq 0.66\sqrt{a}$, while, when $a \geq 130 \cdot 4\pi$, we find that $\sqrt{\pi/2}|T(z;a)| \leq 0.5013\sqrt{a}$. This establishes the first bound. When a is large, it is better to simply use

$$\begin{aligned}|T(z;a)| &\leq \frac{|J_2(a\sqrt{1+z})| + |J_2(a)|}{z} + |J_2(a\sqrt{1+z})| \\ &\leq \frac{2}{z|a^2 - 7/4|^{1/4}} + \frac{1}{|a^2(1+z) - 7/4|^{1/4}} \leq \frac{7}{3z\sqrt{a}}\end{aligned}$$

which we prove first for $z \leq 1/4$ by using $|a^2(1+z) - 7/4|^{1/4} \geq |a^2 - 7/4|^{1/4}$ and then on discretizing the interval (such precision is not required, it only leads to be better looking estimate). When $z \leq 1/10$ and $a \geq 130 \cdot 4\pi$, we find that $|T(z;a)| \leq 2.1/(z\sqrt{a})$. □

3. Some a priori estimates

Let us start with two well-known estimates.

Lemma 3.1. *When s is a positive real number, $s \neq 1$ and $M \geq 1$, we have*

$$\sum_{m \leq M} \frac{1}{m^s} = \frac{M^{1-s}}{1-s} + \zeta(s) + \mathcal{O}^*(1/M^s).$$

When $s \in [1/2, 1)$, we have $\sum_{m \leq M} 1/m^s \leq M^{1-s}/(1-s)$.

Proof. Indeed, we find that

$$\begin{aligned}\sum_{m \leq M} \frac{1}{m^s} &= s\int_1^M [t]\frac{dt}{t^{s+1}} + \frac{[M]}{M^s} = \frac{s}{s-1} - s\int_1^M \{t\}\frac{dt}{t^{s+1}} + \frac{M^{1-s}}{1-s} - \frac{\{M\}}{M^s} \\ &= \frac{s}{s-1} - s\int_1^\infty \{t\}\frac{dt}{t^{s+1}} + \frac{M^{1-s}}{1-s} + s\int_M^\infty \{t\}\frac{dt}{t^{s+1}} - \frac{\{M\}}{M^s} \\ &= \zeta(s) + \frac{M^{1-s}}{1-s} + s\int_M^\infty \{t\}\frac{dt}{t^{s+1}} - \frac{\{M\}}{M^s}.\end{aligned}$$

We finally check that

$$s\int_M^\infty \{t\}\frac{dt}{t^{s+1}} - \frac{\{M\}}{M^s} = s\int_M^\infty (\{t\} - \{M\})\frac{dt}{t^{s+1}}$$

and the first part of the lemma follows readily. For the second part, we notice that $\zeta(s)$ is negative and decreasing over $[0,1)$. In $1/2$, GP/Pari tells us that $\zeta(1/2) = -1.460\,354\cdots$. The proof is complete. □

Lemma 3.2. *When s is a positive real number, χ is any non-principal Dirichlet character and $L' \geq L \geq 1$, we have*

$$\sum_{L \leq \ell \leq L'} \frac{\chi(\ell)}{\ell^s} = \mathcal{O}^*(\Omega(\chi)/L^s)$$

where

$$\Omega(\chi) = \max_{L' \geq L \geq 1}\left|\sum_{L \leq \ell \leq L'} \chi(\ell)\right|. \tag{6}$$

We also have $\displaystyle\sum_{\ell \leq L} \frac{\chi(\ell)}{\ell^s} = L(s,\chi) + \mathcal{O}^*(\Omega(\chi)/L^s)$.

Proof. The first estimate follows by Abel summation. It may also be proved by using:

$$\sum_{L\le\ell\le L'}\frac{\chi(\ell)}{\ell^s}=s\int_L^{L'}\sum_{L\le\ell\le t}\chi(\ell)\frac{dt}{t^{s+1}}.$$

The second part of the lemma is a trivial consequence of the first one together with the expression $L(s,\chi)=\sum_{\ell\ge1}\chi(\ell)/\ell^s$. □

The next lemma gives an upper bound as well as a mean to approximate $w(s)$ defined in (3).

Lemma 3.3. *When $s\in[1/2,1)$ and χ is a non-principal quadratic Dirichlet character, we have*

$$\sum_{n\le N}\frac{(1\star\chi)(n)}{n^sN^{1-s}}\le\frac{L(1,\chi)}{1-s}+\frac{|\zeta(s)L(s,\chi)|}{N^{1-s}}+\frac{(\frac14+|\zeta(s)|+\frac{2}{1-s})\Omega(\chi)}{\sqrt{N}}.$$

Moreover, the quantity considered is asymptotic to $L(1,\chi)/(1-s)$.

