Chowla's conjecture: from the Liouville function to the Moebius function

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Abstract

This aim of this short note is to try to clarify the links between different version of the "Chowla conjecture".

1 Historical setting

In Problem 57 of the book [1] (see equation (341) therein), S. Chowla formulated the following conjecture, where, by "integer polynomial", we understand "an element of $\mathbb{Z}[X]$".

Conjecture (Chowla's Conjecture for the Liouville function)**.** *For any integer polynomial $f(x)$ that is not of the form $cg(x)^2$ for some integer polynomial $g(x)$, one has*

$$\sum_{n\le x}\lambda(f(n)) = o(x)$$

where λ is the Liouville function.

This conjecture has then been stated and formulated in many different forms, very often by restricting f to be a product of linear factors as by T. Tao in [5], and even more often to be a product of monic linear factors. Furthermore, the Liouville function λ is sometimes replaced by the Moebius function μ as by P. Sarnak in [4]. The aim of this note is to establish some links between these conjectures.

We should also mention that some authors, like A. Hildebrand in [2] or K. Matomäki, M. Radziwiłł & T. Tao in [3], refer to another closely connected conjecture of Chowla that states that the sequence $(\lambda(n),\lambda(n+1),\cdots,\lambda(n+k))$ may take any value in $\{\pm 1\}^k$. This is indeed Problem 56 of [1].

Here are different forms that can be called "Chowla's conjecture".

Conjecture 1. *For any finite tuple $((a_i,b_i))_{i\in I}$ of positive integers such that $a_ib_j - a_jb_i \neq 0$ as soon as $i\neq j$, we have*

$$\sum_{n\le x}\prod_{i\in I}\lambda(a_in+b_i) = o(x).$$

Remark 1. Note that when the condition of the $((a_i,b_i))_{i\in I}$ is verified, it is also verified for the coefficients after the subtitution $n\mapsto pn+q$.

Conjecture 2. *For any positive integer a and any strictly increasing finite sequence $(b_i)_{i\in I}$ of non-negative integers, we have*

$$\sum_{n\le x}\prod_{i\in I}\lambda(an+b_i)=o(x).$$

Conjecture 3. *For any finite tuple $((a_i,b_i))_{i\in I}$ of positive integers such that $a_ib_j-a_jb_i\neq 0$ as soon as $i\neq j$, we have*

$$\sum_{n\le x}\prod_{i\in I}\mu(a_in+b_i)=o(x).$$

Conjecture 4. *For any positive integer a and any strictly increasing finite sequence $(b_i)_{i\in I}$ of non-negative integers, we have*

$$\sum_{n\le x}\prod_{i\in I}\mu(an+b_i)=o(x).$$

Conjecture 5. *For any finite tuple $((a_i,b_i))_{i\in I}$ of positive integers such that $a_ib_j-a_jb_i\neq 0$ as soon as $i\neq j$, and any additional finite tuple $((c_k,d_k))_{k\in K}$ we have*

$$\sum_{n\le x}\prod_{i\in I}\mu(a_in+b_i)\prod_{k\in K}\mu^2(c_kn+d_k)=o(x)$$

provided the set I be non-empty.

Conjecture 6. *For any positive integer a and any strictly increasing finite sequences $(b_i)_{i\in I}$ and $(d_k)_{k\in K}$ of non-negative integers, we have*

$$\sum_{n\le x}\prod_{i\in I}\mu(an+b_i)\prod_{k\in K}\mu^2(an+d_k)=o(x)$$

provided the set I be non-empty.

In the ergodic context, the hypothesis $\sum_{n\le x}\prod_{i\in I}\mu(n+b_i)\prod_{k\in K}\mu^2(n+d_k)=o(x)$ is often seen, with some natural conditions on the parameters. Since it is applied to powers T^a of a same operator T, the proper statements are really Conjectures 2, 4 and 6.

The reader may wonder whether the non-negativity condition concerning the parameters b_i is restrictive or not. It is not and we can reduce to this case by a suitable shift of the variable n. This would implies discarding finitely many terms is the diverse sums we consider and increase the initial b_i by $b_i+(N_0-1)a_i$, assuming we replace n by $n+N_0-1$; there clearly exists an N_0 for which all the $b_i+(N_0-1)a_i$ are positive.

Theorem 1.

- *Conjecture 1 implies Conjecture 3.*
- *Conjecture 3 implies Conjecture 5.*
- *Conjecture 2 implies Conjecture 4.*
- *Conjecture 4 implies Conjecture 6.*
- *Conjecture 5 implies Conjecture 1.*

Note that we have not been able to prove that Conjecture 6 implies Conjecture 2.

2 Lemmas

Lemma 2. *Let $(f(n))_n$ be a complex sequence such that $|f(n)| \leq 1$, let a and b be fixed and assume that, for every u and w, one has*

$$\sum_{n \leq y} \lambda(a(u^2 n + w) + b) f(u^2 n + w) = o_{u,w}(y). \tag{1}$$

Then $\sum_{n \leq x} \mu(an+b) f(n) = o(x)$.

