Local models and pseudo-caracters

An introduction And an application

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Lectures given at IMSc

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Introduction

The first aim of these notes is to explain the notions of *local model* and of *pseudo-character*. The latter one originates from the work of Selberg, Motohashi and Jutila in the seventies. The first notion can be traced back to the book of (Motohashi, 1983) and is really introduced in (Ramaré, 2009). The second aim of these notes is to show how analytical techniques can be used to great effect to enhance the power of the large sieve.

We shall first concentrate in proving Hoheisel Theorem*, i.e.

Theorem 0.1. *There exist two constants x_0 and $a \in [\frac{1}{2}, 1)$*† *such that there is at least one prime in any interval $(x, x + x^a]$, provided $x \geq x_0$.*

This Theorem has been extended in (Gallagher, 1970) and (Motohashi, 1978) to include the celebrated Theorem of Linnik on the least prime in an arithmetic progression, proven in (Linnik, 1944a) and (Linnik, 1944b). We shall use heavily the paper (Motohashi, 1978) as well as the subsequent book (Motohashi, 1983).‡. We shall avoid the notion of *zero-density estimate* and use only the (de la Vallée-Poussin, 1899) zero-free region, while the original proof used (at least) the region of (Littlewood, 1922)§.

The proof we follow is a simplication¶ of the method to Motohashi. In the course of this proof, we shall introduce analytical material: a hybrid large sieve inequality adapted to Dirichlet polynomials and the Mellin transform. We mention here that Selberg gave in 1973-74 a series of lectures at Princeton in which pseudo-characters were introduced. See the remark at the bottom of page 164 of (Motohashi, 1978).

Our next step will be to generalize this. We develop the material required to use Motohashi's method in a general context. This is where general *pseudo-characters* are being used.

Let us say rapidly here the notion of local models applies to general sieving situation, like when sieving an interval to get twin primes, while the notion of pseudo-characters applies when sieving the initial interval weighted by a non-negative multiplicative function to get the prime numbers only. Using the terminology of chapter 12, the host sequence in the chacteristic function of the interval $[1, N]$ multiplied by a non-negative multiplicative function while the compact set is the set of invertible elements. We will further show

*See (Hoheisel, 1930)

†The important thing is that a should be < 1.

‡The reader may also look at (Iwaniec & Kowalski, 2004, chapter 18).

§See also (Weyl, 1921) and (Titchmarsh, 1951, Theorem 5.17).

¶In this simpler context!

that the pseudo-characters can in fact be seen as local models but modified local scalar products

Let us mention here a difference in the approach we present with the one proposed by Motohashi. Given a non-negative multiplicative function, this author seeks to minimize the quadratic form

$$\sum_n f(n)\Big(\sum_{d|n}\lambda_d\Big)^2$$

under condition $\lambda_1 = 1$. Minimum is asked for simply to get as small an upper bound as possible, while condition $\lambda_1 = 1$ is asked for to get (10.5). In the approach we propose, we want to approximate (and we will only be able to majorize) the quantity $\sum_p f(p)$, where the sum ranges only over prime numbers. We get the same result.

Recently (Kowalski & Michel, 2002) have introduced analogues of these pseudo-characters in the frame of modular forms and we hope that these notes will enable the reader to better understand this nascent notion by making the usual one clearer.

For a general Hoheisel like Theorem, see (Ramachandra, 1976).

This introduction gets carried further at beginning of chapter 14 where we will have all the necessary vocabulary.

Overview of the lectures:

Friday 22/01 Introductory lecture

(a) The Hoheisel Theorem.
(b) The large sieve inequality.
(c) Brun-Tichmarsh Theorem via local models (and $G(R) \geq \operatorname{Log} R$).

Monday 25/01 Some analytical tools

(a) Integral version of an inequality of Gallagher.
(b) On the Perron summation formula.
(c) Defining Λ.
(d) ψ in terms of $-\zeta'/\zeta$.

Wednesday 27/01 Barban & Vehov weights

(a) The $\lambda_d^{(1)}$ and Graham's Lemma. Writing with $L(y,d)$.
(b) Proof of Theorem 10.1 and (10.7).

1-6 1 Hoheisel Theorem via pseudo characters

(a) V_r and its factorization.
(b) The companions
(c) $V^*(s;Z)$.
(d) Choosing the parameters.

1-6 2 Compacts sets and local models

1. The compact set linked with the prime numbers and the one associated to the prime twins.
2. Going to $\mathbb{Z}/M\mathbb{Z}$ and local models.
3. Large sieve inequalities.

8-13 1 Extension to multiplicative functions.

(a) Hypothesis on f.
(b) Local scalar products and orthogonal basis.
(c) Pseudo-characters,
(d) A large sieve inequality for pseudo-characters.

8-13 2 Extending the Motohashi setting

Contents

Chapter 1

Setting the background

Here is one of the Theorem we are going to prove in these lectures.

Hoheisel Theorem. *There exist two constants* x_0 *and* $a \in [\frac{1}{2}, 1)^*$ *such that there is at least one prime in any interval* $(x, x + x^a]$*, provided* $x \geq x_0$*.*

This is proven in (Hoheisel, 1930). Let me tell you why this result created quite a shock when it was published. To do so, let us introduce some notations and introduce the Tchebysheff ϑ function:

$$\vartheta(x) = \sum_{p \leq x} \operatorname{Log} p \tag{1.1}$$

where p runs through prime numbers. The above Theorem says that

$$\vartheta(x + h) - \vartheta(x) > 0$$

with $h = x^a$. We have the prime number Theorem at our disposal, that says that

$$\vartheta(y) = y + \mathcal{O}(R(y)) \qquad (y \geq 2) \tag{1.2}$$

where $R(y) \geq 0$ is[†] a remainder term and we can take for instance

$$R(y) = y/(\operatorname{Log} y)^{100}. \tag{1.3}$$

Let us use (1.2) on our problem. It yields (when $0 \leq h \leq x$)

$$\vartheta(x + h) - \vartheta(x) = h + \mathcal{O}\big(R(x + h) + R(x)\big)$$

which we want to be positive. On taking R as in (1.3), we find that

$$R(x + h) + R(x) \ll x/(\operatorname{Log} x)^{100}.$$

We can thus prove that $\vartheta(x + h) - \vartheta(x) > 0$ when h is larger than a constant time $x/(\operatorname{Log} x)^{100}$. This is way larger than what we want! So, what is the error term this

[*] The important thing is that a should be < 1.
[†] ...an upper bound of...

approach requires to work? Well, it is not hard to see that we would need something like $R(y) \leq y^a/3$. Such an estimate is way out of reach for the following reason. The function

$$D(s) = s \int_1^\infty \vartheta(y) dy / y^{s+1} \tag{1.4}$$

is in fact equal to* (when $\Re s > 1$)

$$D(s) = -\frac{\zeta'(s)}{\zeta(s)} + g(s) \tag{1.5}$$

where the function $g(s)$ is analytical for at least $\Re s > 1/2$. The estimate $R(y) \leq y^a/3$ implies, on using (1.4), that $D(s)$ has an meromorphic continuation on the half-plane $\Re s > a$ with a simple pole at $s = 1$. But this implies in turn that $-\zeta'(s)/\zeta(s)$ has the same property, and thus the Riemann ζ-function cannot vanish in this region! In short, we are simply asking for a quasi-Riemann hypothesis† –

Looking closer, we see that the above proof would not only give the existence of a prime in the given interval, but also prove that the number of primes in this interval is what it should be. It can be proven that the two statements:

- the Riemann zeta function does not vanish for $\Re s > a$,
- $\vartheta(x) = x + \mathcal{O}(x^b)$ for any $b > a$

are equivalent.

This is why Hoheisel Theorem was so surprising. So what were the ingredients of his proof? He needed two strong arguments:

ZRF: A strong zero-free region for ζ‡;

ZDE: A zero density estimate, i.e. a tool that says that the Riemann zeta function may have zeros close to the line $\Re s = 1$ but not too many.

The first one belongs to the classical stream ("prove that the zeta function does not vanish as far as one can from the line $\Re s = 1$") and its proof shares this same property§, but the second one is more surprising. This line of thoughts, in terms of density estimates, has been initiated by Bohr. Here I shorten the history¶.

The method developed by Selberg (see (Bombieri, 1987/1974a)[Theorem ?]), (Jutila, 1977b) and (Motohashi, 1978) replaces these two arguments by a single one and this is

*See section 1.1

†The Riemann hypothesis asserts that the Riemann zeta function does not vanish for $\Re s > 1/2$; the quasi-Riemann hypothesis that there exists $a < 1$ such that this function does not vanish on the half plane $\Re s > a$.

‡At least of the level of the Littlewood-Weyl one from(Littlewood, 1922). See also (Weyl, 1921) and (Titchmarsh, 1951, Theorem 5.17).

§It goes by studying exponential sums.

¶And noticeably skip Linnik's gigantic contribution in the forties, see (Linnik, 1944a) and (Linnik, 1944b). He discovered in particular that both arguments could be replaced by two strong density estimates (one of them being Linnik's density Lemma). I have to mention also (Gallagher, 1970).

one of the object of these lectures. It is difficult to continue without having much more material, also because the philosophy of the method at stake is still not well understood, or at least, I feel my understanding of it to be very incomplete. The main new ingredient seems however to be a large sieve estimate.

Let me simply add some remarks at this level:

- The method was somehow forgotten, and can be found in full in (Motohashi, 1978) and (Motohashi, 1983). Recently (Kowalski & Michel, 2002) adapted a raw form of this method to the context of modular form. A good amount of arithmetical information is however still missing to duplicate the original proof.
- One of the main argument comes from the large sieve inequality and in its capability in producing a sieve effect.
- Once the previous point has been seen, one can prove density estimates or go directly to the original problem. This is what we do here, following (Motohashi, 1978).
- On the special problem of the Hoheisel Theorem, we simply mention that the next step is due to (Iwaniec & Jutila, 1979), where they developped much more the sieve argument.

1.1 Some details on (1.4) and (1.5)

The Riemann zeta function is given by

$$\zeta(s) = \sum_{n\geq 1} 1/n^s = \prod_{p\geq 2} \frac{1}{1-p^{-s}} \tag{1.6}$$

and s is a complex numbers such that $\Re s > 1$. On taking the logarithmic derivative, we find that

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{p} \frac{\operatorname{Log} p}{p^s - 1} = \sum_{n\geq 1} \frac{\Lambda(n)}{n^s} \tag{1.7}$$

where Λ is the van Mangoldt function defined by

$$\Lambda(n) = \begin{cases} \operatorname{Log} p & \text{when } n = p^\nu \text{ for some prime } p \text{ and integer } \nu \geq 1, \\ 0 & \text{otherwise.} \end{cases} \tag{1.8}$$

This function essentially puts a weight $\operatorname{Log} p$ at every prime number p and the reader may overlook* what happens for prime powers. This expression of $-\zeta'/\zeta$ is valid only for $\Re s > 1$. It is now fairly easy to prove that

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{p\geq 2} \frac{\operatorname{Log} p}{p^s} + g(s) \tag{1.9}$$

*In first approximation! Since their contribution is very small.

where $g(s)$ is analytic for $\Re s > 1/2$. On the other hand, a classical summation by parts gives us:

$$\begin{aligned}\sum_{p\geq 2}\frac{\operatorname{Log} p}{p^s} &= s\sum_{p\geq 2}\operatorname{Log} p\int_p^\infty dy/y^{s+1}\\ &= s\int_2^\infty\sum_{p\leq y}\operatorname{Log} p\, dy/y^{s+1} = D(s)\end{aligned}$$

by (1.4). On combining this with (1.9), we prove (1.5).

Chapter 2

Large sieve ingredients

2.1 The large sieve inequality

We do not have time to present the large sieve inequality in full details. The reader will find proofs, historical details and further developments in (Montgomery, 1971) and in (Bombieri, 1987/1974a). Here is the large sieve inequality in the special form we will require.

Theorem 2.1. *Let* $(u_n)_{y<n\le y+N}$ *be a sequence of complex numbers. We have*

$$\sum_{r\le R}\sum_{a \bmod r}\Bigl|\sum_{y<n\le y+N} u_n e(na/r)\Bigr|^2 \le \sum_{y<n\le y+N} |u_n|^2\,(N-1+R^2)$$

where $e(\alpha)=\exp(2i\pi\alpha)$.

This inequality is exactly taken from (Montgomery & Vaughan, 1974).*

2.2 Using the large sieve inequality to sieve the prime numbers

We give here a first example:

*Well, looking closely the inequality in the paper quoted seems to have $N+R^2$ instead of $N-1+R^2$. The $N-1+R^2$ can be obtained in several ways:

1. (Selberg, 1991) in his lectures on sieves proves it. In fact Selberg got the optimal large sieve inequality in a general context at exactly the same time as Montgomery & Vaughan did, except that the -1 was missing in their case.
2. Cohen, a student of Montgomery, showed that one could get this -1 starting from Montgomery & Vaughan's result.
3. All this historical debate is made pointless in our case by the following remark: what occurs is the spacing between two consecutive Farey points a/r and b/r'. But we know that $r+r'\le R$, so in fact the inverse of this spacing is at most $R(R-1)\le R^2-1$, provided $R\ge 2$.

The -1 will be convenient later on.

Theorem 2.2. *Let* $(u_n)_{n\le N}$ *be a sequence of complex numbers. We have, for any* $M\in\mathbb{R}$*,*

$$\sum_{r\le R}\frac{\mu^2(r)}{\phi(r)}\Bigl|\sum_{n\le N}u_n c_r(n+M)\Bigr|^2\le\sum_{n\le N}|u_n|^2(N-1+R^2)$$

where $c_r(m)$ *is the Ramanujan sum modulo* r.*

Note that this also gives the inequality

$$\sum_{r\le R}\frac{\mu^2(r)}{\phi(r)}\Bigl|\sum_{y<n\le y+N}u_n c_r(n)\Bigr|^2\le\sum_{y<n\le y+N}|u_n|^2(N-1+R^2). \tag{2.1}$$

Proof. Indeed, we can write

$$c_r(m)=\sum_{a \bmod^* r}e(am/r) \tag{2.2}$$

and we only need to use the Cauchy-Schwartz inequality followed by Theorem 2.1 to conclude. □

This Theorem is in essence (Wolke, 1974, (8)) (see also (Wolke, 1973)) as noted by Richert in the notes of (Richert, 1976, Chapter 2). The coefficients $c_r(n)$ are the first instance of what Selberg called *pseudo-characters*. In this special context, they are also, upto a multiplicative factor independant of r, what we have called *local models* in (Ramaré, 2009). We shall explain that in more details in subsequent lectures.

A first application

Let us take for (u_n) is the characteristic function of the primes in the interval $(M,M+N]$, where we additionnally assume that $M>\sqrt{N}$. In that case, and provided $R\le\sqrt{N}$, we see that $u_n c_r(n)=u_n\mu(r)$ when $r\le R$ since either n is not a prime number and both side vanish, or it is one and it is prime to r. We have thus reached

$$\sum_{r\le R}\frac{\mu^2(r)}{\phi(r)}\Bigl|\sum_n u_n\Bigr|^2\le\sum_n|u_n|^2(N-1+R^2).$$

On setting $Z=\sum_n u_n=\sum_n|u_n|^2$, this yields

$$Z\le(N-1+R^2)/\sum_{r\le R}\frac{\mu^2(r)}{\phi(r)}.$$

We are left with showing that

$$\sum_{r\le R}\frac{\mu^2(r)}{\phi(r)}\ge\operatorname{Log}R. \tag{2.3}$$

*See (2.2) and (10.8).

Proof. Let us first note that

$$\frac{\mu^2(r)}{\phi(r)} = \frac{\mu^2(r)}{r}\prod_{p|r}\frac{1}{1-\frac{1}{p}} = \frac{\mu^2(r)}{r}\prod_{p|r}\Big(1+\frac{1}{p}+\frac{1}{p^2}+\cdots\Big)$$
$$= \sum_{r|\ell\subset r}\frac{1}{\ell}$$

where the summation ranges over every integer ℓ that are divisible by r and all whose prime factors are the ones of r. As a consequence

$$\sum_{r\le R}\frac{\mu^2(r)}{\phi(r)} = \sum_{\ell}\frac{1}{\ell}$$

where ℓ ranges the integers all whose prime factors are not more than R. The range $[1,R]$ is included thus covered, which ends the proof provided one remembers that

$$\sum_{\ell\le R} 1/\ell \ge \operatorname{Log} R \quad (R\ge 1).$$

□

On choosing $R=\sqrt{N}/\operatorname{Log} N$, we get the following version of the Brun-Titchmarsh Theorem:

Theorem 2.3. *The number Z of prime numbers in the interval $(y, y+N]$ is at most*

$$2(1+o(1))N/\operatorname{Log} N.$$

We have established this Theorem under the condition $y>\sqrt{N}$, and we let the reader complete the proof.

This proof shows clearly the sieving effect (in a somewhat vague sense) of the factor $c_r(m)$. The reader will find in (Elliott, 1992) a very same use of this factor.

This example shows that inequality (2.1) "contains" informations on the distribution of primes in short intervals, since in the Brun-Titchmarsh Theorem, N can be much smaller than M. Note however that we have access to an upper bound, and this is typical of sieve results for primes. We shall combine this approach with some analytic material (Mellin transforms and Cauchy residue Theorem) to get a lower bound, as required for Hoheisel Theorem.

2.3 An integral hybrid

Let us start by a Lemma due to Gallagher (this is (Gallagher, 1970, Lemma 1), as well as (Bombieri, 1987/1974a, Theorem 9)).

Theorem 2.4. *Let $c > 1$ be a real parameter. With $\tau = e^{2\pi/(cT)}$, we have*

$$\int_{-T}^{T} \Bigl|\sum_n w_n n^{it}\Bigr|^2 dt \le \frac{\pi^2}{\sin(\pi/c)^2} T^2 \int_0^\infty \Bigl|\sum_{y<n\le\tau y} w_n\Bigr|^2 dy/y$$

for every absolutely convergent series $\sum_n w_n$.

We give a very explicit version of this Lemma but we shall only use it with $c = 2$ in the sequel.