In this lemma as well as in the next one, we remember that $(1\star\chi)(n)$ is real and non-negative.

Proof. By using Lemma 3.1 and 3.2, and the Dirichlet hyperbola formula, we find that the sum S to be computed equals (with parameters $L\ge1$ and $M\ge1$ such that $LM=N$)

$$\begin{aligned}S&=\sum_{\ell\le L}\frac{\chi(\ell)}{\ell^s}\biggl(\zeta(s)+\frac{(N/\ell)^{1-s}}{1-s}+\mathcal{O}^*\Bigl(\frac{\ell^s}{4N^s}\Bigr)\biggr)+\sum_{m\le M}\frac{1}{m^s}\mathcal{O}^*\bigl(\Omega(\chi)/L^s\bigr)\\&=\zeta(s)\sum_{\ell\le L}\frac{\chi(\ell)}{\ell^s}+\frac{N^{1-s}}{1-s}\sum_{\ell\le L}\frac{\chi(\ell)}{\ell}+\mathcal{O}^*\biggl(\frac{L}{4N^s}+\frac{\Omega(\chi)M^{1-s}}{(1-s)L^s}\biggr)\\&=\zeta(s)L(s,\chi)+\frac{N^{1-s}L(1,\chi)}{1-s}\\&\quad+\mathcal{O}^*\biggl(\frac{N^{1-s}\Omega(\chi)}{(1-s)L}+\frac{L}{4N^s}+\frac{\Omega(\chi)M^{1-s}}{(1-s)L^s}+|\zeta(s)|\frac{\Omega(\chi)}{L^s}\biggr)\end{aligned}$$

so that

$$\begin{aligned}S/N^{1-s}\le&\frac{|\zeta(s)L(s,\chi)|}{N^{1-s}}+\frac{L(1,\chi)}{1-s}+\frac{\Omega(\chi)}{(1-s)L}+\frac{L}{4N}\\&+\Omega(\chi)\frac{M^{1-s}}{(1-s)L^sN^{1-s}}+|\zeta(s)|\frac{\Omega(\chi)}{L^sN^{1-s}}.\end{aligned}$$

The simplistic choice $L=M=N^{1/2}$ leads to

$$S/N^{1-s}\le\frac{|\zeta(s)L(s,\chi)|}{N^{1-s}}+\frac{L(1,\chi)}{1-s}+\frac{2\Omega(\chi)+\frac14}{(1-s)\sqrt{N}}+|\zeta(s)|\frac{\Omega(\chi)}{N^{1-s/2}}$$

as we have announced. □

Our final tool in this section is the next lemma which gives an upper bound as well as a mean to approximate $W(s)$ defined in (4).

Lemma 3.4. *When $s>1$ and χ is a non-principal quadratic Dirichlet character, we have*

$$N^{s-1}\sum_{n>N}\frac{(1\star\chi)(n)}{n^s}\le\frac{L(1,\chi)}{s-1}+\frac{(2\zeta(s)+\frac{1}{s-1})\Omega(\chi)+\frac14}{\sqrt{N}}.$$

Moreover, the quantity considered is asymptotic to $L(1,\chi)/(s-1)$.

Proof. We proceed as in Lemma 3.3. On setting $S = \sum_{n\le N}(1\star\chi)(n)/n^s$, we find that (with parameters $L \ge 1$ and $M \ge 1$ such that $LM = N$)

$$\begin{aligned} S &= \sum_{\ell\le L}\frac{\chi(\ell)}{\ell^s}\biggl(\zeta(s) - \frac{(\ell/N)^{s-1}}{s-1} + \mathcal{O}^*\Bigl(\frac{\ell^s}{4N^s}\Bigr)\biggr) + \sum_{m\le M}\frac{1}{m^s}\mathcal{O}^*\left(\Omega(\chi)/L^s\right) \\ &= \zeta(s)\sum_{\ell\le L}\frac{\chi(\ell)}{\ell^s} - \frac{1}{(s-1)N^{s-1}}\sum_{\ell\le L}\frac{\chi(\ell)}{\ell} + \mathcal{O}^*\Bigl(\frac{L}{4N^s} + \Omega(\chi)\frac{\zeta(s)}{L^s}\Bigr) \\ &= \zeta(s)L(s,\chi) - \frac{L(1,\chi)}{(s-1)N^{s-1}} + \mathcal{O}^*\Bigl(\frac{\Omega(\chi)}{(s-1)LN^{1-s}} + \frac{L}{4N^s} + 2\Omega(\chi)\frac{\zeta(s)}{L^s}\Bigr)\end{aligned}$$

so that

$$(\zeta(s)L(s,\chi) - S)N^{s-1} \le \frac{L(1,\chi)}{s-1} + \frac{\Omega(\chi)}{(s-1)L} + \frac{L}{4N} + 2\Omega(\chi)\frac{\zeta(s)N^{s-1}}{L^s}.$$