Proof. We readily check on the Dirichlet series that

$$\mu(m) = \sum_{u^2 v = m} \mu(u) \lambda(v) \tag{2}$$

We define $X = ax + b$. We infer from this identity that, for any positive integer U, one has

$$\begin{aligned}
\sum_{n \leq x} \mu(an+b) f(n) &= \sum_{m \leq ax+b} \mu(m) f\Big(\frac{m-b}{a}\Big) 1_{m \equiv b[a]} \\
&= \sum_{u \leq \sqrt{ax+b}} \mu(u) \sum_{v \leq (ax+b)/u^2} \lambda(v) f\Big(\frac{u^2 v - b}{a}\Big) 1_{u^2 v \equiv b[a]} \\
&= \sum_{u \leq U} \mu(u) \sum_{v \leq (ax+b)/u^2} \lambda(v) f\Big(\frac{u^2 v - b}{a}\Big) 1_{u^2 v \equiv b[a]} + \mathcal{O}^*(\sum_{u > U} X/u^2)
\end{aligned}$$

since

$$\Big| \sum_{v \leq (ax+b)/u^2} \lambda(v) f\Big(\frac{u^2 v - b}{a}\Big) 1_{u^2 v \equiv b[a]} \Big| \leq \sum_{v \leq X/u^2} 1 \leq X/u^2.$$

We next recall that a comparison to an integral ensures us that $\sum_{u > U} u^{-2} \leq 1/U$ (since U is an integer). Since we want the reader to follow as closely as possible the argument, we also use the notation $f = \mathcal{O}^*(g)$ to mean that $|f| \leq g$. We have reached

$$\begin{aligned}
\sum_{n \leq x} \mu(an+b) f(n) &= \sum_{u \leq U} \mu(u) \sum_{v \leq (ax+b)/u^2} \lambda(u^2 v) f\Big(\frac{u^2 v - b}{a}\Big) 1_{u^2 v \equiv b[a]} + \mathcal{O}^*(X/U) \\
&= \sum_{u \leq U} \mu(u) \sum_{\substack{m \leq x, \\ am+b \equiv 0[u^2]}} \lambda(am+b) f(m) + \mathcal{O}^*(X/U)
\end{aligned}$$

(with $am = u^2 v - b$) from which we infer that

$$\Big| \sum_{n \leq x} \mu(an+b) f(n) \Big| \leq \sum_{u \leq U} \Big| \sum_{\substack{m \leq x, \\ am+b \equiv 0[u^2]}} \lambda(am+b) f(m) \Big| + \frac{X}{U}.$$

The set $\{m / am + b \equiv 0[u^2]\}$ is a finite union of arithmetic progressions modulo u^2, say $\mathcal{W}$, hence

$$\sum_{\substack{m \leq x, \\ am+b \equiv 0[u^2]}} \lambda(am+b) f(m) = \sum_{\substack{0 \leq w < u^2, \\ w \in \mathcal{W}}} \sum_{k \leq \frac{ax+b-w}{u^2}} \lambda(a(u^2 k + w) + b) f(u^2 k + w).$$

Our hypothesis applies to the inner sum. The remainder of the proof is mechanical. First note that $X \le (a+b)x$. Let $\varepsilon > 0$ be fixed. We select

$$U = \left[\frac{2}{(a+b)\varepsilon}\right] + 1 \le \frac{3}{(a+b)\varepsilon}.$$

There exists $y_0(a,b,\varepsilon)$ such that, for every $u \le U$, every $w \in \mathcal{W}$ and every $y \ge y_0(a,b,\varepsilon)$, one has

$$\Big|\sum_{n\le y} f(a^2 n)\Big| \le \tfrac{1}{6}\varepsilon^2 y.$$

We thus assume that $x \ge y_0(a,b,\varepsilon)/(a+b)$ and get, for such an x that

$$\Big|\sum_{n\le x} \mu(an+b) f(n)\Big| \le x\left(\frac{1}{6}\frac{3}{\varepsilon}\varepsilon^2 + \frac{\varepsilon}{2}\right) \le \varepsilon x$$

(where we have bounded above $|\mathcal{W}|$ by u^2) as required. □

Lemma 3. *Let $(f(n))_n$ be a sequence sur that $|f(n)| \le 1$, let a and b be fixed and, assume that, for every u and w, one has*

$$\sum_{n\le y} f(u^2 n + w) = o_{u,w}(y). \tag{3}$$

Then $\sum_{n\le x} \mu^2(an+b) f(n) = o(x)$.

Proof. We use the identity

$$\mu^2(n) = \sum_{d^2|n} \mu(d).$$

After this initial step, the proof runs as the one of Lemma 2. □

Lemma 4. *Let $(f(n))_n$ be a sequence sur that $|f(n)| \le 1$, let a and b be fixed and, assume that, for every u and c such that $cu^2 \equiv b[a]$, one has*

$$\sum_{n\le y} \mu(an+c) f\Big(u^2 n + \frac{cu^2 - b}{a}\Big) = o_{u,w}(y). \tag{4}$$

Then $\sum_{n\le x} \lambda(an+b) f(n) = o(x)$.