Proof. We follow (Bombieri, 1987/1974a, Théorème 9) closely. Let $\delta > 0$ be a parameter to be chosen. We define

$$F_\delta(x) = \begin{cases} \delta^{-1} & \text{when } |x| \le \delta/2, \\ 0 & \text{else,} \end{cases}$$

whose Fourier transform is given by

$$\hat{F}_\delta(t) = \int_{-\infty}^{\infty} F_\delta(x) e(xt) dt = \frac{\sin \pi\delta t}{\pi\delta t}. \tag{2.4}$$

We obtain

$$\sum_n w_n e^{2i\pi t(\operatorname{Log} n)/(2\pi)} \hat{F}_\delta(t) = \sum_n w_n \widehat{F_\delta\Bigl(t - \frac{\operatorname{Log} n}{2\pi}\Bigr)}.$$

Parseval identity yields

$$\int_{-\infty}^{\infty} \Bigl|\sum_n w_n e^{2i\pi t(\operatorname{Log} n)/(2\pi)} \frac{\sin \pi\delta t}{\pi\delta t}\Bigr|^2 dt = \int_{-\infty}^{\infty} \Bigl|\sum_{|\operatorname{Log} n - 2\pi x| \le \pi\delta} w_n \delta^{-1}\Bigr|^2 dx.$$

Let us recall that the function $t \mapsto (\sin \pi\delta t)/(\pi\delta t)$ is non-increasing for $\delta|t| \le 1$, which enables us to write

$$\begin{aligned} \Bigl(\frac{\sin(\pi/c)}{\pi/c}\Bigr)^2 \int_{-(c\delta)^{-1}}^{(c\delta)^{-1}} \Bigl|\sum_n w_n n^{it}\Bigr|^2 &\le \int_0^\infty \Bigl|\delta^{-1} \sum_{ze^{-\pi\delta} < n \le e^{\pi\delta} z} w_n\Bigr|^2 dz/z \\ &\le \int_0^\infty \Bigl|\delta^{-1} \sum_{y < n \le e^{2\pi\delta} y} w_n\Bigr|^2 dy/y \end{aligned}$$

with $2\pi x = \operatorname{Log} z$. We take $\delta = 1/(cT)$ to conclude. □

Theorem 2.5. *When $T \ge \pi$, we have*

$$\sum_{r \le R} \sum_{a \bmod^* r} \int_{-T}^{T} \Bigl|\sum_{n \le N} u_n n^{it} e(an/r)\Bigr|^2 dt \le 500 \sum_n |u_n|^2 (n + R^2 T).$$

The constant 500 is indeed very large, but see Theorem 2.6 below for an improvement.

Proof. We use Lemma 2.4 together with Theorem 2.1:

$$\sum_{r\le R}\sum_{a \bmod^* r}\int_{-T}^{T}\Big|\sum_n u_n n^{it} e(an/r)\Big|^2 dt$$
$$\le \pi^2T^2\int_0^\infty \sum_{r\le R}\sum_{a \bmod^* r}\Big|\sum_{y<n\le \tau y} u_n e(an/r)\Big|^2 dy/y$$
$$\le \pi^2T^2\int_0^\infty \sum_{y<n\le\tau y}|u_n|^2[(\tau-1)y+R^2]dy/y$$
$$\le \pi^2T^2\sum_{n\ge1}|u_n|^2\int_{n/\tau}^{n}[(\tau-1)y+R^2]dy/y.$$

We rewrite this upper bounde as

$$\pi^2T^2\sum_{n\ge1}|u_n|^2\big((\tau+\tau^{-1}-2)n+R^2\operatorname{Log}\tau\big).$$

We conclude by noticing that $(e^x+e^{-x}-2)\le\frac{11}{10}x^2$ as soon as $|x|\le 1$. □

Corollary 2.1. *When $T\ge\pi$ and $M\in\mathbb{R}$, we have*

$$\sum_{r\le R}\frac{\mu^2(r)}{\phi(r)}\int_{-T}^{T}\Big|\sum_{n\le N}u_n n^{it}c_r(n+M)\Big|^2 dt \le 500\sum_n |u_n|^2(n+R^2T).$$

2.4 A numerical improvement on the Theorem of Gallagher

This section is still very much in progress.

We investigate here the proof of Theorem 2.4 with the aim of reducing the constant 500. Here is a “generic” proof.

Let F be a function to be chosen later. We assume that $F(t)=0$ as soon as $|t|\ge 1/2$. Let $\delta>0$ be a parameter that we shall also chose later. We define

$$F_\delta(x)=F(x/\delta).$$

We thus get

$$\sum_n u_n e^{2i\pi t(\operatorname{Log} n)/(2\pi)}\hat F_\delta(t)=\sum_n u_n \widehat{F_\delta\Big(t-\frac{\operatorname{Log} n}{2\pi}\Big)}.$$

Parseval indentity yields

$$\int_{-\infty}^{\infty}\Big|\sum_n u_n e^{2i\pi t(\operatorname{Log} n)/(2\pi)}\hat F_\delta(t)\Big|^2 dt=\int_{-\infty}^{\infty}\Big|\sum_n u_n F_\delta\Big(x-\frac{\operatorname{Log} n}{2\pi}\Big)\Big|^2dx$$
$$=\int_0^\infty\Big|\sum_n u_n F_\delta\Big(\frac{\operatorname{Log}(y/n)}{2\pi}\Big)\Big|^2 dy/y.$$

Our hypothesis on F implies that the y's in the relevant range verify $e^{-\pi\delta} \le y/n \le e^{\pi\delta}$. As a consequence

$$\begin{aligned}
\sum_{r\le R}\sum_{a\,mod^* r}\int_{-\infty}^{\infty}\Bigl|\sum_n u_n n^{it} e(na/r)\hat{F}_\delta(t)\Bigr|^2 dt
&\le \int_0^\infty \sum_n |u_n|^2\Bigl|F_\delta\Bigl(\frac{\mathrm{Log}(y/n)}{2\pi}\Bigr)\Bigr|^2\bigl(y(e^{\pi\delta}-e^{-\pi\delta})+R^2\bigr)dy/y\\
&\le \sum_n |u_n|^2\int_0^\infty\Bigl|F_\delta\Bigl(\frac{\mathrm{Log}(y/n)}{2\pi}\Bigr)\Bigr|^2\bigl(e^{\pi\delta}-e^{-\pi\delta}+R^2y^{-1}\bigr)dy\\
&\le \sum_n |u_n|^2\int_0^\infty\Bigl|F_\delta\Bigl(\frac{\mathrm{Log}\,u}{2\pi}\Bigr)\Bigr|^2\bigl(n(e^{\pi\delta}-e^{-\pi\delta})+R^2u^{-1}\bigr)du\\
&\le \sum_n |u_n|^2 n(e^{\pi\delta}-e^{-\pi\delta})\int_0^\infty\Bigl|F\Bigl(\frac{\mathrm{Log}\,u}{2\pi\delta}\Bigr)\Bigr|^2 du + R^2\sum_n|u_n|^2\int_0^\infty\Bigl|F\Bigl(\frac{\mathrm{Log}\,u}{2\pi\delta}\Bigr)\Bigr|^2 du/u.
\end{aligned}$$

We change variable by setting $u = \exp(2\pi\delta w)$. We get

$$\begin{aligned}
&\sum_{r\le R}\sum_{a\,mod^* r}\int_{-\infty}^{\infty}\Bigl|\sum_n u_n n^{it} e(na/r)\hat{F}_\delta(t)\Bigr|^2 dt\\
&\le 2\pi\delta\sum_n|u_n|^2 n(e^{\pi\delta}-e^{-\pi\delta})\int_{-\infty}^{\infty}|F(w)|^2e^{2\pi\delta w}dw + 2\pi\delta R^2\sum_n|u_n|^2\int_{-\infty}^{\infty}|F(w)|^2dw\\
&\qquad\le 2\pi\delta\sum_n|u_n|^2\bigl(n(e^{\pi\delta}-e^{-\pi\delta})e^{\pi\delta}+R^2\bigr)\int_{-\infty}^{\infty}|F(w)|^2dw.
\end{aligned}$$

Since $\hat{F}_\delta(t) = \delta\hat{F}(\delta t)$, we have finally reached

$$\sum_{r\le R}\sum_{a\,mod^* r}\int_{-\infty}^{\infty}\Bigl|\sum_n u_n n^{it} e(na/r)\hat{F}(\delta t)\Bigr|^2 dt \le 2\pi\sum_n|u_n|^2\bigl(n\frac{e^{2\pi\delta}-1}{\delta}+R^2\delta^{-1}\bigr)\int_{-\infty}^{\infty}|F(w)|^2dw.$$

Now that we have this generic proof at our disposal, we simple have to optimise the choice of the function F. We want $\hat{F}(\delta t) \ge 1$ when $|t| \le T$ as well as $F(x) = 0$ when $|x| \ge 1/2$. It is more usual to set $G = \hat{F}$ and to ask that $G(t) \ge 1$ when $|t| \le \delta T$, and $\hat{G}(x) = 0$ when $|x| \ge 1/2$ (we have to minimize $\int_{-\infty}^{\infty}|G(t)|^2dt$).

2.4.1 Some material taken from Vaaler

We consider the two functions

$$H(z) = \Bigl(\frac{\sin\pi z}{z}\Bigr)^2\Bigl(\sum_{\substack{m\in\mathbb{Z},\\ m\neq 0}}\frac{\mathrm{sgn}\,m}{(z-m)^2}+\frac{2}{z}\Bigr) \tag{2.5}$$

and

$$K(z) = \Bigl(\frac{\sin\pi z}{z}\Bigr)^2. \tag{2.6}$$

We build from them the following function

$$G(z) = \tfrac{1}{2}\big(H(\theta(z+x_0)) + K(\theta(z+x_0)) - H(\theta(z-x_0)) + K(\theta(z-x_0))\big). \tag{2.7}$$

We can also write

$$\begin{aligned}2G(z) = H(\theta(z+x_0)) - \operatorname{sgn}(\theta(z+x_0)) + K(\theta(z+x_0))\\ - H(\theta(z-x_0)) + \operatorname{sgn}(\theta(z-x_0)) + K(\theta(z-x_0)) + 2\mathbb{1}_{|z|\le x_0}\end{aligned}$$

which is $\ge 2\mathbb{1}_{|z|\le x_0}$ thanks to (Vaaler, 1985, Lemma 5). This same paper helps us in computing its Fourier transform (see (Vaaler, 1985, (2.29)) and (Vaaler, 1985, Corollary 7)) :

$$\begin{aligned}2\hat{G}(t) = \theta^{-1}\frac{\hat{J}(t/\theta)-1}{i\pi t/\theta}e(tx_0) + \theta^{-1}(1-|t/\theta|)^+ e(tx_0)\\ - \theta^{-1}\frac{\hat{J}(t/\theta)-1}{i\pi t/\theta}e(-tx_0) + \theta^{-1}(1-|t/\theta|)^+ e(-tx_0) + 2\frac{e(x_0t)-e(-x_0t)}{2i\pi t}\\ = \frac{\hat{J}(t/\theta)}{\pi t}2\sin(2\pi t x_0) + 2\theta^{-1}(1-|t/\theta|)^+\cos(2\pi t x_0).\end{aligned}$$

(Vaaler, 1985, (2.32)) computes the Fourier transform of J and shows in particular that $\hat{J}(t) = 0$ when $|t| \ge 1$. We thus take $\theta = 1/2$ and $x_0 = \delta T$. We have reached

$$\hat{G}(t) = \frac{\hat{J}(2t)}{\pi t}\sin(2\pi t x_0) + 2(1-|2t|)^+\cos(2\pi t x_0).$$

We now have to compute its L^2-norm. We have

$$\begin{aligned}\int_{-\infty}^{\infty}|\hat{G}(t)|^2dt &= 2\int_0^{1/2}\Big|\frac{\hat{J}(2t)}{\pi t}\sin(2\pi t x_0) + 2(1-2t)\cos(2\pi t x_0)\Big|^2 dt\\ &= 2\int_0^{1/2}\Big|\frac{2\pi t(1-2t)\cot(2\pi t)+2t}{\pi t}\sin(2\pi t x_0) + 2(1-2t)\cos(2\pi t x_0)\Big|^2 dt\\ &= \int_0^{1}\Big|\frac{2\pi(1-t)\cot(\pi t)+2}{\pi}\sin(\pi t x_0) + 2(1-t)\cos(\pi t x_0)\Big|^2 dt = \rho(x_0).\end{aligned}$$

Here is the inequality we have proven so far

$$\sum_{r\le R}\sum_{a \bmod^* r}\int_{-T}^{T}\Big|\sum_n u_n n^{it} e(na/r)\Big|^2 dt \le 2\pi\sum_n |u_n|^2\Big(n\frac{e^{2\pi\delta}-1}{\delta} + R^2\delta^{-1}\Big)\rho(\delta T).$$

We take $\delta = 2\pi/T$ and compute that $\rho(1/(2\pi)) = 1.7762 + \mathcal{O}^*(10^{-4})$, which yields

Theorem 2.6. *When $T \ge 2\,000$, we have*

$$\sum_{r\le R}\sum_{a \bmod^* r}\int_{-T}^{T}\Big|\sum_n u_n n^{it} e(na/r)\Big|^2 dt \le 71\sum_n |u_n|^2(n + R^2T).$$

The best coefficient with respect to $\sum_n |u_n|^2 n$ seems to be $16\pi^2/3 = 52.6378 + \mathcal{O}^*(10^{-4})$, got when $x_0 = 0$.

Chapter 3

On the Mellin transform

3.1 Some smoothed formulae

Let us start by computing some complex integrals.

Lemma 3.1. *We have, when $x > 0$:*

$$\frac{1}{2i\pi}\int_{2-i\infty}^{2+i\infty}\frac{x^{-z}dz}{z^2} = \begin{cases} -\operatorname{Log} x & \textit{when } x < 1 \\ 0 & \textit{when } x \geq 1. \end{cases}$$

Lemma 3.2. *We have, when $x > 0$:*

$$\frac{1}{2i\pi}\int_{2-i\infty}^{2+i\infty}\frac{2x^{-z}dz}{z(z+1)(z+2)} = \begin{cases} (1-x)^2 & \textit{when } x < 1 \\ 0 & \textit{when } x \geq 1. \end{cases}$$

Proof. Indeed, when $x \geq 1$, it is enough to shift the line of integration towards the right. We have to note that in this case, the polar contributions are to be multiplied by a coefficient -1 and thate the horizontal segments give no contribution. This sentence may look mysterious at first glance, so we give now a complete proof. First the integral on a infinite path is the limit of the integral on a finite path:

$$\frac{1}{2i\pi}\int_{2-i\infty}^{2+i\infty}\frac{x^{-z}dz}{z(z+1)} = \lim_{T,T'\to\infty}\frac{1}{2i\pi}\int_{2-iT}^{2+iT'}\frac{x^{-z}dz}{z(z+1)}$$

where z runs overs the segments from bottom to top, and where T and T' tend independantly towards infinity. We then compare the integral on a segment to

$$\frac{-1}{2i\pi}\int_{A-iT}^{A+iT'}\frac{x^{-z}dz}{z(z+1)}$$

for some $A \geq 2$ that we shall take large but which is just now fixed. We do this comparison by applying the Cauchy residue Theorem to the rectangle with corners $2-iT$, $2-iT'$, $A-iT$ and $A-iT'$. The integrals on the horizontal and vertical segments are bounded above respectively by A/T^2 and by A/T'^2 which both tend to 0 when T and T' tend to ∞. On another side, no residues belongs to the enclosed area, and thus

$$\frac{1}{2i\pi}\int_{2-iT}^{2+iT'}\frac{x^{-z}dz}{z(z+1)} = \frac{-1}{2i\pi}\int_{A-iT}^{A+iT'}\frac{x^{-z}dz}{z(z+1)} + o(1)$$

where this $o(1)$ tends to 0 when T and T' tend to infinity. We next note that the integral over $\Re s = A$ is $\mathcal{O}(x^{-A})$ since

$$\frac{1}{2\pi}\int_{A-iT}^{A+iT'}\left|\frac{dz}{z(z+1)}\right|$$

is bounded above independantly of T, T' and $A \geq 2$. We then just have to let A tends to infinity. We shall summarize this process by saying that "we shift the line of integration to the right".

When $x < 1$, we shift this time the line of integration to the left. We get polar contribution at $z = 0$, $z = -1$ and $z = -2$ amounting to :

$$1 - 2x + x^2 = (1-x)^2$$

as expected. We prove the first Lemma is a similar fashion but we have to study the contribution of a double pole at 0. □

Here is another classical transform.

Lemma 3.3 (The Cahen-Millen formula)**.** *We have, when $x > 0$:*

$$e^{-x} = \frac{1}{2i\pi}\int_{2-i\infty}^{2+i\infty}\Gamma(z)x^{-z}dz$$

Proof. We shift the line of integration to the LHS. The Γ function has a simple pole at each $-n$ where $-n$ is a non-positive integer. Its residue there is $(-1)^n/n!$ as shown for instance by the complements formula

$$\frac{1}{\Gamma(z)\Gamma(1-z)} = \frac{\sin \pi z}{\pi}.$$

This leads to the contribution $\sum_{n\geq 0}(-x)^n/n! = e^{-x}$ as expected. □

3.2 Mellin transforms

Here is how we will use these lemmas. Let f be an arithmetical function, say the divisor function $n \mapsto \tau(n)$ that counts the number of (positive) divisors of n. We have

$$\begin{aligned}\sum_{n\leq x}\tau(n)\,\mathrm{Log}\,\frac{x}{n} &= \sum_{n\geq 1}\tau(n)\frac{1}{2i\pi}\int_{2-i\infty}^{2+i\infty}\frac{(n/x)^{-z}dz}{z^2}\\ &= \frac{1}{2i\pi}\int_{2-i\infty}^{2+i\infty}\sum_{n\geq 1}\frac{\tau(n)}{n^z}\frac{x^z dz}{z^2} = \frac{1}{2i\pi}\int_{2-i\infty}^{2+i\infty}\zeta(z)^2\,\frac{x^z dz}{z^2}.\end{aligned}$$

Formally, we start with a function f defined over $]0, \infty[$ and we want to write it in the shape

$$f(x) = \frac{1}{2i\pi} \int_{c-i\infty}^{c+i\infty} Mf(z)x^{-z}dz. \tag{3.1}$$

On writing $z = c + 2i\pi y$ and $x = e^u$, we get

$$f(e^u) = \int_{-\infty}^{\infty} Mf(c + 2i\pi y)e^{-uc}e^{-2i\pi uy}dy$$

that is to say that $Mf(c + 2i\pi y)$ is simply the Fourier transform of $e^{uc}f(e^u)$, i.e.

$$Mf(c + 2i\pi y) = \int_{-\infty}^{\infty} f(e^u)e^{uc}e^{2i\pi uy}du$$

or also

$$Mf(z) = \int_0^{\infty} f(x)x^{z-1}dx. \tag{3.2}$$

We call Mf the *Mellin transform* of the function f. The above line of derivation is formal and the convergence problems may be delicate and are to addressed seriously. It is not our aim to write a treaty on these questions but to show the reader how to reach (3.1) and thus how to guess Mf. A usual process consists in using the parameter c to ensure convergence, but the conditions at 0 and at ∞ are often antagonistic. A way to do is then to write f as $f_1 + f_2$, where f_1 vanishes when $x > 1$ and f_2 vanishes $x < 1$, with, for instance $f_1(1) = f_2(1) = f(1)/2$. We can then compute the Melin transforms of f_1 and f_2 but in distinct domains of z. *If* these transforms admit an analytical continuation to a common domain then $Mf_1 + Mf_2$ is a candidate for Mf.