We take $L = M = N^{1/2}$ and get

$$(\zeta(s)L(s,\chi) - S)N^{s-1} \le \frac{L(1,\chi)}{s-1} + \frac{(2\zeta(s) + \frac{1}{s-1})\Omega(\chi) + \frac14}{\sqrt{N}}.$$

We finally notice that

$$\zeta(s)L(s,\chi) - S = \sum_{n>N}\frac{(1\star\chi)(n)}{n^s}.$$

□

Notice that we could improve on Lemma 3.3 and 3.4 by using our final result! This is not required as we shall use these two lemmas only when N is large to reduce the maxima in the definitions of w and W by a finite bound.

4. Around the Voronoï Summation Formula

In [2, Theorem 10.2.17], H. Cohen proposes a functional approach to the Voronoï Summation Formula: first prove the formula for smooth functions and then argue by density. Convergence issues inherent to this method are well-known. In particular, the paper [5] by Hardy & Landau investigates closely what happens when taking for f in the next lemma to be the characteristic function of the initial interval. The approach of [2] works however perfectly well is the restricted context that follows.

Lemma 4.1. *Let $f : [0,\infty) \mapsto \mathbb{C}$ be a continuous function such that $f(t) \ll 1/(1+|t|^2)$. We define $\mathscr{F}(f,m) = \int_0^\infty f(t)J_0\bigl(4\pi\sqrt{mt/|\Delta_d|}\bigr)dt$. Assume that the series $\sum_{m\ge1}(1\star\chi)(m)\mathscr{F}(f,m)$ converges. Then, we have*

$$\begin{aligned}\sum_{n\ge1}(1\star\chi)(n)f(n) =& L(1,\chi)\check{f}(0) + \frac{f(0)}{2|\Delta_d|}\sum_{1\le r\le|\Delta_d|} r\chi(r) \\ &+ \frac{2\pi}{\sqrt{|\Delta_d|}}\sum_{m\ge1}(1\star\chi)(m)\mathscr{F}(f,m).\end{aligned}$$

Proof. Let us denote by $\zeta_{\mathbb{Q}[\sqrt{d}]}(s)$ the Dedekind zeta-function of the imaginary quadratic field $\mathbb{Q}[\sqrt{d}]$. It satisfies the functional equation

$$\gamma_d(1-s)\zeta_{\mathbb{Q}[\sqrt{d}]}(s)(1-s) = \gamma_d(s)\zeta_{\mathbb{Q}[\sqrt{d}]}(s) \quad\text{where}\quad \gamma_d(s) = \left(\frac{\sqrt{|\Delta_d|}}{2\pi}\right)^s\Gamma(s). \tag{7}$$

We use [2, Theorem 10.2.17] by H. Cohen. The kernel to be considered is

$$
\begin{aligned}
K_d(x) &= \frac{1}{2i\pi}\int_{\frac12-i\infty}^{\frac12+i\infty}\left(\frac{\sqrt{|\Delta_d|}}{2\pi}\right)^{s}\left(\frac{\sqrt{|\Delta_d|}}{2\pi}\right)^{s-1}\frac{\Gamma(s)}{\Gamma(1-s)}x^{-s}ds\\
&= \frac{1}{2i\pi}\frac{2\pi}{\sqrt{|\Delta_d|}}\int_{\frac12-i\infty}^{\frac12+i\infty}\frac{4^s\Gamma(s)}{\Gamma(1-s)}\left(\frac{16\pi^2x}{|\Delta_d|}\right)^{-s}ds.
\end{aligned}
$$

The formula

$$
4^s\frac{\Gamma(s)}{\Gamma(1-s)} = \int_0^\infty t^{s-1}J_0(\sqrt{t})dt
$$

gives us

$$
K_d(x) = \frac{2\pi}{\sqrt{|\Delta_d|}}J_0\bigl(4\pi\sqrt{x/|\Delta_d|}\bigr).
$$