Proof. We readily check on the Dirichlet series that

$$\lambda(m) = \sum_{u^2 v = m} \mu(v) \tag{5}$$

We infer from this identity that, for any positive integer U, one has

$$\begin{aligned}\sum_{n\le x} \lambda(an+b) f(n) &= \sum_{m \le ax+b} \lambda(m) f\Big(\frac{m-b}{a}\Big) 1_{m\equiv b[a]} \\ &= \sum_{u\le\sqrt{ax+b}} \sum_{v \le (ax+b)/u^2} \mu(v) f\Big(\frac{u^2 v - b}{a}\Big) 1_{u^2 v \equiv b[a]} \\ &= \sum_{u\le U} \sum_{v\le (ax+b)/u^2} \mu(v) f\Big(\frac{u^2 v - b}{a}\Big) 1_{u^2 v\equiv b[a]} + \mathcal{O}^*(X/U)\end{aligned}$$

again with $X = ax + b$. Once u is fixed, the set $\{v/u^2v \equiv b[a]\}$ is a finite union of arithmetic progressions modulo a, say $\mathcal{W}$, hence

$$\sum_{v \le (ax+b)/u^2} \mu(v) f\Big(\frac{u^2v - b}{a}\Big) 1_{u^2v\equiv b[a]}$$
$$= \sum_{\substack{0\le v_0<a,\\ v_0\in\mathcal{W}}} \sum_{k \le \frac{ax+b}{au^2} - \frac{v_0}{a}} \mu(ak + v_0) f\Big(u^2k + \frac{u^2v_0 - b}{a}\Big).$$

Our hypothesis applies to the inner sum. The remainder of the proof is mechanical. □

3 Proof of the Theorem 1

Proof. To prove that Conjecture 1 implies Conjecture 3, we prove that, assuming Conjecture 1 and following its notation and hypotheses, we have (with the shortcut $a_{i_0} = a$ and $b_{i_0} = b$)

$$\sum_{n\le x} \mu(an + b) \prod_{i\in I'} \mu(a_i n + b_i) \prod_{i\in I''\setminus\{i_0\}} \lambda(a_i n + b_i) = o(x)$$

assuming that

$$\sum_{n\le x} \lambda(an + b) \prod_{i\in I'} \mu(A_i n + B_i) \prod_{i\in I''\setminus\{i_0\}} \lambda(A_i n + B_i) = o(x)$$

where none of the vectors (A_i, B_i) and (a, b) are colinear. We use a recursion on the cardinality of I' to do so. Lemma 2 is tailored for this purpose, the only part that needs checking is that no two vectors of new set of parameters $(au^2, aw + b)$ and $(a_iu^2, a_iw + b_i)$ are colinear, which is immediate:

$$\begin{vmatrix} au^2 & a_iu^2 \\ aw + b & a_iw + b_i \end{vmatrix} = au^2(a_iw + b_i) - a_iu^2(aw + b) = u^2 \begin{vmatrix} a & a_i \\ b & b_i \end{vmatrix} \neq 0.$$

The same proof shows that Conjecture 2 implies Conjecture 4: we simply have to note that the required coefficients a and a_i's remain the same.

The same proof again shows that Conjecture 3 implies Conjecture 5, we simply have to replace λ by 1, and Lemma 2 by Lemma 3, in its proof and similarly that Conjecture 4 implies Conjecture 6.

Let us finally turn towards the proof that Conjecture 3 implies Conjecture 1, a task for which we will use Lemma 4. We aim at proving that (again with the shortcut $a_{i_0} = a$ and $b_{i_0} = b$)

$$\sum_{n\le x} \lambda(an + b) \prod_{i\in I'} \lambda(a_i n + b_i) \prod_{i\in I''\setminus\{i_0\}} \mu(a_i n + b_i) = o(x)$$

assuming that

$$\sum_{n\le x} \mu(an + b) \prod_{i\in I'} \lambda(A_i n + B_i) \prod_{i\in I''\setminus\{i_0\}} \mu(A_i n + B_i) = o(x)$$

where none of the vectors (A_i, B_i) and (a, b) are colinear. We use a recursion on the cardinality of I' to do so. The only part that needs checking is the hypothesis in Lemma 4, namely that no two vectors of new set of parameters (a, c) and $(a_i u^2, a_i \frac{cu^2-b}{a} + b_i)$ are colinear, which is immediate:

$$\begin{vmatrix} a & a_i u^2 \\ c & a_i \frac{cu^2-b}{a} + b_i \end{vmatrix} = a\Big(a_i \frac{cu^2 - b}{a} + b_i\Big) - a_i u^2 c = \begin{vmatrix} a & a_i \\ b & b_i \end{vmatrix} \neq 0.$$

This concludes the proof of our Theorem. □

References

[1] Sarvadaman Chowla. *The Riemann hypothesis and Hilbert's tenth problem.* Mathematics and Its Applications, Vol. 4. Gordon and Breach Science Publishers, New York-London-Paris, 1965.

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[5] Terence Tao. The logarithmically averaged Chowla and Elliott conjectures for two-point correlations. *Forum Math. Pi*, 4:e8, 36, 2016.