3.3 A truncated Mellin transform

The Mellin transform of the Heavyside function Y which takes the value 0 on $(0, 1)$, then $1/2$ at 1 and 1 afterwards is simply $1/z$. This transform usually does not decrease in absolute value sufficiently fast in vertical strips and this leads to convergence problem.* The easiest path is to use a truncated transform. The necessary material is contained in the following Lemma:

Lemma 3.4. *When $\kappa > 0$ and $x > 0$, we have*

$$\left| Y(x) - \frac{1}{2i\pi} \int_{\kappa-iT}^{\kappa+iT} \frac{x^z dz}{z} \right| \le \frac{x^\kappa}{\pi} \min\left(\tfrac{7}{2}, \frac{1}{T|\operatorname{Log} x|}\right).$$

The proof will show that we have exactly the same bounds when we take for $Y(1)$ any other value belonging to $[0, 1]$.

*These problems are not mere technicalities and concerns deep properties of the functions under scrutiny.

Proof. When $x < 1$, we select $K > \kappa$ tending to infinity and write

$$\left(\int_{\kappa-iT}^{\kappa+iT} + \int_{\kappa+iT}^{K+iT} + \int_{K+iT}^{K-iT} + \int_{K-iT}^{\kappa-iT}\right) \frac{x^z dz}{z} = 0.$$

The third integral tends to 0 when K tends to infinity. Both integrals on the horizontal segments are majorised by $x^\kappa/(T|\operatorname{Log} x|)$. This yields

$$\left| Y(x) - \frac{1}{2i\pi}\int_{\kappa-iT}^{\kappa+iT} \frac{x^z dz}{z}\right| \leq \frac{x^\kappa}{\pi T|\operatorname{Log} x|} \quad (0 < x < 1).$$

The same upper bound holds when $x > 1$, which we prove as above but by shifting the integration line to the LHS. These bounds are efficient when $T|\operatorname{Log} x|$ is large enough; otherwise we write

$$\int_{\kappa-iT}^{\kappa+iT} \frac{x^z dz}{z} = x^\kappa \int_{\kappa-iT}^{\kappa+iT} \frac{dz}{z} + x^\kappa \int_{-T}^{T} \frac{(x^{it}-1)idt}{\kappa+it}.$$

The first integral takes value $2\arctan(T/\kappa) \leq \pi$ while we use

$$\left|\frac{x^{it}-1}{it\operatorname{Log} x}\right| = \left|\int_0^1 e^{iut\operatorname{Log} x} du\right| \leq 1$$

for the second one. This enables us to bound it by $2T|\operatorname{Log} x|$ (even if $x = 1$), and thus

$$\left|\frac{1}{2i\pi}\int_{\kappa-iT}^{\kappa+iT} \frac{x^z dz}{z}\right| \leq \frac{x^\kappa}{\pi}\left(\frac{\pi}{2} + T|\operatorname{Log} x|\right).$$

This is enough when $x < 1$. When $x > 1$, we note that

$$1 - \frac{x^\kappa}{2i\pi}\int_{\kappa-iT}^{\kappa+iT} \frac{dz}{z} = 1 - \frac{x^\kappa}{\pi}\arctan(T/\kappa)$$

which is $\geq -x^\kappa/2$ and $\leq 1 \leq x^\kappa$. In conclusion we have proved that

$$\left| Y(x) - \frac{1}{2i\pi}\int_{\kappa-iT}^{\kappa+iT} \frac{x^z dz}{z}\right| \leq \frac{x^\kappa}{\pi}\min\left(\pi + T|\operatorname{Log} x|, \frac{1}{T|\operatorname{Log} x|}\right).$$

In order to simplify this bound, let us note that

$$\min(\pi + u, 1/u) \leq \min(\alpha, 1/u)$$

where $\alpha = 1/u_0 = \pi + u_0$. Since $\alpha \leq 7/2$, the Lemma is proved. □

We deduce from the previous Lemma the following classical formula.

Theorem 3.1 (Truncated Perron summation formula)**.** *Let* $F(z) = \sum_n a_n/n^z$ *be a Dirichlet series absolutely convergent for* $\Re z > \kappa_a$*, and let* $\kappa > \kappa_a$*. When* $x \geq 1$ *and* $T \geq 1$*, we have*

$$\sum_{n\leq x} a_n = \frac{1}{2i\pi}\int_{\kappa-iT}^{\kappa+iT} F(z)\frac{x^z dz}{z} + \mathcal{O}^*\left(\int_{1/T}^{\infty} \sum_{|\operatorname{Log}(x/n)|\leq u} \frac{|a_n|}{n^\kappa}\frac{2x^\kappa du}{Tu^2}\right).$$

The Theorem in this form comes from (Ramaré, 2007).

In this formula, the error term is essentially raw. Let us note that the range of summation in

$$\sum_{|\operatorname{Log}(x/n)|\le u} |a_n|/n^\kappa$$

can be rather short. The interval in n can be rewritten as $e^{-u}x \le n \le e^u x$. When $u \ge 1$, the upper bound $\sum_{n\ge1}|a_n|/n^\kappa$ is usually enough. When u is smaller, we will normally use a bound of the shape $ux^{\kappa_a}B/x^\kappa$ for some *decent* B (a constant times $\operatorname{Log} x$ for instance), leading to the error term

$$\mathcal{O}\biggl(\frac{Bx^{\kappa_a}\operatorname{Log} T}{T} + \frac{x^\kappa}{T}\sum_{n\ge1}|a_n|/n^\kappa\biggr).$$

Notice further that the shorter sums we will have to consider are of size $\simeq x/T$.

Proof. We start Lemma 3.4 and write

$$\sum_{n\le x} a_n = \sum_{n\ge1} a_n Y(x/n) = \sum_{n\ge1} a_n \frac{1}{2i\pi}\int_{\kappa-iT}^{\kappa+iT}\frac{(x/n)^z dz}{z} + \mathcal{O}^*\biggl(\sum_{n\ge1}\frac{|a_n|x^\kappa}{\pi n^\kappa}\min\Bigl(\tfrac{7}{2},\frac{1}{T|\operatorname{Log}(x/n)|}\Bigr)\biggr).$$

Let us set $\varepsilon = 1/T$. We keep the contribution to the error term of the integers n such that $|\operatorname{Log}(x/n)| \le \varepsilon$ as such. Otherwise, we write

$$\sum_{\varepsilon\le|\operatorname{Log}(x/n)|}\frac{|a_n|x^\kappa}{n^\kappa|\operatorname{Log}(x/n)|} = \sum_{\varepsilon\le|\operatorname{Log}(x/n)|}\frac{|a_n|x^\kappa}{n^\kappa}\int_{|\operatorname{Log}(x/n)|}^\infty\frac{du}{u^2}$$
$$= \int_\varepsilon^\infty\sum_{|\operatorname{Log}(x/n)|\le u}\frac{|a_n|x^\kappa}{n^\kappa}\frac{du}{u^2} - \int_\varepsilon^\infty\sum_{|\operatorname{Log}(x/n)|\le\varepsilon}\frac{|a_n|x^\kappa}{n^\kappa}\frac{du}{u^2}$$

which is enough. □

To see the relative strength of this Theorem, let us try to compute the number of integers not more that x ... The generating series is of course the Riemann ζ function which has $\kappa_a = 1$. We select $\kappa = 1 + 1/\operatorname{Log} x$ and reach the error term $\mathcal{O}(\operatorname{Log}(xT)/T)$ provided $T \le x$. Concerning the integral, we shift the line of integration to the line $\kappa = 0$ where $\zeta(it)$ is $\mathcal{O}(\sqrt{|t|+2}\operatorname{Log}(|t|+2))$. This finally gives us

$$\sum_{n\le x} 1 = x + \mathcal{O}\bigl(\sqrt{T}\operatorname{Log} T + x\operatorname{Log}(xT)/T\bigr).$$

On taking $T = x^{2/3}$, we reach an error term of size ... $x^{1/3}\operatorname{Log} x$. This will help the reader getting an idea of the loss incurred in using the Perron formula, since the error term $\mathcal{O}(1)$

is here possible. One can in fact get $\mathcal{O}_\varepsilon(x^\varepsilon)$ for every $\varepsilon > 0$ by using a smoothed sum instead of truncing brutally at $n \le x$. The reader will find in (Ramaré, 2007) a way to reduce this error term to a power of logarithm.

The loss is however largely compensated by the fact that we can now use informations on the Mellin transforms of our initial sequence.

3.4 Expressing ψ in terms of $\zeta'(s)/\zeta(s)$

We have defined the Tchebyschef function ϑ but it is easier to work with a different function, namely

$$\psi(x) = \sum_{n \le x} \Lambda(n). \tag{3.3}$$

We express it in terms of its Mellin transform via Theorem 3.1 and this yields:

$$\psi(x) = \frac{1}{2i\pi} \int_{\sigma_0 - iT}^{\sigma_0 + iT} \frac{-\zeta'(s)}{\zeta(s)} \frac{x^s}{s} ds + \mathcal{O}(xT^{-1} \operatorname{Log}^2 x) \tag{3.4}$$

provided $2 \le T \le x/2$ and with

$$\sigma_0 = 1 + (\operatorname{Log} x)^{-1}. \tag{3.5}$$

Proof. In Theorem 3.1, we take $\kappa = \sigma_0$ and notice that $\kappa_a = 1$. Let us study the error term, i.e. let us find an upper bound for

$$I = \int_{1/T}^{\infty} \sum_{|\operatorname{Log}(x/n)| \le u} \frac{\Lambda(n)}{n^\kappa} \frac{2x^\kappa du}{Tu^2} = \int_{1/T}^{1} \cdots + \int_{1}^{\infty} \cdots = I_1 + I_2$$

say. In case of I_1, the range of summation reads $e^{-u}x \le n \le e^u x$. Note that $e^{-u} \ge 1 - u$ for u, and thus $e^{-u}x \ge x - (x/T) \ge x/2$. As a consequence, $n^\kappa \gg x$ in this range of n (and while $1/T \le u \le 1$). The integer n runs over an interval of length* $(e^u - e^{-u})x \le 3ux$. Thus

$$\sum_{|\operatorname{Log}(x/n)| \le u} \frac{\Lambda(n)}{n^\kappa} \ll \frac{\operatorname{Log} x}{x}(ux + 1) \ll u \operatorname{Log} x.$$

† As a consequence

$$I_1 \ll \int_{1/T}^{1} \frac{xu \operatorname{Log} x}{Tu^2} du \ll \frac{x \operatorname{Log}^2 x}{T}.$$

The treatment of I_2 is easier since we simply say that

$$\sum_{|\operatorname{Log}(x/n)| \le u} \frac{\Lambda(n)}{n^\kappa} \le \sum_{n \ge 1} \frac{\Lambda(n)}{n^\kappa} = \frac{-\zeta'(\kappa)}{\zeta(\kappa)} \ll 1/(\kappa - 1) = \operatorname{Log} x.$$

The claimed expression follows readily. □

*The function sh is convex for $u \ge 0$. This gives us $(\operatorname{sh} u)/u \le (\operatorname{sh} 1)/1$, hence the inequality. We can replace the constant 3 by $2 \operatorname{sh} 1$.

†The number of integer points in an interval of length $\mathcal{O}(ux)$ is $\mathcal{O}(ux + 1)$ and this is $\mathcal{O}(ux)$ here.

Chapter 4

The local method of Landau et alia

We shall require some facts on the analytic continuation of the Riemann zeta function and most of them are recalled in the next chapter. As it turns out, the bound $|-\zeta'/\zeta(s)| \ll \mathrm{Log}\,|\Im s|$ valid when $|\Im s| \geq 2$ and $|1-\Re s|\,\mathrm{Log}\,|\Im s|$ is small enough will be of particular importance. We take this opportunity to develop some material and to expand on the historical side.

4.1 The Borel-Caratheodory Theorem

The Borel-Caratheodory Theorem which bounds the modulus of an analytic function in term of a bound for its real part is of fundamental importance in what follows. Caratheodory did not publish it anywhere: Landau says he owns a series a results in this vein from letters exchanged with him. See in particular (Landau, 1908, Satz I, section 5) and (Landau, 1926, Lemma 1). (Titchmarsh, 1932, section 5.5) attributes this Theorem to (Borel, 1897, page 365)* and to Caratheodory.

Theorem 4.1 (Borel-Caratheodory)**.** *Let F be an analytic function on $|s-s_0| \leq R$ such that $\Re F(s) \leq A$ in this disc. For any $r < R$, positive, we have*

$$\max_{|s-s_0|\leq r} |F(s) - F(s_0)| \leq \frac{Ar}{R-r}$$

and, for any $k \geq 1$,

$$\max_{|s-s_0|\leq r} |F^{(k)}(s)| \leq \frac{2k!R}{(R-r)^{k+1}}(A - \Re F(s_0)).$$

In fact (Landau, 1908, Satz I, section 5) has a slightly distinct version of the upper bound. See also (Titchmarsh, 1932, section 5.51), where the author follows roughly Landau's proof. This latter one relies on Schwarz's Lemma, though Landau does not cite the later (the proof is anyway quite obvious). I do not know whether Schwarz has anteriority or not. The proof we follow here is essentially the one due Borel, and can be found in (Tenenbaum, 1995, Theorem 11 and corollaries).

*Note that this proof has to be somewhat modified to meet our needs.

Proof. R We can assume with no loss of generality that $s_0 = 0$ and that $F(s_0) = 0$ (otherwise consider $F(s) - F(s_0)$). Note that under these assumptions, the corresponding A, namely $A - \Re F(s_0)$ is non-negative. We expand F is power series:

$$F(s) = \sum_{n\geq 1} a_n s^n$$

and write $a_n = |a_n|e^{i\theta_n}$. We have

$$\Re F(Re^{i\theta}) = \sum_{n\geq 1} |a_n| R^n \cos(n\theta + \theta_n).$$

We multiply this expression by $\cos(m\theta + \theta_m)$ and integrate it termwise to get

$$\pi|a_m|R^m = \int_0^{2\pi} \Re F(Re^{i\theta}) \cos(m\theta + \theta_m) d\theta.$$

When $m = 0$, this reads

$$0 = \int_0^{2\pi} \Re F(Re^{i\theta}) d\theta$$

which we combine with the above to obtain

$$\pi|a_m|R^m = \int_0^{2\pi} \Re F(Re^{i\theta})(1 + \cos(m\theta + \theta_m)) d\theta \leq 2\pi(A - \Re F(s_0)).$$

This readily yields, when $k \geq 1$,

$$\begin{aligned} |F^{(k)}(re^{i\theta})| &\leq \sum_{n\geq k} n(n-1)\cdots(n-k+1)|a_n| r^{n-k} \\ &\leq \frac{2(A - \Re F(s_0))}{R^k} \sum_{n\geq k} n(n-1)\cdots(n-k+1)(r/R)^{n-k} = 2\frac{Rk!(A - \Re F(s_0))}{(R-r)^{k+1}}. \end{aligned}$$

When $k = 0$, we use the additional fact that $a_0 = 0$ to get the claimed bound. □

4.2 The Landau local method

The proof and the statement of the following Lemma has taken some years to find a proper shape. One can find traces of it in (Landau, 1908), between equations (92) and (93), see the definition of F. It will evolve until (Landau, 1926, Lemma 1) to yield a bound on $\zeta'/\zeta(s)$ next to the line $\Re s = 1$. At the time, Gronwall and Landau were improving each other's bound. See also (Titchmarsh, 1951, section 3.9, Lemma α).

The circle of ideas we present below belongs to this realm.

Lemma 4.1. *Let M be an upper bound for the holomorphic function F in $|s - s_0| \leq R$. Assume we know of a lower bound $m > 0$ for $|F(s_0)|$. Then*

$$\frac{F'(s)}{F(s)} = \sum_{|\rho - s_0|\leq R/2} \frac{1}{s-\rho} + \mathcal{O}^*\left(16\frac{\operatorname{Log}(M/m)}{R}\right)$$

for every s such that $|s-s_0| \le R/4$ and where the summation variable ρ ranges the zeros ρ of F in the region $|\rho - s_0| \le R/2$, repeated according to multiplicity.

This Lemma will be our main tool in what follows. The reader should look at the very interesting section 3 of (Heath-Brown, 1992a), and more precisely to (Heath-Brown, 1992a, Lemma 3.2). An expression for the real part of $F'(s)/F(s)$ in terms os the possible zeros is obtained there. In fact, the proof therein contains an expression for $F'(s)/F(s)$, but it seems necessary to take the real part to bound it solely in term of M/m (notations as above). See however subsection 4.4.3 of this chapter.

See also (Heath-Brown, 1992b) as well as (Ford, 2000, Lemma 2.1 and Lemma 2.2).