Concerning the value at 0, we use $\zeta(0) = -1/2$ and

$$
L(0,\chi) = \frac{-1}{|\Delta_d|}\sum_{1\le r\le|\Delta_d|} r\chi(r) \tag{8}
$$

as per [2, Corollary 10.3.2]. □

Lemma 4.2. *We have*

$$
\begin{aligned}
\sum_{n\le X}\left(1-\frac{n}{X}\right)(1\star\chi)(n) =& \frac{XL(1,\chi)}{2} + \frac{1}{2|\Delta_d|}\sum_{1\le r\le|\Delta_d|} r\chi(r)\\
&+ \frac{\sqrt{|\Delta_d|}}{2\pi}\sum_{m\ge1}\frac{(1\star\chi)(m)}{m}J_2\bigl(4\pi\sqrt{mX/|\Delta_d|}\bigr).
\end{aligned}
$$

Proof. By using Lemma 4.1, we readily find that

$$
\begin{aligned}
\sum_{n\le X}\left(1-\frac{n}{X}\right)(1\star\chi)(n) =& \frac{XL(1,\chi)}{2} + \frac{1}{2|\Delta_d|}\sum_{1\le r\le|\Delta_d|} r\chi(r)\\
&+ \frac{2\pi}{\sqrt{|\Delta_d|}}\sum_{m\ge1}(1\star\chi)(m)\int_0^X\left(1-\frac{t}{X}\right)J_0\bigl(4\pi\sqrt{mt/|\Delta_d|}\bigr)dt.
\end{aligned}
$$

By using Lemma 2.1, we find that

$$
\begin{aligned}
\sum_{n\le X}\left(1-\frac{n}{X}\right)(1\star\chi)(n) =& \frac{XL(1,\chi)}{2} + \frac{1}{2|\Delta_d|}\sum_{1\le r\le|\Delta_d|} r\chi(r)\\
&+ \frac{2\pi}{\sqrt{|\Delta_d|}}\sum_{m\ge1}(1\star\chi)(m)\frac{|\Delta_d|}{4\pi^2 m}J_2\bigl(4\pi\sqrt{mX/|\Delta_d|}\bigr).
\end{aligned}
$$

Our lemma follows swiftly from this last expression. □

5. Main engine

Lemma 5.1. *When $Y\in[0,X/3]$ and $X\ge|\Delta_d|$, we have*

$$
\begin{aligned}
&\sum_{n\le X}(1\star\chi)(n) + \sum_{X<n\le X+Y}\frac{X+Y-n}{Y}(1\star\chi)(n)\\
&\qquad= \frac{(2X+Y)L(1,\chi)}{2} + \frac{1}{2|\Delta_d|}\sum_{1\le r\le|\Delta_d|} r\chi(r) + \mathcal{O}^*\Bigl(0.36\,C_0(d)\sqrt{X|\Delta_d|/Y}\Bigr)
\end{aligned}
$$

where $C_0(d) = L(1,\chi)c_0(d)$. When $Y\in[0,X/10]$ and $X\ge 130^2|\Delta_d|$, the constant 0.36 *may be reduced to* 0.292.

Proof. By using Lemma 4.2 twice, we find that

$$(1/Y)\sum_{n\le X}\left[(X+Y)\left(1-\frac{n}{X+Y}\right)-X\left(1-\frac{n}{X}\right)\right](1\star\chi)(n)$$
$$=\frac{(2X+Y)L(1,\chi)}{2}+\frac{1}{2|\Delta_d|}\sum_{1\le r\le|\Delta_d|}r\chi(r)$$
$$+\frac{\sqrt{|\Delta_d|}}{2\pi}\sum_{m\ge1}\frac{(1\star\chi)(m)}{m}T(Y/X;4\pi\sqrt{mX/|\Delta_d|}).$$

We now majorize the last sum by appealing to Lemma 2.3. Recall that we assume that $X\ge|\Delta_d|$ (resp. $X\ge 130^2|\Delta_d|$). We use the first estimate of Lemma 2.3 when

$$4\pi\sqrt{mX/|\Delta_d|}\le\frac{7X}{3Y\cdot0.53}\qquad\left(\text{resp. }4\pi\sqrt{mX/|\Delta_d|}\le\frac{2.1X}{Y\cdot0.4}\right)$$

i.e. when

$$m\ge M=\frac{49|\Delta_d|X}{144(0.53\pi)^2Y^2}\qquad\left(\text{resp. }m\ge M=\frac{2.1^2|\Delta_d|X}{16(0.4\pi)^2Y^2}\right).$$