Proof. Let us consider

$$G(s) = \frac{F(s)}{\prod_{|\rho - s_0| \le R/2}(s-\rho)}.$$

When $|s-s_0| = R$, we have $|s-\rho| \ge |s-s_0| - |\rho - s_0| \ge R/2 \ge |\rho - s_0|$ for the zeros under consideration, and thus, by the maximum principle, when $|s - s_0| \le R$, we have

$$\left|\frac{G(s)}{G(s_0)}\right| = \left|\frac{F(s)}{F(s_0)} \prod_{|\rho - s_0| \le R/2} \frac{s_0 - \rho}{s-\rho}\right| \le M/m.$$

Since this function has no zeros inside $|s-s_0| \le R/2$, we can write

$$G(s)/G(s_0) = e^{H(s)} \quad (|s-s_0| \le R/2)$$

for an analytic function H that verifies $H(s_0) = 0$. Furthermore $\Re H(s) \le \operatorname{Log}(M/m)$. By the Borel-Caratheodory Theorem, we deduce that

$$\left|\frac{G'(s)}{G(s)}\right| = |H'(s)| \le \frac{8R}{(R-2r)^2} \operatorname{Log}(M/m) \quad (|s-s_0| \le r < R/2).$$

We have thus proved our assertion. □

4.3 Consequence for the Riemann zeta function

Here is the main consequence of Lemma 4.1:

Lemma 4.2. *Let $t_0 \ge 4$. We have*

$$\frac{\zeta'(s)}{\zeta(s)} = \sum_{|\rho - 1 - it_0| \le 1} \frac{1}{s-\rho} + \mathcal{O}(\operatorname{Log} t_0) \quad (|s-1-it_0| \le 1/2) \tag{4.1}$$

We use $F = \zeta$, $s_0 = 1 + it_0$ and $R = 2$ We only have to get some polynomial upper bound for $|\zeta(s)|$ when $-1 \le \Re s \le 3$ and $\Im s \ge 2$, as well as a lower bound for $|\zeta(s_0)|$. Concerning the upper bound, here is an expedient way to get one:

$$\zeta(s) = s\int_1^\infty [t]\frac{dt}{t^{s+1}} = \frac{s}{s-1} - \frac{1}{2} - s\int_1^\infty B_1(t)\frac{dt}{t^{s+1}}$$

where $[t]$ denotes the integer part of t, $\{t\}$ its fractional part, and $B_1(t)$ is the first Bernoulli function, defined by $B_1(t) = \{t\} - \frac{1}{2}$. We consider the higher order Bernoulli functions, B_2 and B_3:

$$\tfrac{1}{2}B_2(t) = \int_1^t B_1(u)du + \frac{1}{12} = \tfrac{1}{2}\{t\}^2 - \tfrac{1}{2}\{t\} + \frac{1}{12}.$$

This function is periodical of period 1 when $t \geq 1$ and has mean value 0 over a period (i.e. $\int_1^2 B_2(u)du = 0$). As a consequence

$$\tfrac{1}{3}B_3(t) = \int_1^t B_2(u)du = \tfrac{1}{3}\{t\}^3 - \tfrac{1}{2}\{t\}^2 + \tfrac{1}{6}\{t\}$$

is bounded. As it turns out, it is also of zero mean value over a period. Here is why we have introduced this set of functions:

$$\begin{aligned}\zeta(s) &= \frac{s}{s-1} - \frac{1}{2} - s\int_1^\infty B_1(t)\frac{dt}{t^{s+1}} \\ &= \frac{s}{s-1} - \frac{1}{2} + \frac{s}{12} + \frac{s(s+1)}{2}\int_1^\infty B_2(t)\frac{dt}{t^{s+2}} \\ &= \frac{s}{s-1} - \frac{1}{2} + \frac{s}{12} - \frac{s(s+1)(s+2)}{6}\int_1^\infty B_3(t)\frac{dt}{t^{s+3}}\end{aligned}$$

which gives us an expression of the continuation of ζ to $\Re s > -2$. In particular, when $|s| \geq 2$ and $\Re s \geq -1$, we find that

$$|\zeta(s)| \leq \tfrac{13}{12}|s| + \frac{1}{2} + \tfrac{1}{20}\frac{|s|^3}{2} \leq \tfrac{2}{5}|s|^3.$$

Better bounds are available (in fact, $|\zeta(s)| \ll |s|^{3/2}$ under our conditions), but this will be enough for us.

We also need a lower bound for $1/|\zeta(1+it_0)|$. This is readily obtained as follows. We can modify the above proof to show that

$$|\zeta'(s)| \ll (\operatorname{Log} t)^2 \quad (\Re s \geq 1, t = \Im s \geq 2).$$

We use this bound to shift $\zeta(1+it_0)$ to $\zeta(\sigma+it_0)$ at a cost of $\mathcal{O}((\sigma-1)(\operatorname{Log} t_0)^2)$. We next recall the classical Mertens's inequality*:

$$1 \leq |\zeta(\sigma)|^3|\zeta(\sigma+it_0)|^4|\zeta(\sigma+2it_0)|.$$

*To prove it, notice that $-\zeta'/\zeta$ has a Dirichlet expansion with non-negative coefficients. Since $3 + 4\cos\theta + \cos(2\theta) = 2(1+\cos\theta)^2 \geq 0$, we find that

$$0 \leq 3\Re\frac{-\zeta'}{\zeta}(\sigma) + 4\Re\frac{-\zeta'}{\zeta}(\sigma+it_0) + \Re\frac{-\zeta'}{\zeta}(\sigma+2it_0).$$

Integrating this inequality in σ between σ_0 and ∞ yields the desired result. Alternatively, one could use directly the Dirichlet expansion of $\operatorname{Log}\zeta$, which also has non-negative coefficients.

Since $|\zeta(\sigma)| \ll 1/(\sigma-1)$ and $|\zeta(\sigma+2it_0)| \ll \operatorname{Log} t_0$, we get

$$(\sigma-1)^{3/4}(\operatorname{Log} t_0)^{-1/4} \ll |\zeta(\sigma+it_0)|.$$

We then take $\sigma = 1 + C(\operatorname{Log} t_0)^{-9}$ for a large enough constant C and get

$$|\zeta(\sigma+it_0)| \gg 1/(\operatorname{Log} t_0)^7.$$

This is an apriori bound that is enough for our purpose, but much better is known.

The reader has now all the elements to end the proof of Lemma 4.2. Note that the proof of

$$\frac{\zeta'(s)}{\zeta(s)} = \sum_{|\rho-1-(\operatorname{Log} t_0)^{-1}-it_0|\le 1} \frac{1}{s-\rho} + \mathcal{O}(\operatorname{Log} t_0) \quad (|s-1-it_0| \le 1/2)$$

would have been slightly simpler since the bound

$$|\zeta(1+(\operatorname{Log} t_0)^{-1}+it_0)|^{-1} \le |\zeta(1+(\operatorname{Log} t_0)^{-1})| \ll (\operatorname{Log} t_0)$$

would have been enough. It is however simply easier to write down the expression we have chosen!

4.4 Bounding $|\zeta'/\zeta|$ next to the line $\Re s = 1$

We do not reproduce here a proof of a zero-free region, though we have all the ingredients, but we content ourselves with citing (Kadiri, 2005):

Theorem 4.2 (Kadiri)**.** *The Riemann ζ-function has no zeros in the region*

$$\Re s \ge 1 - \frac{1}{R_0 \operatorname{Log}|\Im s|}, |\Im s| \ge 2, \quad \textit{with } R_0 = 5.69693.$$

The proof of this result is very intricate, but it is fairly easy to get a similar result with a much larger value of R_0, and even easier to prove it without specifying any admissible value for R_0.

Let s be in the region

$$\Re s \ge 1 - \frac{1}{2R_0 \operatorname{Log}(1+|\Im s|)}, \quad |\Im s| \ge 2. \tag{4.2}$$

Lemma 4.3. *We have*

$$\left|\zeta'/\zeta(s)\right| \ll \operatorname{Log} t$$

when $\Re s \ge 1 - \frac{1}{2R_0}(1+\operatorname{Log} t)^{-1}$ *where* $R_0 > 0$ *is the constant of the zero free region given in Theorem 4.2.*

There exists essentially three ways to bound $\zeta'/\zeta(s)$ when $s = \sigma + it_0$ is well within the zero free region, i.e. in the region given by (4.2). One is due to Landau, another one to Linnik and a third one to Titchmarsh. We present the three of them. Let us set

$$s_1 = 1 + \frac{4}{R_0 \operatorname{Log}(1+t_0)}.$$

We apply Lemma 4.2 to s and s_1 and substract:

$$\frac{\zeta'}{\zeta}(s) - \frac{\zeta'}{\zeta}(s_1) = \sum_\rho \frac{s - s_1}{(s-\rho)(s_1-\rho)} + \mathcal{O}(\operatorname{Log} t_0) \tag{4.3}$$

When s in the region given by (4.2), we have, for any zero $\rho = \beta + i\gamma$ verifying $|s - \rho| \le 1$,

$$(1 - \tfrac{1}{3})(1-\beta) \ge \frac{1-\frac{1}{3}}{R_0 \operatorname{Log}(1+t_0)} \ge \tfrac{1}{3}(\sigma_1 - 1) + (1-\sigma)$$

and thus $\sigma - \beta \ge (\sigma_1 - \beta)/3$. As a consequence

$$\Big|\sum_\rho \frac{s-s_1}{(s-\rho)(s_1-\rho)}\Big| \ll \sum_\rho \frac{1/\operatorname{Log} t_0}{|s_1-\rho|^2}.$$

4.4.1 Landau's way

(Landau, 1926, Hilfsatz 1) remarks that

$$\sum_\rho \frac{1/\operatorname{Log} t_0}{|s_1-\rho|^2} \ll \sum_\rho \frac{\sigma_1 - \beta}{|s_1-\rho|^2} = \sum_\rho \Re \frac{1}{s_1-\rho} = \Re\frac{\zeta'}{\zeta}(s_1) + \mathcal{O}(\operatorname{Log} t_0).$$

Note that $\Re\frac{\zeta'}{\zeta}(s_1) < 0$ (a fact that Landau does not use). But to end the proof, we will anyway have to use

$$\Big|\frac{\zeta'}{\zeta}(s_1)\Big| \le \Big|\frac{\zeta'}{\zeta}(\sigma_1)\Big| \ll \operatorname{Log} t_0.$$

4.4.2 Linnik's way

(Linnik, 1944a) proposes a different conclusion that shed some more lights as to what happens. This is the one followed in (Ramaré, 2009). We first deduce from the above Linnik's density Lemma:

Lemma 4.4. *Let $n(t_0; r)$ be the number of zeros ρ of ζ such that $|\rho - 1 - it_0| \le r$. We have*

$$n(t_0; r) \ll 1 + r \operatorname{Log} t_0.$$

Proof. Assume $r > 0$. We use the formula (4.1) with $s = 1 + r + it_0$ and take real part. We get

$$\sum_{|\rho - 1 - it_0| \le 1} \frac{1 + r - \beta}{|s-\rho|^2} \ll r^{-1} + \operatorname{Log} t_0$$

by using

$$|\zeta'/\zeta(s)| \le -\zeta'/\zeta(1+r) \ll r^{-1}.$$

We can discard any zeros, by positivity, from the left and thus restrict the summation to the zeros counted in $n(t_0; r)$. Note that

$$\frac{1+r-\beta}{|s-\rho|^2} = \frac{r}{(1-\beta+r)^2+(\gamma-t_0)^2} \ge \frac{r}{2(1-\beta)^2+2r^2+(\gamma-t_0)^2} \ge \frac{r}{4r^2} = \frac{1}{4r}.$$

The Lemma follows readily. □

Proof of Lemma 4.3 by Linnik. We again exploit (4.3) but we now remark that $|s-\rho| \gg |1+it_0-\rho|$ and that $|s_1-\rho| \gg |1+it_0-\rho|$. Thus

$$(\operatorname{Log} t_0)\left|\frac{\zeta'(s)}{\zeta(s)} - \frac{\zeta'(s_1)}{\zeta(s_1)}\right| \ll \sum_{|\rho-1-it_0|\le 1} \frac{1}{|1+it_0-\rho|^2}.$$

Let us use diadic decomposition on this last sum. Set $r_k = 2^k/(R_0 \operatorname{Log}(t_0+1))$ when $k \ge 0$, so that:

$$\sum_{|\rho-1-it_0|\le 1} \frac{1}{|1+it_0-\rho|^2} = \sum_{k\ge 0}\sum_{r_k<|\rho-1-it_0|\le r_{k+1}} \frac{1}{|1+it_0-\rho|^2}$$
$$\ll \sum_{k\ge 0}\sum_{r_k<|\rho-1-it_0|\le r_{k+1}} \frac{1+r_{k+1}\operatorname{Log} t_0}{r_k^2} \ll \sum_{k\ge 0}\frac{(\operatorname{Log} t_0)^2}{2^k} \ll (\operatorname{Log} t_0)^2.$$

We conclude the proof as before. □

4.4.3 Titchmarsh's way

(Titchmarsh, 1951, Lemma γ, section 3.9) uses yet another way: first get an upper bound for $-\Re\zeta'(s)/\zeta(s)$ and then apply the Borel-Caratheodory Theorem to this function.

4.5 Bounding $|1/\zeta|$ next to the line $\Re s = 1$

The analytical upper bound we produce for the mean of the Barban & Vehov weights relies on a lower bound for $|\zeta(s)|$ when s is in the vicinity of the line $\Re s = 1$. This is again a consequence our upper bound for $|-\zeta'/\zeta(s)|$. The reader will first notice that it is obvious when $\Re s \ge 1 + 1/(\operatorname{Log} 2 + |\Im s|)$. Otherwise, let $s = \sigma_0 + it_0$ be in the region given by (4.2). We have

$$\operatorname{Log}\frac{\zeta(\sigma_0+it_0)}{\zeta(\sigma_1+it_0)} = \int_{\sigma_1}^{\sigma_0} \frac{\zeta'}{\zeta}(\sigma+it_0)d\sigma$$

with $\sigma_1 = 1 + (\operatorname{Log} t_0)^{-1} + it_0$. We bound the integrand by $\mathcal{O}(\operatorname{Log} t_0)$, and thus, on taking real parts:

$$\operatorname{Log}|\zeta(\sigma_0+it_0)| = \operatorname{Log}|\zeta(\sigma_1+it_0)| + \mathcal{O}(1). \quad (t_0 \ge 2) \tag{4.4}$$

The right-hand side is $\leq \operatorname{Log}\operatorname{Log} t_0 + \mathcal{O}(1)$ in absolute value, hence our bound for $1/|\zeta(\sigma_0 + it_0)|$. Note that we could have taken real parts earlier and thus relied only on an upper bound for $-\Re\zeta'/\zeta(s)^*$.

4.6 Some other consequences

As a matter of fact, (4.4) gives more information than just a lower bound. Let us start by recalling the following Lemma, which is a direct consequence of (Montgomery & Vaughan, 1974, Corollary 2).

Lemma 4.5.

$$\int_0^T \Big|\sum_n a_n n^{it}\Big|^2 dt = \sum_n |a_n|^2 \big(T + \mathcal{O}^*(3\pi n)\big).$$

Lemma 4.5 together with (4.4) lead directly to:

Lemma 4.6. *Let $T \geq 2$ and $\sigma \geq 1 - (12\operatorname{Log}(1+T))^{-1}$. We have*

$$\int_1^T |\zeta(\sigma+it)|^{\pm 1} dt \asymp T, \qquad \int_1^T |\zeta(\sigma+it)|^{\pm 2} dt \asymp T.$$

Proof. We use (4.4) with $\sigma_0 = \sigma$ and σ_1. This leads immediately to

$$\int_0^T |\zeta(\sigma+it)|^{\pm 1} dt \asymp \int_0^T |\zeta(\sigma_1+it)|^{\pm 1} dt$$

and the same with ± 2.† By Cauchy inequality, we have

$$\left|\int_0^T |\zeta(\sigma_1+it)|^{\pm 1} dt\right|^2 \leq T\int_0^T |\zeta(\sigma_1+it)|^{\pm 2} dt = T\sum_{n\geq 1} \frac{T+\mathcal{O}(n)}{n^{2\sigma_1}} \asymp T^2$$

by appealing to Lemma 4.5 below. Furthermore

$$\left|\int_1^T dt\right|^2 \leq \int_1^T |\zeta(\sigma+it)| dt \int_1^T |\zeta(\sigma+it)|^{-1} dt$$

hence the lower bounds. □

4.7 Better bounds

Using Vinogradov-Korobov – Full-fledged density estimates –

*The minus sign is here because $\sigma_0 \leq \sigma_1$.

†Here $\pm$ means we choose a sign + or − and stick to it on both sides of the relation.

Chapter 5

Some known facts about the Riemann zeta function

Theorem 5.1. *There exists a positive constant $c_1 \leq 1/(2\operatorname{Log} 2)$ such that the Riemann zeta function has no zero in the region*

$$\Re s \geq 1 - \frac{c_1}{\operatorname{Log}(2 + |\Im s|)}$$

and further satisfies, for $s = \sigma + it$ in this region the following bounds:

$$\left|\frac{\zeta'}{\zeta}(s)\right| \ll \operatorname{Log}(2+|t|), \quad \left|\zeta(s) - \frac{1}{s-1}\right| \ll \operatorname{Log}(2+|t|), \quad |1/\zeta(s)| \ll \operatorname{Log}(2+|t|).$$

The bound on c_1 ensures that the region given by this Theorem stays within the half plane $\Re s \geq 1/2$. It is even its sole purpose. See also (McCurley, 1984), (Kadiri, 2002), (Kadiri, 2005) and (Ford, 2000).

Note that this Theorem in particular yields the bounds

$$\left|\frac{\zeta'}{\zeta}(s)\right| \ll \operatorname{Log}(2+|t|), \quad \left|\zeta(s) - \frac{1}{s-1}\right| \ll \operatorname{Log}(2+|t|), \quad |1/\zeta(s)| \ll \operatorname{Log}(2+|t|)$$

for any $\sigma \geq 1$. See also (Cheng & Graham, 2004) and (Cheng, 1999).

Theorem 5.2. *We have, when $\sigma \geq 1/2$,*

$$|\zeta(\sigma+it)| + |\zeta'(\sigma+it)| \ll (1+|t|)^{\frac{1}{6}-0.000001}.$$

Theorem 5.3. *We have, when $\sigma \geq 0$,*

$$|\zeta(\sigma+it)| \ll (1+|t|)^{1/2}\operatorname{Log}(2+|t|).$$

Chapter 6

Barban & Vehov weights

While studying an optimisation problem close to the one that is the classical initial founding of the Selberg sieve for prime numbers, Barban & Vehov* noticed in (Barban & Vehov, 1968) the property

$$\sum_{n\le N}\biggl(\sum_{\substack{d|n,\\ d\le z}}\mu(d)\frac{\operatorname{Log}(z/d)}{\operatorname{Log} z}\biggr)^2 \ll N/\operatorname{Log} z. \tag{6.1}$$

They sketched a proof and later proofs were given later by Motohashi (in 1974, see (Motohashi, 1983)[section 1.3]) and (Graham, 1978). The novelty of this estimate is that no additionnal $+\mathcal{O}(z^2)$ arises, as it does when using a direct approach. This enables us to avoid the condition $N \ge z^2 \operatorname{Log} z$.

The second novelty in (Barban & Vehov, 1968) comes from the fact that they consider the weight

$$\lambda_d^{(1)} = \begin{cases} \mu(d) & \text{when } d \le z,\\ \mu(d)\frac{\operatorname{Log}(z^2/d)}{\operatorname{Log} z} & \text{when } z < d \le z^2,\\ 0 & \text{when } z^2 < d. \end{cases} \tag{6.2}$$

(They consider in fact slightly more general weights with a y instead of the z^2 that we use here). Motohashi (see (Motohashi, 1983, section 1.3)) and (Graham, 1978) continued this study. The Theorem we need is the following.

Theorem 6.1. *We have, when $x \ge z \ge 2$,*

$$\sum_{n\le x}\Bigl(\sum_{d|n}\lambda_d^{(1)}\Bigr)^2/n \ll \frac{\operatorname{Log} x}{\operatorname{Log} z}.$$

When $x \le z$, the sum simply vanishes for every summand does! We will use this Lemma with $h(n) = 1$ in which case $H(\omega) = \zeta(\omega) \ll 1/(\omega - 1)$ since ω is real and close to 1. The proof of this Theorem will be given at the end of chapter 7.

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*For a reason unknown to me but which stems almost surely from older transliteration rules, Vehov is spelled Vekhov in Zentralblatt

A reduction step

We can decompose the $\lambda_d^{(1)}$'s as follows

$$\lambda_d^{(1)} = \mu(d)\frac{\operatorname{Log}(z^2/d)}{\operatorname{Log} z}\,\mathbb{1}_{d\le z^2} - \mu(d)\frac{\operatorname{Log}(z/d)}{\operatorname{Log} z}\,\mathbb{1}_{d\le z}. \tag{6.3}$$

Chapter 7

An analytical proof of Barban & Vehov bound

We have chosen to give an analytical proof of Theorem 6.1 that goes through Mellin transforms because of the flexibility of this method. This is very much the path followed by (Motohashi, 1978). We give full details.

7.1 A preliminary estimate

Lemma 7.1. *The quantity* $\mathfrak{M}_1(r,y,\omega)$ *defined in* (8.3) *verifies*

$$|\mathfrak{M}_1(r,y,\omega)| \ll \frac{r}{\phi(r)}\Big(\sum_{d|r}\frac{\mu^2(d)}{\sqrt{d}} + \zeta^{-1}(\omega)\operatorname{Log} y\Big)$$

as soon as $\omega \geq 1$. *We understand* $\zeta^{-1}(1)$ *as being* $= 0$.

Proof. On using Lemma 3.1, we get:

$$\mathfrak{M}_1(r,y,\omega) = \frac{1}{2i\pi}\int_{2-i\infty}^{2+i\infty}\sum_{\substack{d\geq 1,\\ (d,r)=1}}\frac{\mu(d)}{d^{s+\omega}}\frac{y^s ds}{s^2} = \frac{1}{2i\pi}\int_{2-i\infty}^{2+i\infty}\mathscr{H}_r(s+\omega)\frac{y^s ds}{\zeta(s+\omega)s^2}$$

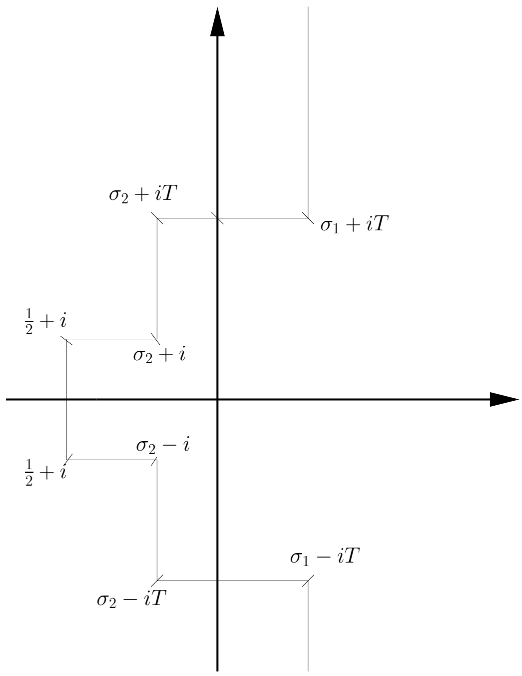
where

$$\mathscr{H}_r(z) = \prod_{p|r}(1-p^{-z})^{-1}. \tag{7.1}$$

We shift the line of integration to $\Re s = \sigma_1 = 1/\operatorname{Log} z$. Let $c_1 > 0$ be the constant given by Theorem 5.1. We have by this Theorem

$$1/|\zeta(\sigma+it)| \leq c_2 \operatorname{Log}(2+|t|) \quad \text{pour} \quad \sigma \geq 1 - c_1(\operatorname{Log}(2+T))^{-1} \text{ et } |t| \leq T. \tag{7.2}$$

We choose $T = y$ (a much smaller value would do!) and modify our contour integration into the following one



where $\sigma_2 = 1 - \omega - c_1(\mathrm{Log}(2+T))^{-1}$ is $\geq -\omega - 1/2$ by the hypothesis on ω and c_1. We meet a double pole at $s = 0$ which we will study later. In the half-plane $\Re z \geq \omega + \sigma_2$, we obviously have

$$|\mathscr{H}_r(z)| \leq \prod_{p|r} (1 - p^{-1/2})^{-1} = \mathscr{H}_r(1/2).$$

On each half-line from $\sigma_1 + iT$ to $\sigma_1 + i\infty$ and from $\sigma_1 - i\infty$ to $\sigma_1 - iT$, we bound above $1/|\zeta(s+\omega)|$ by $\zeta(\omega) \leq \omega/(\omega - 1) \ll \mathrm{Log}\, z$, so that the total contribution from these two half-lines is

$$\ll \mathscr{H}_r(1/2)(\mathrm{Log}\, z)/z \ll \mathscr{H}_r(1/2).$$

On the segment going from $\sigma_2 + iT$ to $\sigma_1 + iT$ and symmetically from $\sigma_1 - iT$ to $\sigma_2 - iT$, we use $1/|\zeta(s+\omega)| \ll \mathrm{Log}\, T = \mathrm{Log}\, z$, which gives rise to a contribution of a same order as the preceding one,since x^s is here bounded. On the segment going from $\sigma_2 + i$ to $\sigma_2 + iT$ and from $\sigma_2 - iT$ to $\sigma_2 - i$, we bound above $1/|\zeta(s+\omega)|$ by $\mathrm{Log}(|t| + 2)$ where $t = \Im s$. The factor y^s is again bounded, so that the contribution to the integral is

$$\ll \mathscr{H}_r(1/2).$$

The same upper bound holds true for the segments going from $\frac{1}{2} + i$ to $\sigma_2 + i$ and from $\sigma_2 - i$ to $\frac{1}{2} - i$ since this time, every factor save $\mathscr{H}_r(s+\omega)$ is bounded above by constants. We treat the contribution of the segment going from $\frac{1}{2} - i$ to $\frac{1}{2} + i$ in the same fashion.

Now that we have bounded the residual contribution, let us estimate the contribution of the double pole at $s = 0$. We have, near $s = 0$,

$$\mathscr{H}_r(s+\omega)\frac{x^s}{\zeta(s+\omega)} = \frac{\mathscr{H}_r(\omega)}{\zeta(\omega)} + \Big(\frac{\mathscr{H}_r(\omega)}{\zeta(\omega)}\,\mathrm{Log}\, x + \frac{\mathscr{H}_r'(\omega)}{\zeta(\omega)} - \frac{\mathscr{H}_r(\omega)\zeta'(\omega)}{\zeta(\omega)^2}\Big)s + \mathcal{O}(s^2)$$

so that the residue we are interested in reads

$$\mathfrak{R}(r,\omega,y) = \frac{\mathscr{H}_r(\omega)}{\zeta(\omega)}\operatorname{Log} y + \frac{\mathscr{H}_r'(\omega)}{\zeta(\omega)} - \frac{\mathscr{H}_r(\omega)\zeta'(\omega)}{\zeta(\omega)^2}. \tag{7.3}$$

We bound above $\mathscr{H}_r(\omega)$ by $\mathscr{H}_r(1) = r/\phi(r)$, and* $-\zeta'(\omega)/\zeta(\omega)$ by $1/(\omega-1)$ when y is large enough.

We still have to treat the derivative $\mathscr{H}_r'(\omega)$. We do so by looking at the logarithmic derivative

$$\frac{\mathscr{H}_r'(\omega)}{\mathscr{H}_r(\omega)} = -\sum_{p|r}\frac{\operatorname{Log} p}{p^\omega - 1}$$

whose absolute value is not more than $-\zeta'(\omega)/\zeta(\omega)$. As a consequence

$$|\mathfrak{R}(r,\omega,y)| \le \frac{r}{\phi(r)}((\omega-1)\operatorname{Log} y + 2).$$

We conclude by noticing that

$$\mathscr{H}_r(1/2) = \frac{r}{\phi(r)}\prod_{p|r}(1+p^{-1/2}) = \frac{r}{\phi(r)}\sum_{d|r}\mu^2(d)/\sqrt{d} \tag{7.4}$$

The lemma follows readily. □

7.2 Proof of the main theorem

We start with a lemma.

Lemma 7.2. *We have*

$$\sum_{n\ge1}\Bigg(\sum_{\substack{d|n,\\ d\le y}}\mu(d)\operatorname{Log}\frac{y}{d}\Bigg)^2/n^\omega \ll \frac{\operatorname{Log} y}{\omega-1}$$

as soon as $1<\omega\le 1+\frac12(\operatorname{Log} y)^{-1}$.

Proof. Let us denote by S the quantity to evaluate. We expand the square and find that

$$S = \sum_{d_1,d_2\le y}\frac{\mu(d_1)\operatorname{Log}\frac{y}{d_1}\mu(d_2)\ \operatorname{Log}\frac{y}{d_2}}{[d_1,d_2]^\omega}\zeta(\omega) = S_1\zeta(\omega)$$

and we are left with bounding S_1. We achieve that in two steps. First we use Selberg diagonalization process. We start by writing

$$S_1 = \sum_{d_1,d_2\le y}\frac{\mu(d_1)\operatorname{Log}\frac{y}{d_1}\ \mu(d_2)\operatorname{Log}\frac{y}{d_2}\ (d_1,d_2)^\omega}{d_1^\omega d_2^\omega}.$$

*See (Ford, 2000, Lemma 2.3). This very same estimate has been proved before, but I forgot the reference ...

We next define

$$f_\omega(r) = \prod_{p|r}(p^\omega - 1)$$

so that, when d is squarefree, we have $d^\omega = (f_\omega \star \mathbb{1})(d)$. From this we infer that

$$S_1 = \sum_{r \le y} \mu^2(r) \frac{f_\omega(r)}{r^{2\omega}} \mathfrak{M}_1(r, y/r, \omega)^2$$

by using the notations of Lemma 7.1. To use this Lemma, we bound $1/\zeta(\omega)$ by* $\mathcal{O}(\omega - 1)$ and this implies here that $\zeta(\omega)^{-1} \operatorname{Log}(y/r) \ll 1$. We thus get

$$|S_1| \ll \sum_{r \le y} \mu^2(r) \frac{f_\omega(r)}{r^{2\omega}} \Big(\frac{r}{\phi(r)} \sum_{d|r} \frac{\mu^2(d)}{\sqrt{d}} \Big)^2 \ll \sum_{r \le y} \frac{\mu^2(r)}{\phi(r)} \Big(\sum_{d|r} \frac{\mu^2(d)}{\sqrt{d}} \Big)^2$$

since, when r is squarefree,

$$\frac{f_\omega(r)}{r^{2\omega}} \Big(\frac{r}{\phi(r)} \Big)^2 = \prod_{p|r} \frac{p^\omega - 1}{p^{2\omega}} \frac{p^2}{(p-1)^2} \le \prod_{p|r} \frac{p-1}{p^2} \frac{p^2}{(p-1)^2} = \frac{1}{\phi(r)}.$$

By a routine calculation, we infer

$$\begin{aligned} |S_1| &\ll \sum_{r \le y} \frac{\mu^2(r)}{\phi(r)} \sum_{d_1, d_2 | r} \frac{\mu^2(d_1)\mu^2(d_2)}{\sqrt{d_1 d_2}} \\ &\ll \sum_{d_1, d_2 \le y} \frac{\mu^2(d_1)\mu^2(d_2)}{\sqrt{d_1 d_2} \phi([d_1, d_2])} \sum_{r \le y/[d_1 d_2]} \frac{\mu^2(r)}{\phi(r)} \ll \sum_{r \le y} \frac{\mu^2(r)}{\phi(r)} \ll \operatorname{Log} y \end{aligned}$$

since, by multiplicativity,

$$\begin{aligned} \sum_{d_1, d_2 \ge 1} \frac{\mu^2(d_1)\mu^2(d_2)}{\sqrt{d_1 d_2} \phi([d_1, d_2])} &= \prod_{p \ge 2} \sum_{\substack{\alpha, \beta \ge 0, \\ d_1 = p^\alpha, d_2 = p^\beta}} \frac{\mu^2(d_1)\mu^2(d_2)}{\sqrt{d_1 d_2} \phi([d_1, d_2])} \\ &= \prod_{p \ge 2} \Big(1 + \frac{2}{(p-1)\sqrt{p}} + \frac{1}{(p-1)p} \Big) \ll 1. \end{aligned}$$

We use $\zeta(\omega) \ll 1/(\omega - 1)$, and the Lemma follows readily □

Here is what we were aiming at.

*See (Bastien & Rogalski, 2002, Corollary 1) as well as (Ford, 2000, Lemma 2.3).

Proof of Theorem 6.1. We start with the decomposition (6.3). This leads us to[*]

$$(\mathrm{Log}\, z)^2 \sum_{n\le x}\Big(\sum_{d|n}\lambda_d^{(1)}\Big)^2/n \le 2\sum_{n\le x}\bigg(\sum_{\substack{d|n,\\ d\le z^2}}\mu(d)\,\mathrm{Log}\,\frac{z^2}{d}\bigg)^2/n + 2\sum_{n\le x}\bigg(\sum_{\substack{d|n,\\ d\le z}}\mu(d)\,\mathrm{Log}\,\frac{z}{d}\bigg)^2/n$$

so that the result will be implied by the corresponding one on each summand, i.e., when $x \ge \sqrt{y}$,

$$\sum_{n\le x}\Big(\sum_{\substack{d|n,\\ d\le y}}\mu(d)\,\mathrm{Log}\,\frac{y}{d}\Big)^2/n \ll (\mathrm{Log}\, x)(\mathrm{Log}\, y)$$

which we now establish. We notice that

$$\sum_{n\le x}\Big(\sum_{\substack{d|n,\\ d\le y}}\mu(d)\,\mathrm{Log}\,\frac{y}{d}\Big)^2/n \le e\sum_{n\le x}\Big(\sum_{\substack{d|n,\\ d\le y}}\mu(d)\,\mathrm{Log}\,\frac{y}{d}\Big)^2/n^{\omega}$$

with $\omega = 1 + (\mathrm{Log}\, x)^{-1}$. We can now drop the condition $n \le x$ and apply Lemma 7.2. The Theorem follows readily. □

7.3 An asymptotic formula

We have presented a proof as simple as possible, but it is important to know what is the best result one can expect. (Graham, 1978) also presents more complete asymptotic results. It is an easy task to modify[†] the proof we have presented of Lemma 7.1 to yield

$$\mathfrak{M}_1(r, y, \omega) = \mathfrak{R}(r, \omega, y) + \mathcal{O}\big(\mathscr{H}_r(1/2)\exp(-c\sqrt{\mathrm{Log}(y+2)})\big) \tag{7.5}$$

for some positive constant c and where $\mathfrak{R}$ is defined at equation (7.3) and $\mathscr{H}_r(1/2)$ is given by (7.4). To do so, select

$$T = \exp(\sqrt{\mathrm{Log}(z+2)}).$$

Note that $-\zeta'/\zeta^2(\omega)$ tends to 1 when ω tends to one, and this implies that

$$\mathfrak{R}(r, 1, y) = \mathscr{H}_r(1) = r/\phi(r).$$

[*]via $|a+b|^2 \le 2(|a|^2 + |b|^2)$

[†]Since this part is reserved for more advanced readers, we shall give much less details concerning the evaluations.

Chapter 8

An elementary proof of Barban & Vehov bound

We have introduced and studied the weights $\lambda_d^{(1)}$ in the preceding two chapters. We prove here a Lemma that is going to be a substitute for Lemma 7.1 by elementary means. The path taken is different from the one followed by (Graham, 1978) and bears similarities with (Granville & Ramaré, 1996, Lemme 10.2).

8.1 A general Theorem

The Theorem we now present is very efficient when the average mean of the non-negative multiplicative function g on the primes is close to constant. There are several version of it, the most precise one being due to (Halberstam & Richert, 1979). The one we give is a slight modification of the one proposed in (Tenenbaum, 1995) and comes from (Ramaré, 2009, Theorem 9.2).

The starting idea comes from the celebrated paper (Levin & Fainleib, 1967).

Theorem 8.1. *Let $D \geq 2$ be a fixed real parameter. Assume that g is a non-negative multiplicative function and that*

$$\sum_{\substack{p \geq 2, \nu \geq 1 \\ p^\nu \leq Q}} g(p^\nu) \operatorname{Log}(p^\nu) \leq KQ + K' \qquad (\forall Q \in [1, D])$$

for two constants $K, K' \geq 0$. Then

$$\sum_{d \leq D} g(d) \leq (K+1) \frac{D}{\operatorname{Log} D - K'} \sum_{d \leq D} g(d)/d$$

whenever $D > \exp K'$.