We thus get

$$\frac{\sqrt{|\Delta_d|}}{2\pi}\sum_{m\ge1}\frac{(1\star\chi)(m)}{m}|T(Y/X;4\pi\sqrt{mX/|\Delta_d|})|$$
$$\le 0.53\frac{|\Delta_d|^{1/4}X^{1/4}}{\sqrt{\pi}}\sum_{m\le M}\frac{(1\star\chi)(m)}{m^{3/4}}+\frac{7X^{3/4}}{3Y}\frac{|\Delta_d|^{3/4}}{4\pi^{3/2}}\sum_{m>M}\frac{(1\star\chi)(m)}{m^{5/4}}.$$

We recall that $C_0(d)=L(1,\chi)c_0(d)$ where $c_0(d)$ is being defined in (2). On appealing to Lemmas 3.3 and 3.4, this leads to

$$\frac{\sqrt{|\Delta_d|}}{2\pi}\sum_{m\ge1}\frac{(1\star\chi)(m)}{m}|T(Y/X;4\pi\sqrt{mX/|\Delta_d|})|$$
$$\le 0.53C_0(d)\frac{|\Delta_d|^{1/4}X^{1/4}}{\sqrt{\pi}}M^{1/4}+C_0(d)\frac{7X^{3/4}}{3Y}\frac{|\Delta_d|^{3/4}}{4\pi^{3/2}M^{1/4}}$$
$$\le C_0(d)\frac{X^{1/2}\sqrt{0.53}\sqrt{7/3}}{Y^{1/2}\pi}|\Delta_d|^{1/2}\le 0.36\,C_0(d)\sqrt{X|\Delta_d|/Y}.$$

When $Y\in[0,X/10]$ and $X\ge130^2|\Delta_d|$, the constant 0.36 can be replaced by an upper bound for $\sqrt{2.1\cdot0.4/\pi}$, e.g. 0.292. The proof is complete. □

Proof of Theorem 1.1. We use Lemma 5.1 with $X-Y+Y$ and $X+Y$. Thus

$$\frac{-Y}{2}L(1,\chi)-0.36\,C_0(d)\sqrt{(X-Y)|\Delta_d|/Y}\le S-XL(1,\chi)+\frac{1}{2|\Delta_d|}\sum_{1\le r\le|\Delta_d|}r\chi(r)$$
$$\le\frac{Y}{2}L(1,\chi)+0.36\,C_0(d)\sqrt{X|\Delta_d|/Y}.$$

We select

$$Y=\left(\frac{0.36\,C_0(d)\sqrt{X}}{L(1,\chi)}\right)^{2/3}\tag{9}$$

getting the error term

$$X^{1/3}C_0(d)^{2/3}L(1,\chi)^{1/3}\left(\frac{0.36^{2/3}}{2}+0.36^{2/3}\right).$$

We readily find that $Y/X = \big(0.36\, c_0(d)/X\big)^{2/3}$. Therefore, when $X \geq 2c_0(d)$, we find that $Y \leq X/3$. The theorem follows readily in that case. The case $X \geq \max(10c_0(d), 130^2|\Delta_d|)$ is treated similarly as we get the hypothesis on Y is ensured by the inequality $Y/X \leq \big(0.292\, c_0(d)/X\big)^{2/3} \leq 0.095$. □

6. Computing $C_0(d)$

We use the script `ConvolutionAndVoronoi-01.gp` and the function `run(1000000, d)` therein to build the next table. We compute the maxima in $w(3/4)$ and $W(5/4)$ exactly until $M = 10^6$, and then rely on Lemma 3.3 and 3.4. The last column with title "best" is the best constant we could hope to reach if we were to compute up to $M = \infty$.

d	Δ_d	$\Omega(\chi)$	$L(1,\chi)c_0(d) \leq$	best
-1	-4	1	2.02	2.01
-2	-8	2	2.86	2.84
-3	-3	1	1.56	1.56
-5	-20	4	3.63	3.58
-6	-24	4	3.32	3.26
-7	-7	2	3.06	3.05
-10	-40	4	2.58	2.56
-11	-11	3	2.45	2.44
-13	-52	6	2.27	2.25
-14	-56	8	4.37	4.26
-15	-15	3	4.19	4.16
-17	-68	8	3.97	3.90
-19	-19	4	1.87	1.87
-21	-84	8	3.57	3.49
-22	-88	8	1.76	1.72
-23	-23	5	5.08	5.04
-26	-104	12	4.83	4.69
-29	-116	12	4.57	4.45
-30	-120	8	3.00	2.93
-31	-31	6	4.39	4.32
-33	-132	8	2.86	2.79
-34	-136	8	2.82	2.75
-35	-35	6	2.76	2.75
-37	-148	10	1.38	1.34
-38	-152	12	4.00	3.90
-39	-39	6	5.21	5.13

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