Proof. Let us define $G(D) = \sum_{d \le D} g(d)$ and $\tilde{G}(D) = \sum_{d \le D} g(d)/d$. On using $\operatorname{Log} \frac{D}{d} \le \frac{D}{d}$, we get

$$\begin{aligned} G(D) \operatorname{Log} D &= \sum_{d \le D} g(d) \operatorname{Log} \frac{D}{d} + \sum_{d \le D} g(d) \operatorname{Log} d \\ &\le D \sum_{d \le D} \frac{g(d)}{d} + \sum_{\substack{p \ge 2, \nu \ge 1 \\ p^\nu \le D}} g\big(p^\nu\big) \operatorname{Log}\big(p^\nu\big) \sum_{\substack{\ell \le D/p^\nu \\ (\ell,p)=1}} g(\ell). \end{aligned}$$

The second summand has been obtained by using

$$\operatorname{Log} d = \sum_{p^\nu \| d} \operatorname{Log}\big(p^\nu\big).$$

Finaly

$$\begin{aligned} \sum_{\substack{p \ge 2, \nu \ge 1 \\ p^\nu \le D}} g\big(p^\nu\big) \operatorname{Log}\big(rp^\nu\big) \sum_{\substack{\ell \le D/p^\nu \\ (\ell,p)=1}} g(\ell) &= \sum_{\ell \le D} g(\ell) \sum_{\substack{p \ge 2, \nu \ge 1 \\ p^\nu \le D/\ell \\ (p,\ell)=1}} g\big(p^\nu\big) \operatorname{Log}\big(p^\nu\big) \\ &\le \sum_{\ell \le D} g(\ell) K \frac{D}{\ell} \end{aligned}$$

and the Theorem follows readily. □

8.2 Preliminary Lemmas

Here are the Lemmas on which the method relies.

Lemma 8.1. *We have, for any $\eta \ge 0$,*

$$\sum_{p \le x} \frac{1}{p^{1+\eta}} = (1+\eta) \operatorname{Log} \frac{1 - 2^{-\eta}}{1 - x^{-\eta}} + \mathcal{O}(1).$$

Proof. A partial summation readily yields

$$\begin{aligned} \sum_{p \le x} \frac{1}{p^{1+\eta}} &= (1+\eta) \int_2^x \sum_{p \le x} 1 \frac{dt}{t^{2+\eta}} + \mathcal{O}(1/\operatorname{Log} x) \\ &= (1+\eta) \int_2^x \frac{dt}{t^{1+\eta} \operatorname{Log} t} + \mathcal{O}(1) = -(1+\eta) \int_{x^{-\eta}}^{2^{-\eta}} \frac{du}{\operatorname{Log} u} + \mathcal{O}(1) \end{aligned}$$

by* setting $u = t^{-\eta}$. We continue:

$$\begin{aligned} \sum_{p \le x} \frac{1}{p^{1+\eta}} &= -(1+\eta) \int_{x^{-\eta}}^{2^{-\eta}} \frac{du}{u-1} - (1+\eta) \int_{x^{-\eta}}^{2^{-\eta}} \Big(\frac{1}{\operatorname{Log} u} - \frac{1}{u-1} \Big) du + \mathcal{O}(1) \\ &= (1+\eta) \operatorname{Log} \frac{1 - 2^{-\eta}}{1 - x^{-\eta}} + \mathcal{O}(1) \end{aligned}$$

*We could dispense with the prime number Theorem as announced. □

Lemma 8.2. *Let $c \geq 0$ be a constant. When $c(\operatorname{Log} x)^{-1} \geq \varepsilon \geq 0$ and $r \geq 1$, we have*

$$\sum_{n\leq x}\sum_{\substack{d|n,\\(d,r)=1}}\mu(d)(n/d)^{\varepsilon} \ll \frac{x}{\operatorname{Log} x}.$$

Proof. The multiplicative function to study can also be written as

$$g_\varepsilon(n) = \prod_{\substack{p^\nu\|n,\\ p\nmid r}}(p^{\nu\varepsilon}-p^{(\nu-1)\varepsilon})\prod_{\substack{p^\nu\|n,\\ p|r}}p^{\nu\varepsilon} = n^\varepsilon\prod_{\substack{p^\nu\|n,\\ p\nmid r}}(1-p^{-\varepsilon}).$$

We use Theorem 8.1 with $g = g_\varepsilon$ and $D = x$. We have to bound, when $Q \leq x$,

$$\sum_{\substack{p^\nu\leq Q,\\ p|r}}p^{\nu\varepsilon}\operatorname{Log}(p^\nu) + \sum_{\substack{p^\nu\leq Q,\\ p\nmid r}}p^{\nu\varepsilon}(1-p^{-\varepsilon})\operatorname{Log}(p^\nu) \leq e^c\sum_{p^\nu\leq Q}\operatorname{Log}(p^\nu) \ll Q$$

as required. We get

$$\sum_{n\leq x}g_\varepsilon(n) \ll \frac{x}{\operatorname{Log} x}\sum_{n\leq x}g_\varepsilon(n)/n \ll \frac{x}{\operatorname{Log} x}\sum_{n\subset P(x)}\frac{g_\varepsilon(n)}{n}\Big(\frac{x}{n}\Big)^{\varepsilon'+\varepsilon}$$

where $n \subset P(x)$ means that* every prime factor of n is mot more than x and where

$$\varepsilon' = 1/\operatorname{Log} x. \tag{8.1}$$

We infer from the above that

$$\begin{aligned}\sum_{n\leq x}g_\varepsilon(n) &\ll \frac{x^{1+\varepsilon+\varepsilon'}}{\operatorname{Log} x}\prod_{\substack{p\leq x,\\ p\nmid r}}\Big(1+\sum_{\nu\geq1}\frac{(1+\nu(1-p^{-\varepsilon}))}{p^{\nu(1+\varepsilon')}}\Big)\prod_{p|r}\Big(1+\sum_{\nu\geq1}\frac{1}{p^{\nu(1+\varepsilon')}}\Big)\\ &\ll \frac{x^{1+\varepsilon+\varepsilon'}}{\operatorname{Log} x}\prod_{p\leq x}\Big(1+\frac{1-p^{-\varepsilon}}{p^{1+\varepsilon'}}+\frac{1}{1-p^{-1}}-1-\frac{1}{p}\Big)\\ &\ll \frac{x^{1+\varepsilon+\varepsilon'}}{\operatorname{Log} x}\prod_{p\leq x}\Big(1+\frac{1-p^{-\varepsilon}}{p^{1+\varepsilon'}}+\frac{2}{p^2}\Big)\end{aligned}$$

since

$$\frac{1}{1-x}-1-x = \frac{x^2}{1-x} \leq 2x^2$$

when $x \leq 1/2$. We employ Lemma 8.1. In our case, we have $\eta = \varepsilon'$ or $\eta = \varepsilon' + \varepsilon$, and thus $1 - x^{-\eta}$ is a constant that is > 0 and < 1. As a consequence

$$\sum_{p\leq x}\frac{1}{p^{1+\eta}} = -\operatorname{Log}\eta + \mathcal{O}(1) = \operatorname{Log}\operatorname{Log} x + \mathcal{O}(1). \tag{8.2}$$

The Lemma follows readily. □

*We have omitted the coprimality to r condition.

Lemma 8.3. *Let $c \geq 0$ be a constant. When $c(\operatorname{Log} x)^{-1} \geq \varepsilon \geq 0$ and $r \geq 1$, we have*

$$\sum_{n\le x}\sum_{\substack{d|n,\\(d,r)=1}}\mu(d)(n/d)^\varepsilon\tau(n/d)\ll\frac{r}{\phi(r)}x.$$

Proof. We proceed as in the preceding proof. The multiplicative function we want to sum reads

$$g_\varepsilon(n)=\prod_{\substack{p^\nu\|n,\\p\nmid r}}(p^{\nu\varepsilon}(\nu+1)-p^{(\nu-1)\varepsilon}\nu)\prod_{\substack{p^\nu\|n,\\p|r}}p^{\nu\varepsilon}(\nu+1)=n^\varepsilon\prod_{\substack{p^\nu\|n,\\p\nmid r}}(1+\nu(1-p^{-\varepsilon}))\prod_{\substack{p^\nu\|n,\\p|r}}(1+\nu).$$

We use Theorem 8.1 with $g=g_\varepsilon$ and $D=x$. We have to bound, when $Q\le x$,

$$\sum_{\substack{p^\nu\le Q,\\p|r}}p^{\nu\varepsilon}(1+\nu)\operatorname{Log}(p^\nu)+\sum_{\substack{p^\nu\le Q,\\p\nmid r}}p^{\nu\varepsilon}(1+\nu(1-p^{-\varepsilon}))\operatorname{Log}(p^\nu)\le e^c\sum_{p^\nu\le Q}(1+\nu)\operatorname{Log}(p^\nu)\ll Q$$

as required. We thus get

$$\sum_{n\le x}g_\varepsilon(n)\ll\frac{x}{\operatorname{Log}x}\sum_{n\le x}g_\varepsilon(n)/n\ll\frac{x}{\operatorname{Log}x}\sum_{n\subset P(x)}\frac{g_\varepsilon(n)}{n}\Big(\frac{x}{n}\Big)^{\varepsilon+\varepsilon'}$$

where ε' is defined in (8.1). We thus get

$$\sum_{n\le x}g_\varepsilon(n)\ll\frac{x^{1+\varepsilon+\varepsilon'}}{\operatorname{Log}x}\prod_{\substack{p\le x,\\p\nmid r}}\Big(1+\sum_{\nu\ge1}\frac{(1+\nu(1-p^{-\varepsilon}))}{p^{\nu(1+\varepsilon')}}\Big)\prod_{\substack{p\le x,\\p|r}}\Big(1+\sum_{\nu\ge1}\frac{(1+\nu)}{p^{\nu(1+\varepsilon')}}\Big).$$

Note that

$$\frac{\sum_{\nu\ge0}(1+\nu)p^{-\nu(1+\varepsilon')}}{\sum_{\nu\ge0}(1+\nu(1-p^{-\varepsilon}))p^{-\nu(1+\varepsilon')}}\le\frac{\sum_{\nu\ge0}(1+\nu)p^{-\nu(1+\varepsilon')}}{\sum_{\nu\ge0}p^{-\nu(1+\varepsilon')}}=\sum_{\nu\ge0}\frac{1}{p^{\nu(1+\varepsilon')}}\le\frac{p}{p-1}$$

so that

$$\begin{aligned}\sum_{n\le x}g_\varepsilon(n)&\ll\frac{r}{\phi(r)}\frac{x^{1+\varepsilon+\varepsilon'}}{\operatorname{Log}x}\prod_{p\le x}\Big(1+\frac{(2-p^{-\varepsilon})}{p^{1+\varepsilon'}}+\Big(\frac{1}{1-p^{-1}}\Big)^2-1-\frac{1}{p}\Big)\\&\ll\frac{r}{\phi(r)}\frac{x^{1+\varepsilon+\varepsilon'}}{\operatorname{Log}x}\prod_{p\le x}\Big(1+\frac{(2-p^{-\varepsilon})}{p^{1+\varepsilon'}}+\frac{4}{p^2}\Big)\end{aligned}$$

since

$$\Big(\frac{1}{1-x}\Big)^2-1-x=\frac{x^2}{(1-x)^2}\le4x^2$$

when $x\le1/2$. We invoke again Lemma 8.1, and in fact, equation (8.2) is enough to end the proof. □

Lemma 8.4. *When $\varepsilon \geq 0$, we have*

$$\sum_{h\leq H} h^\varepsilon = \frac{H^{1+\varepsilon}}{1+\varepsilon} + \mathcal{O}^*\bigl(H^\varepsilon\bigr)$$

Proof. Indeed, a summation by parts gves us directly

$$\sum_{h\leq H} h^\varepsilon = \sum_{h\leq H} \varepsilon\int_0^h dt/t^{1-\varepsilon} = \varepsilon\int_0^h \sum_{t<h\leq H} 1\,dt/t^{1-\varepsilon}$$
$$= \varepsilon\int_0^h (H-t)\,dt/t^{1-\varepsilon} + \mathcal{O}^*(H^\varepsilon).$$

□

Let us recall the following classical Lemma.

Lemma 8.5. *We have*

$$\sum_{\ell\leq L} 1/\ell = \operatorname{Log} L + \gamma + \mathcal{O}^*\Bigl(\frac{7}{12L}\Bigr)$$

Lemma 8.6. *When $\varepsilon \geq 0$ and $r \geq 1$, we have*

$$\sum_{m\leq x} m^\varepsilon \tau(m) = \frac{x^{1+\varepsilon}}{1+\varepsilon}\Bigl(\operatorname{Log} x + 2\gamma - \frac{1}{1+\varepsilon}\Bigr) + \mathcal{O}^*\bigl(6x^{\frac12+\varepsilon}\bigr).$$

Proof. Let us denote by S the sum to evaluate. We use Dirichlet hyperbola principle to write

$$\begin{aligned}
S &= 2\sum_{\ell\leq\sqrt{x}} \ell^\varepsilon \sum_{h\leq x/\ell} h^\varepsilon - \Bigl(\sum_{h\leq\sqrt{x}} h^\varepsilon\Bigr)^2\\
&= 2\frac{x^{1+\varepsilon}}{1+\varepsilon}\sum_{\ell\leq\sqrt{x}}\frac{1}{\ell} + \mathcal{O}^*(x^\varepsilon\sqrt{x}) - \frac{x^{1+\varepsilon}}{(1+\varepsilon)^2} + \mathcal{O}^*(x^\varepsilon(2\sqrt{x}+1))\\
&= 2\frac{x^{1+\varepsilon}}{1+\varepsilon}\sum_{\ell\leq\sqrt{x}}\frac{1}{\ell} - \frac{x^{1+\varepsilon}}{(1+\varepsilon)^2} + \mathcal{O}^*\bigl(4x^{\frac12+\varepsilon}\bigr)\\
&= \frac{x^{1+\varepsilon}}{1+\varepsilon}(\operatorname{Log} x + 2\gamma) - \frac{x^{1+\varepsilon}}{(1+\varepsilon)^2} + \mathcal{O}^*\bigl(6x^{\frac12+\varepsilon}\bigr)
\end{aligned}$$

with the help of Lemma 8.5. □

8.3 The method

We define here

$$\mathfrak{M}_k(r,y,\omega) = \sum_{\substack{d\leq y,\\(d,r)=1}} \frac{\mu(d)\operatorname{Log}^k(y/d)}{d^\omega}. \tag{8.3}$$

Here is the main Lemma :

Lemma 8.7. *Let $c \geq 0$ be a constant. When $c(\operatorname{Log} x)^{-1} \geq \varepsilon \geq 0$ and $r \geq 1$, we have*

$$|\mathfrak{M}_0(r, x, \omega)| \ll_c 1, \quad |\mathfrak{M}_1(r, x, \omega)| \ll_c r/\phi(r).$$

Proof. It is easier to write $\omega = 1 + \varepsilon$ and to discuss in terms of ε. Let us start with $\mathfrak{M}_0(r, x, \omega)$. We start with

$$S_0 = \sum_{n \leq x} \sum_{\substack{d|n,\\ (d,r)=1}} \mu(d)(n/d)^{\varepsilon}$$

which is $\mathcal{O}(x/\operatorname{Log} x)$ by Lemma 8.2. Let us write this sum differently:

$$S_0 = \sum_{\substack{d \leq x,\\ (d,r)=1}} \mu(d) \sum_{m \leq x/d} m^{\varepsilon}$$

and we employ Lemma 8.4 to reach

$$S_0 = \frac{x^{1+\varepsilon}}{1+\varepsilon} \sum_{\substack{d \leq x,\\ (d,r)=1}} \frac{\mu(d)}{d^{1+\varepsilon}} + \mathcal{O}(x).$$

By conjugating both estimates, we indeed get

$$\mathfrak{M}_0(r, x, \omega) \ll 1.$$

Let us now consider $\mathfrak{M}_1(r, x, \omega)$. We start from

$$S_1 = \sum_{n \leq x} \sum_{\substack{d|n,\\ (d,r)=1}} \mu(d)(n/d)^{\varepsilon} \tau(n/d)$$

which is $\mathcal{O}(xr/\phi(r))$ by Lemma 8.3. Let us write this sum differently:

$$S_1 = \sum_{\substack{d \leq x,\\ (d,r)=1}} \mu(d) \sum_{m \leq x/d} m^{\varepsilon} \tau(m)$$

and we use Lemma 8.6 to reach

$$S_1 = \frac{x^{1+\varepsilon}}{1+\varepsilon} \sum_{\substack{d \leq x,\\ (d,r)=1}} \frac{\mu(d)}{d^{1+\varepsilon}} \Big(\operatorname{Log} \frac{x}{d} + 2\gamma - \frac{1}{1+\varepsilon} \Big) + \mathcal{O}^*\Big(18 x^{1+\varepsilon} \Big)$$

since $\sum_{d \leq x} 1/\sqrt{d} \leq 2\sqrt{x}$. So more work using the bound on of $\mathfrak{M}_0(r, x, \omega)$ leads to the claimed result. □

It is not difficult to get along these lines the following Lemma:

Lemma 8.8. *Let $c \geq 0$ be a constant. When $c(\operatorname{Log} x)^{-1} \geq \varepsilon \geq 0$ and $r \geq 1$, we have*

$$|\mathfrak{M}_k(r, x, \omega)| \ll_{c,k} \left(\frac{r}{\phi(r)}\right)^k (\operatorname{Log} x)^{k-1}.$$

Proof. Indeed, take instead of Lemma 8.3, we prove that

$$\sum_{n \leq x} \sum_{\substack{d|n,\\(d,r)=1}} \mu(d)(n/d)^\varepsilon \tau_{k+1}(n/d) \ll \left(\frac{r}{\phi(r)}\right)^k x (\operatorname{Log} x)^{k-1}.$$

We then continue as above. □

Here is a surprising elementary consequence of the estimate for $\mathfrak{M}_1(r, k, \omega)$.

Lemma 8.9. *We have*

$$\Big|\sum_{\substack{d \leq x,\\(d,r)=1}} \frac{\mu(d)}{d} - x^\varepsilon \sum_{\substack{d \leq x,\\(d,r)=1}} \frac{\mu(d)}{d^{1+\varepsilon}}\Big| \ll_c \varepsilon \frac{r}{\phi(r)}$$

as soon as $0 \leq \varepsilon \leq c(\operatorname{Log} x)^{-1}$.

Proof. It is enough to consider

$$\int_0^\varepsilon \sum_{\substack{d \leq x,\\(d,r)=1}} \frac{\mu(d) x^\eta}{d^{1+\eta}} \operatorname{Log}(x/d) d\eta \ll \varepsilon \frac{r}{\phi(r)}.$$

□

Chapter 9

More general weights

We have consider the initial weights $\lambda_d^{(1)}$ due to Barban & Vehov. As it turns out, some stronger weights may be required, and (Motohashi, 1978, Lemma 5) indeed provides us with such generalizations. Let us describe a particular case: We set

$$\lambda_d^{(2)} = \frac{\mu(d)}{2(\operatorname{Log} z)^2}\biggl(\Bigl(\operatorname{Log}\frac{z}{d}\Bigr)^2 \mathbb{1}_{d\le z} - 2\Bigl(\operatorname{Log}\frac{z^2}{d}\Bigr)^2 \mathbb{1}_{d\le z^2} + \Bigl(\operatorname{Log}\frac{z^3}{d}\Bigr)^2 \mathbb{1}_{d\le z^3}\biggr). \tag{9.1}$$

Incidentally, the reader will have recognized there a linear combination of the weights used by (Goldston *et al.*, 2009). This linear combination has the effect that

$$\lambda_d^{(2)} = \mu(d) \quad \text{when } d \le z \tag{9.2}$$

a property that is fundamental for our approach. These weights verify also:

Lemma 9.1. *We have, when $x \ge y^{1/3}$,*

$$\sum_{n\le x}\biggl(\sum_{\substack{d|n,\\ d\le y}} \mu(d)\Bigl(\operatorname{Log}\frac{y}{d}\Bigr)^2\biggr)^2 / n \ll \operatorname{Log} x (\operatorname{Log} y)^3.$$

And this implies:

Theorem 9.1. *We have, when $x \ge z$,*

$$\sum_{n\le x}\Bigl(\sum_{d|n}\lambda_d^{(2)}\Bigr)^2/n \ll \frac{\operatorname{Log} x}{\operatorname{Log} z}.$$

In fact, much more is true and Motohashi proves that

Theorem 9.2. *We have, when $x \ge z$,*

$$\sum_{n\le x}\tau(n)\Bigl(\sum_{d|n}\lambda_d^{(2)}\Bigr)^2/n \ll \Bigl(\frac{\operatorname{Log} x}{\operatorname{Log} z}\Bigr)^2.$$

Chapter 10

A proof of Hoheisel Theorem

We give a proof of Theorem 0.1. This proof contains many parameters and we give here their priority (see (10.14)):

- Z is a small power of x.
- z is a small power of x, much smaller than Z. We assume that $z^4 \leq x$. See (10.15).
- T is about the size of z though slightly smaller; in fact $z = R^2 T$, where R is small.
- R is a small power of x, much smaller than z or T.

10.1 Introduction of the Mellin transform

We express the function ψ defined by (3.3) in terms of its Mellin transform via (3.4). The quantity we are really interested in is $\psi(x) - \psi(x-h)$ for some $h \in [1, x/2]$. We shift the line of integration from $\Re s = \sigma_0$ to

$$s = \sigma_3 = 1 - c_1 / \operatorname{Log}(T+2) \tag{10.1}$$

where c_1 comes from Theorem 5.1. This yields

$$\psi(x) - \psi(x-h) - h = \frac{1}{2i\pi} \int_{\sigma_3 - iT}^{\sigma_3 + iT} \frac{-\zeta'(s)}{\zeta(s)} \frac{x^s - (x-h)^s}{s} ds + \mathcal{O}(xT^{-1} \operatorname{Log}^2 x). \tag{10.2}$$

This uses the kernel

$$\Delta(x, h; s) = \frac{x^s - (x-h)^s}{s} = \int_{x-h}^{x} u^{s-1} ds. \tag{10.3}$$

We have clearly $|\Delta(x, h; s)| \leq h(x-h)^{\Re s - 1}$ when $\Re s \leq 1$. Note however that the simple bound $|\Delta(x, h; s)| \leq 2x^{\Re s}/|s|$ is more efficient when $|s|$ is large.

10.2 Building the companions

We now consider

$$V_r(s) = \sum_{n\ge 2} c_r(n)\Big(\sum_{d|n}\lambda_d^{(1)}\Big)/n^s. \tag{10.4}$$

This function has the good idea to (almost) factor. Note that the summation can be restricted to integers $n > z$.

Theorem 10.1.

$$1 + V_r(s) = \zeta(s) M_r(s, \lambda_d^{(1)}) \tag{10.5}$$

where

$$M_r(s, \lambda_d^{(1)}) = \sum_{\substack{u|r,\\ d\le z^2}} \frac{u\mu(r/u)\lambda_d^{(1)}}{[u,d]^s}. \tag{10.6}$$

Here is the upper bound we shall need

$$|M_r(s, (\xi_d)_{d\le D})| \ll_{c'} \|\xi_d\|_\infty r^2 D^{1-\sigma}\operatorname{Log} D, \quad (0 \le \Re s = \sigma \le 1 + c'/\operatorname{Log} D). \tag{10.7}$$

Here $(\xi_d)_{d\le D}$ is any sequence of complex numbers bounded in absolute value by $\|\xi_d\|_\infty$ and whose support is included in $[1, D]$, where $D \ge 2$. The $\operatorname{Log} D$ can be omited when $\sigma \le 1/2$.

Proof. Let us start by recalling that

$$c_r(n) = \sum_{\substack{u|r,\\ u|n}} u\mu(r/u). \tag{10.8}$$

As a consequence

$$1 + V_r(s) = \sum_{\substack{u|r,\\ d\le z^2}} u\mu(r/u)\lambda_d^{(1)} \sum_{[u,d]|n} 1/n^s = \zeta(s) \sum_{\substack{u|r,\\ d\le z^2}} \frac{u\mu(r/u)\lambda_d^{(1)}}{[u,d]^s}$$

as announced. Let us prove (10.7). We write

$$|M_r(s, (\xi_d)_{d\le D})| \le \|\xi_d\|_\infty \sum_{\substack{u|r,\\ d\le D}} \frac{u}{[u,d]^\sigma} = \|\xi_d\|_\infty r^2 \sum_{d\le D} \frac{1}{d^\sigma} \ll_{c'} \|\xi_d\|_\infty r^2 D^{1-\sigma}\operatorname{Log} D$$

where $\sigma = \Re s$.* □

*When $|\sigma - 1| \le 1/\operatorname{Log} D$, the summation is $\ll \delta^{-\sigma}\sum_{d\le D/\delta} 1/n \ll \delta^{-\sigma}\operatorname{Log} D$. Otherwise, when $\sigma \ge 1 + (\operatorname{Log} D)^{-1}$, it is $\ll \delta^{-\sigma}\zeta(\sigma) \ll \delta^{-\sigma}\operatorname{Log} z$. Finally, when $\sigma \ge 1 - (\operatorname{Log} D)^{-1}$, a summation by parts is enough.

Here is a formal identity that gives us a decomposition of 1:

$$1 = V_r^2 + 2(1+V_r) - (1+V_r)^2.$$

There follows a decomposition of $-\zeta'/\zeta$ which we modify with the help of (10.5), until we reach

$$-\frac{\zeta'}{\zeta} = -\frac{\zeta'}{\zeta}V_r^2 - 2\zeta' M_r + \zeta'\zeta M_r^2. \tag{10.9}$$

The next step is encompassed in the next lemma.*

Lemma 10.1. *Assume that* $1 \le h \le x/2$. *We have*

$$\psi(x) - \psi(x-h) - h = \frac{-1}{2i\pi}\int_{\sigma_3 - iT}^{\sigma_3 + iT} \frac{\zeta'(s)}{\zeta(s)} V_r^2(s) \frac{x^s - (x-h)^s}{s} ds + \mathcal{O}(r^4 x T^{-2/3} \operatorname{Log} x + r^4 z^2 T^{1/3} h x^{-1/2}).$$

We could improve considerably the dependence in r and z in the error term but we prefer to stick to the simplest line of argument.

Proof. We start from (10.9). We shift the line of integration for the integral of the last two summands in $\Re s = 1/2$. The horizontal integrals that occur ar

$$\ll r^4 T^{1/3} x \max_{1/2 \le \sigma \le 1} (z^4/x)^{1-\sigma} (\operatorname{Log} x)/T \ll r^4 T^{1/3} x (\operatorname{Log} x)/T$$

by Theorem 5.2. The initial error term $\mathcal{O}(xT^{-1}(\operatorname{Log} x)^2)$ gets incorporated by this one. On using this same result, we prove that the integrals on the segment $\Re s = 1/2$ are

$$\ll r^4 z^2 T^{1/3} h x^{-1/2}.$$

□

We now have at our disposal a *family* of representations of $\psi(x) - \psi(x-h)$, which reminds of the so called *ampliation* technique of Iwaniec; the same mechanisms are indeed used†. Here is our fundamental inequality

$$|\psi(x) - \psi(x-h) - h| \ll h \exp\Big(-\frac{c_1 \operatorname{Log} x}{\operatorname{Log}(T+2)}\Big) \operatorname{Log} T \int_{\sigma_3 - iT}^{\sigma_3 + iT} |V_r(s)|^2 |ds| + r^4 x T^{-2/3} \operatorname{Log} x + r^4 z^2 T^{1/3} h x^{-1/2}. \tag{10.10}$$

*This kind of decomposition comes from the theory of density estimates, see for instance... It has then been used with success in several places directly on prime numbers, by Linnik and many others.

†The word originally used by Iwaniec is indeed “ampliation” and not “amplification”, though subsequent authors have more used the second form. The global meaning is preserved with one or the other.

10.3 A good representation of $V_r(s)$ when $\Re s$ is close to 1

We need a representation of $V_r(s)$ in the form of an absolutely convergent series. This is the object of this paragraph.

We consider, when $\Re s \ge 1/2$,

$$V_r^\star(s;Z) = \frac{1}{2i\pi}\int_{1-i\infty}^{1+i\infty} \Gamma(w)V_r(s+w)Z^w dw. \tag{10.11}$$

We readily check that

$$V_r^\star(s;Z) = \sum_{n\ge 2} c_r(n)\Bigl(\sum_{d|n}\lambda_d^{(1)}\Bigr)e^{-n/Z}/n^s \tag{10.12}$$

by appealing to Lemma 3.3. On shifting the line of integration to $\Re w = -\Re s$, we get

$$V_r^\star(s;Z) = V_r(s) + \Gamma(1-s)Z^{1-s}M_r(1,\lambda_d^{(1)}) + \mathcal{O}(r^2 z^2 T^{1/2} Z^{-\Re s}\operatorname{Log} T). \tag{10.13}$$

Let us formalise this result in a Lemma.

Lemma 10.2.

$$V_r(s) = V_r^\star(s;Z) - \Gamma(1-s)Z^{1-s}M_r(1,\lambda_d^{(1)}) + \mathcal{O}(r^2 z^2 T^{1/2} Z^{-\Re s}\operatorname{Log} T)$$

when $1/2 \le \Re s \le 1$.

Proof. We have to bound

$$\int_{-\Re s - i\infty}^{-\Re s + i\infty} |\Gamma(w)\zeta(s+w)Z^w| dw.$$

We split the integral at $\Im w = \pm t$ where $t = |\Im s|$. When $|\Im w| \le T$, we use $|\zeta(s+w)| \ll \sqrt{1+t}\operatorname{Log} T$, while we use $|\zeta(s+w)| \ll \sqrt{1+|\Im w|}\operatorname{Log}(1+|\Im w|)$. □

We will have to bound the absolute value of $M_r(1,\lambda_d^{(1)})$. First note that*

$$\begin{aligned} M_r(1,\lambda_d^{(1)}) &= \sum_{\substack{u|r,\\ d\le z^2}} \frac{(u,d)\mu(r/u)\lambda_d^{(1)}}{d} \\ &= \sum_{\delta|r}\phi(\delta)\sum_{\delta|u|r}\mu(r/u)\sum_{\delta|d\le z^2}\frac{\lambda_d^{(1)}}{d} = \phi(r)\sum_{r|d\le z^2}\frac{\lambda_d^{(1)}}{d}. \end{aligned}$$

The decomposition (6.3) gives us

$$M_r(1,\lambda_d^{(1)}) = \frac{\phi(r)}{r}\,\frac{\mathfrak{M}_1(r,z^2/r,1) - \mathfrak{M}_1(r,z/r,1)}{\operatorname{Log} z}$$

*By first using $[u,d] = ud/(u,d)$ followed by $(u,d) = \sum_{\delta|u,\delta|d}\phi(\delta)$.

on recalling definition (8.3). We thus apply Lemma 7.1 to infer that

$$|M_r(1,\lambda_d^{(1)})| \ll \frac{1}{\operatorname{Log} z}\sum_{d|r}\frac{\mu^2(d)}{\sqrt{d}}.$$

We will also require the following bound:

$$\begin{aligned}\int_{\sigma_3-iT}^{\sigma_3+iT}|\Gamma(1-s)|^2|ds| &= \int_{\sigma_3-iT}^{\sigma_3+iT}|\Gamma(2-s)|^2\frac{dt}{|1-s|^2}\\ &= \frac{1}{1-\sigma_3}\int_{-T/(1-\sigma_3)}^{T/(1-\sigma_3)}|\Gamma(2-\sigma_3+iu(1-\sigma_3))|^2\frac{du}{1+u^2}\\ &\le \frac{\Gamma(2)}{1-\sigma_3}\int_{-\infty}^{\infty}\frac{du}{1+u^2} = \frac{\pi\operatorname{Log} T}{c_1}.\end{aligned}$$

10.4 Using the large sieve inequality

We have built a family of representations of $\psi(x)-\psi(x-h)-h$ and we are now going to use this family to benefit from an averaging affect. We only have to write

$$\begin{aligned}&\sum_{r\le R}\frac{\mu^2(r)}{\phi(r)}|\psi(x)-\psi(x-h)-h|\\ &\qquad\ll h\exp\Big(-\frac{c_1\operatorname{Log} x}{\operatorname{Log}(T+2)}\Big)\operatorname{Log} T\int_{\sigma_3-iT}^{\sigma_3+iT}\sum_{r\le R}\frac{\mu^2(r)}{\phi(r)}|V_r(s)|^2|ds|\\ &\qquad\qquad + R^4xT^{-2/3}\operatorname{Log} x + R^4z^2T^{1/3}hx^{-1/2}\end{aligned}$$

(σ_3 is defined in (10.1)). Lemma 10.2 provides us with a good representation of $V_r(s)$, which we introduce here

$$\begin{aligned}&|\psi(x)-\psi(x-h)-h|\\ &\qquad\ll h\exp\Big(-\frac{c_1\operatorname{Log} x}{\operatorname{Log}(T+2)}\Big)\frac{\operatorname{Log} T}{\operatorname{Log} R}\int_{\sigma_3-iT}^{\sigma_3+iT}\sum_{r\le R}\frac{\mu^2(r)}{\phi(r)}|V_r^\star(s;Z)|^2|ds|\\ &\quad + h\exp\Big(-\frac{c_1\operatorname{Log} x}{\operatorname{Log}(T+2)}\Big)\frac{\operatorname{Log} T}{\operatorname{Log} R} + R^4z^4TZ^{-3/2} + R^4xT^{-2/3}\operatorname{Log} x + R^4z^2T^{1/3}hx^{-1/2}\end{aligned}$$

provided Z be a power of T. Inded, this is used to bound $(\operatorname{Log} T)Z^{-2\sigma_3}$ by $Z^{-3/2}$. We use the large sieve inequality, in fact the hybrid version contained in Theorem 2.5. This gives us

$$\begin{aligned}\int_{\sigma_3-iT}^{\sigma_3+iT}\sum_{r\le R}\frac{\mu^2(r)}{\phi(r)}|V_r^\star(s;Z)|^2|ds| &\ll \sum_{n>z}\Big(\sum_{d|n}\lambda_d^{(1)}\Big)^2e^{-2n/Z}(n+R^2T)n^{-2\sigma_3}\\ &\ll \sum_{n>z}\Big(\sum_{d|n}\lambda_d^{(1)}\Big)^2e^{-2n/Z}(1+R^2Tz^{-1})n^{1-2\sigma_3}.\end{aligned}$$

This is at this level that we use the fact that each factor vanishes when $n \le z$: we need it to control the part containing the factor T. We continue with an integration by parts

$$\sum_{n>z} \frac{\left(\sum_{d|n} \lambda_d^{(1)}\right)^2}{n^{2\sigma_3-1}} e^{-2n/Z} = \sum_{n>z} \frac{\left(\sum_{d|n} \lambda_d^{(1)}\right)^2}{n^{2\sigma_3-1}} \int_n^\infty \frac{2}{Z} e^{-2t/Z} dt$$
$$= \frac{2}{Z} \int_z^\infty \sum_{z<n\le t} \frac{\left(\sum_{d|n} \lambda_d^{(1)}\right)^2}{n^{2\sigma_3-1}} e^{-2t/Z} dt.$$

We continue by simply noticing that, when $n \le t$, we have $n^{2\sigma_3-1} \ge t^{2\sigma_3-2} n$. This puts us in a position to apply Theorem 6.1:

$$\sum_{n>z} \frac{\left(\sum_{d|n} \lambda_d^{(1)}\right)^2}{n^{2\sigma_3-1}} e^{-2n/Z} \le \frac{2}{Z} \int_z^\infty \sum_{z<n\le t} \frac{\left(\sum_{d|n} \lambda_d^{(1)}\right)^2}{n} t^{2(1-\sigma_3)} e^{-2t/Z} dt$$
$$\ll \int_z^\infty \frac{\operatorname{Log} t}{\operatorname{Log} z} t^{2(1-\sigma_3)} e^{-2t/Z} dt/Z$$
$$\ll (Z/2)^{2(1-\sigma_3)} \int_{2z/Z}^\infty \frac{\operatorname{Log} u + \operatorname{Log} Z}{\operatorname{Log} z} u^{2(1-\sigma_3)} e^{-u} du$$

which is then

$$\ll (Z/2)^{2(1-\sigma_3)} \int_1^\infty \frac{\operatorname{Log} u}{\operatorname{Log} z} u^{2(1-\sigma_3)} e^{-u} du + \frac{\operatorname{Log} Z}{\operatorname{Log} z} (Z/2)^{2(1-\sigma_3)} \Gamma\Big(1 + \frac{2c}{\operatorname{Log} T}\Big)$$
$$\ll \exp\Big(\frac{2c_1 \operatorname{Log} Z}{\operatorname{Log}(T+2)}\Big).$$

All this leads to the bound

$$|\psi(x) - \psi(x-h) - h| \ll h \exp\Big(-c_1 \frac{(\operatorname{Log} x - 2\operatorname{Log} Z)}{\operatorname{Log}(T+2)}\Big) \frac{\operatorname{Log} T}{\operatorname{Log} R}$$
$$+ R^4 z^4 T Z^{-3/2} + R^4 x T^{-2/3} \operatorname{Log} x + R^4 z^2 T^{1/3} h x^{-1/2}$$

We then take

$$R = z^{1/100}, T = z^{49/50}, Z = z^{502/300} \tag{10.14}$$

so that the above inequality reduces to

$$|\psi(x) - \psi(x-h) - h|/h \ll \exp\Big(-\frac{50c_1}{49} \frac{\operatorname{Log} x}{\operatorname{Log} z}\Big) + h^{-1} + \frac{x}{h} z^{-46/75} (\operatorname{Log} x) + z^{71/30} x^{-1/2}.$$

We finally select

$$z = (2x/h)^2 \tag{10.15}$$

hence

$$|\psi(x) - \psi(x-h) - h|/h \ll \exp\Big(-\frac{25c_1}{49} \frac{\operatorname{Log} x}{\operatorname{Log}(2x/h)}\Big) + \frac{\operatorname{Log} x}{x/h}.$$

provided that $x/h \le x^{1/10}$. We conclude as follows: when $h = x^\theta$ with θ close enough to 1, but with $x/h \ge \operatorname{Log}^2 x$, the RHS is $\le 1/2$. The Theorem follows.

Chapter 11

Gallagher prime number Theorem

It is time for us to present the Gallagher prime number Theorem. We need some preparations to do so. Let us first recall the following classical result on zeros of L-functions.

Theorem 11.1. *There exists a positive constant c with the following property. For any $Q \geq 2$, any $T \geq 2$ and any primitive Dirichlet character χ modulo $q \leq Q$, the Dirichlet L-function attached to χ does not have any zero in the region (in $\sigma + it$)*

$$\sigma \geq 1 - \frac{c}{\operatorname{Log}(QT)}, \quad |t| \leq T$$

save maybe for single such L-function which is then attached to a real character. This function can have at most one real zero in this region.

This is the classical so called Landau-Page Lemma. Let us mention here, since it seems to be much less known that (Hinz, 1980/81) replaces the $\operatorname{Log}(QT)$ by $\operatorname{Log} Q + (\operatorname{Log} T)^{2/3}(\operatorname{Log}\operatorname{Log} T)^{1/3}$ provided T be ≥ 10. See also (Bartz, 1988). It is possible to work out an explicit value of c, see (McCurley, 1984), (Kadiri, 2002), (Kadiri, 2009) and (Liu & Wang, 2002). Most probably an admissible value should be around $1/8$ or even larger.

When χ is a Dirichlet character, we define

$$\psi(x, \chi) = \sum_{n \leq x} \Lambda(n)\chi(n), \vartheta(x, \chi) = \sum_{p \leq x} \operatorname{Log} p\, \chi(p). \tag{11.1}$$

If an exceptional (with respect to c) character exists, its corresponding zero is called an exceptional zero, or a Siegel zero. Given a primitive character χ to the modulus q, we consider

$$\tilde{\psi}(x, \chi) = \begin{cases} \psi(x) - x & \text{when } q = 1, \\ \psi(x, \chi) & \text{if } \chi \text{ is not exceptional and } q \geq 3, \\ \psi(x, \chi) + \frac{x^\rho}{\rho} & \text{if } \chi \text{ is not exceptional with exceptional zero } \rho. \end{cases}$$

Having these preparations at hand, we can state the main Theorem.

Theorem 11.2 (Gallagher prime number Theorem)**.** *There exists effective constants a_0, a_1 and a_2 such that*

$$\sum_{q\le Q}\sum_{\chi \text{ primitive modulo } q} |\tilde\psi(x,\chi)-\tilde\psi(x-h,\chi)| \le a_1\Delta h \exp\Big(-a_2\frac{\operatorname{Log} x}{\operatorname{Log} Q}\Big)$$

provided

$$Q^{a_0}\le x/Q\le h\le x,\quad \exp(\sqrt{\operatorname{Log} x})\le Q$$

and where $\Delta=1$ when no exceptional character occurs, and $\Delta=(1-\rho)\operatorname{Log} Q$ is there is one (with exceptional zero ρ).

The constants a_0, a_1 and a_2 depend on c from Theorem 11.1.

Let us show how to derive Linnik Theorem on the least prime in an arithmetic progression, namely:

Theorem 11.3. *There exists a constant L with the following property. Let q be a modulus, and a an invertible residue class modulo q. The least prime $\equiv a[q]$ is $\ll q^L$.*

Proof. We start by noticing that

$$\vartheta(x;q,a)=\sum_{\substack{p\le x,\\ n\equiv a[q]}}\operatorname{Log} p=\frac{1}{\phi(q)}\sum_{\chi \bmod q}\overline{\chi(a)}\vartheta(x;\chi)$$

where $\vartheta(x;\chi)$ is defined in (11.1). We have to go from characters to primitive characters. Assume χ is induced by the primitive character χ^* modulo $\mathfrak{f}$. Then we have

$$\vartheta(x;\chi^*)=\vartheta(x;\chi)+\sum_{\substack{p|q,\\ p\nmid\mathfrak{f}}}\chi(p)\operatorname{Log} p=\vartheta(x;\chi)+\mathcal{O}(\operatorname{Log} q)$$

and thus

$$\vartheta(x;q,a)=\frac{1}{\phi(q)}\sum_{\mathfrak{f}|q}\sum_{\substack{\chi^* \bmod q,\\ \chi^* \text{ primitive}}}\overline{\chi(a)}\vartheta(x;\chi^*)+\mathcal{O}(\operatorname{Log} q).$$

We go from ϑ to ψ by

$$\vartheta(x;\chi)=\psi(x;\chi)+\mathcal{O}(\psi(x)-\vartheta(x))=\psi(x;\chi)+\mathcal{O}(\sqrt{x})$$

so that

$$\vartheta(x;q,a)=\frac{1}{\phi(q)}\sum_{\mathfrak{f}|q}\sum_{\substack{\chi^* \bmod q,\\ \chi^* \text{ primitive}}}\overline{\chi(a)}\psi(x;\chi^*)+\mathcal{O}(\sqrt{x}+\operatorname{Log} q).$$

Once these easy things are established, the real proof can start. Assume first that no primitive character either modulo q or to any divisor $\mathfrak{f}$ of q is exceptional. Then

$$\vartheta(x;q,a)=\frac{x}{\phi(q)}+\frac{1}{\phi(q)}\sum_{\mathfrak{f}|q}\sum_{\substack{\chi^* \bmod q,\\ \chi^* \text{ primitive}}}\overline{\chi(a)}\tilde\psi(x;\chi^*)+\mathcal{O}(\sqrt{x}+\operatorname{Log} q). \quad (11.2)$$

$$=\frac{x}{\phi(q)}+\frac{1}{\phi(q)}\mathcal{O}\Big(x\exp\Big(-a_2\frac{\operatorname{Log} x}{\operatorname{Log} q}\Big)\Big)+\mathcal{O}(\sqrt{x}+\operatorname{Log} q) \quad (11.3)$$

on appealing to Theorem 11.2 with $h = x$, $Q = q$ and provided that

$$q^{a_0+1} \le x, \quad \sqrt{\operatorname{Log} x} \le \operatorname{Log} q.$$

The second error term is not more than the main term when $q^3 \le x$. As for the first error term, it is $\le \frac{1}{2}(x/\phi(q))$ when q is a small enough power of x. This proves the Theorem in this case.

Assume now that one of the characters that appears in (11.4) is exceptional. Say χ_1 (which is real-valued) with exceptional $\rho = 1 - \delta$. We then get

$$\vartheta(x; q, a) = \frac{x}{\phi(q)} - \frac{\chi_1(a)x^{1-\delta}}{\phi(q)(1-\delta)} + \frac{1}{\phi(q)} \sum_{\mathfrak{f}|q} \sum_{\substack{\chi^* \bmod q, \\ \chi^* \text{ primitive}}} \overline{\chi(a)}\tilde{\psi}(x; \chi^*) + \mathcal{O}(\sqrt{x} + \operatorname{Log} q) \quad (11.4)$$

$$= \frac{x}{\phi(q)} - \frac{\chi_1(a)x^{1-\delta}}{\phi(q)(1-\delta)} + \frac{1}{\phi(q)} \mathcal{O}\Big(\delta(\operatorname{Log} q)\, x \exp\Big(-a_2 \frac{\operatorname{Log} x}{\operatorname{Log} q}\Big)\Big) + \mathcal{O}(\sqrt{x} + \operatorname{Log} q)$$

under the conditions as above. Note that if $\chi_1(a) = -11$, we are in a even better position as before (the main term is almost doubled), but when $\chi_1(a) = 1$, some cancellation may happen. Let us investigate in which measure:

$$\frac{x}{\phi(q)} - \frac{x^{1-\delta}}{\phi(q)(1-\delta)} = \frac{x(1-x^\delta)}{\phi(q)} - \frac{\delta x^{1-\delta}}{\phi(q)(1-\delta)} \ge \frac{\delta x \operatorname{Log} x}{\phi(q)} - \frac{\delta x^{1-\delta}}{\phi(q)(1-\delta)}$$

$$\ge \frac{\delta x \operatorname{Log} x}{\phi(q)}\Big(1 - \frac{\delta}{x^\delta(1-\delta)}\Big) \ge \frac{\delta x \operatorname{Log} x}{\phi(q)} \frac{1-2\delta}{1-\delta} \ge \frac{\delta x \operatorname{Log} x}{2\phi(q)}$$

since δ is small. Since $\delta \operatorname{Log} x \ge \delta \operatorname{Log} q$, we can conclude as before. □

Chapter 12

A geometrical approach to the sieve

Though the notion of sieve dates back from Erathostenes, and though it has been formalised somewhat by Legendre, it is really the norwegian mathematician Viggo Brun who made it into an effective tool in (Brun, 1919). Its approach is precursory of the modern *combinatorial* sieve. Atle Selberg has then introduced a another line of approach. A third stream was to be initiated at about the same time (1940-1945) by Yu Linnik, but it is only aroung 1970 that the link between Linnik's idea and sieve theory became clear.*

12.1 The sieve problem

Let us start by detecting prime numbers among the integers from $[1, N]$ by a "sieve" process. We use this term with caution for it covers many different processes that differ noticeably, though they have many points in common. Here what we mean by "sieving" is the following: we know that an integer n is a prime if and only if it does not any prime factors its squareroot. Subsequently every prime number $> \sqrt{N}$ is congruent to some invertible element modulo every $q \leq \sqrt{N}$. It is this very property that we want to exploit. Note that the prime numbers below $\sqrt{N}$ are in negligible quantity.

We start more precisely with two objects:

[a] A finite *host sequence* $\mathcal{A}$, for instance the series of integers between $M+1$ et $M+N$.

[b] For every $q \leq Q$, a subset $\mathcal{K}_q \subset \mathbb{Z}/q\mathbb{Z}$.

We of course assume that the sequence $(\mathcal{K}_q)_q$ is *consistant*, by which we mean that $\sigma_{q\to d}(\mathcal{K}_q) = \mathcal{K}_d$ as soon as $d|q$ where $\sigma_{q\to d}$ denotes the canonical projection from $\mathbb{Z}/q\mathbb{Z}$ onto $\mathbb{Z}/d\mathbb{Z}$. More importantly, we assume this sequence to be *multiplicatively split*, i.e. that $\mathcal{K}_{d_1 d_2} \simeq \mathcal{K}_{d_1} \times \mathcal{K}_{d_2}$ whenever $(d_1, d_2) = 1$, via the chinese isomorphism. This is easily seen to be equivalent to the giving, for each prime number p, of a consistant sequence

*Or at least, clearer...

$(\mathcal{K}_{p^\nu})_\nu$ where $\mathcal{K}_{p^\nu} \subset \mathbb{Z}/p^\nu\mathbb{Z}$. Let us remark here that when $\mathcal{B}$ denotes un set of classes modulo d, we use the same symbol $\mathcal{B}$ to denote the set of the integers $\in \mathbb{N}$ that belong to these classes or even for those integers modulo q that are such that $\sigma_{q\rightarrow d}(x)$ falls in $\mathcal{B}$, when $d|q$. Let me add here that the sequence $(\mathcal{K}_q)_q$ indeed defines a compact set $\mathcal{K}$ in $\hat{\mathbb{Z}} = \varprojlim \mathbb{Z}/n\mathbb{Z}$, which explains our terminology.

The object we want to investigate is

$$\mathcal{A}(\mathcal{K}) = \{n \in \mathcal{A} \quad / \quad \forall q \leq Q, \quad n \in \mathcal{K}_q\}. \tag{12.1}$$

One of our main problem is to build a function majorising its characteristic function, or, more simply, to get an upper bound for its cardinality.

This is what we call the sieve problem. This way of doing differs from the points of view taken in (Bombieri, 1987/1974b)or in the lectures on sieves from (Selberg, 1991) on several aspects:

(1) The usual approach considers the classes one removes modulo q and not the ones that are kept. This induces a lack of regularity in the expressions that one has to manipulate. The reader will however find traces of our way of doing in (Bombieri & Davenport, 1968) where the authors prove the Brun-Titchmarsh Theorem in an astonishing fashion.

(2) We consider what happens modulo p, but also what happens modulo $p^2, p^3, \ldots$ and this induces difficulties. (Gallagher, 1974) gives a partial answer on how to deal with this problem when the host sequence is an interval and (Selberg, 1976) a more complete answer, but the complexity of the resulting expressions forces him again to restrict his attention to intervals, so as to be able to use the large sieve inequality to control the error term. This exposition is also somewhat lengthy.

(3) We usually introduce an auxiliary polynomial P and consider the congruence $P(n) \equiv 0[q]$. This is not required here, which renders this approach more "natural"

12.2 Some examples of compacts sets

Here are some examples of compacts sets.

Example 1. First of all, the sequence $(\mathbb{Z}/q\mathbb{Z})_q$ is a compact set. It is even the largest one. grand. —

Example 2. Let us denote by $\mathcal{U}_q$ the set of invertible elements in $\mathbb{Z}/q\mathbb{Z}$; this set is also the *multiplicative group of* $\mathbb{Z}/q\mathbb{Z}$. Then $(\mathcal{U}_q)_{q\in\mathcal{Q}}$ is a compact set, and if we take for $\mathcal{Q}$ the set of all the integers $q \leq \sqrt{N}$, then every prime number between $\sqrt{N}$ and N falls in $\mathcal{U}_q$ modulo q. —

Example 3. Consider the compact set $(\mathcal{U}_q \cap (\mathcal{U}_q - 2))$. On taking $\mathcal{Q}$ as above (i.e. the set of every integers $q \leq \sqrt{N}$), every prime number p such that $p + 2$ is again a prime number falls in $\mathcal{U}_q \cap (\mathcal{U}_q - 2)$ modulo q. —

Example 4. Let us now take, for every integer q, $\mathcal{K}_q$ to be the set of squares modulo q. We readily check that $(\mathcal{K}_q)$ is again a compact set. More gerenally, we can take the set of values taken modulo q by a given polynomial with integer coefficients. —

Example 5. When x lies in $\mathbb{Z}/q\mathbb{Z}$, we have the notion of gcd of x and q at our disposal, and this is simply the gcd of y and q for any y congruent to x modulo q. This definition is easily checked to be independant of the choice of y, so we denote by (x, q) this gcd. We take $\mathcal{K}_q$to be the set of classes modulo q which gcd with q is squarefree. Then $(\mathcal{K}_q)$ is a compact set. Remarkably enough*, for every q, a squarefree integer always falls modulo q inside $\mathcal{K}_q$. —

The compact sets in these five examples (The first three were $(\mathbb{Z}/q\mathbb{Z})$, $(\mathcal{U}_q)$ et $(\mathcal{U}_q \cap (\mathcal{U}_q - 2)))$ are multiplicatively split.

The squarefree case

We shall say that we are in the squarefree case whenever $\mathcal{K}_q = \sigma^{-1}_{q\to d}(\mathcal{K}_d)$ as soon as $d|q$ and they have the same prime factors. Consequently, $\mathcal{K}_{p^\nu}$ contains exactly the same information as does $\mathcal{K}_p$: when we know that x lies in $\mathcal{K}_p$, then this element is automatically in every $\mathcal{K}_{p^\nu}$ for $\nu \geq 1$.

Historically speaking, the first sieve that appeared where squarefree. This condition appears naturally in the framework of the combinatorial sieve and simplifies the usual exposition of the Selberg sieve. The first sieve that were *not* squarefree are found in (Gallagher, 1974) closely followed by (Selberg, 1976). (Motohashi, 1983) adresses also this problem.

*The remarkable fact here lies in that, given a squarefree n, it falls inside $\mathcal{K}_q$ for q. In the case of prime numbers, for instance, one has to add the condition $q < n$.

Chapter 13

Local models

13.1 Local models

Let M be a multiplicatively large integer, say

$$M = \mathrm{lcm}(d, d \le N+2).$$

Let us investigate more closely the compact set

$$\mathcal{K}_M = \mathcal{U}_M \bigcap (\mathcal{U}_M - 2)$$

An integer that verifies $\sqrt{N+2} < n \le N$ is prime, and such that $n+2$ is also prime if and only if n modulo M falls in $\mathcal{K}_M$. This set is however extremely large since* $M = e^{(1+o(1))N}$ (which is to be compared with the length of the interval that n ranges, i.e. $(1+o(1))N$).

Let us expand its characteritic function in Fourier series:

$$\begin{aligned}
1\!\!1_{\mathcal{K}_M}(n) &= \sum_{b \bmod M} \biggl(\frac{1}{M}\sum_{c\in\mathcal{K}_M} e(-bc/M)\biggr) e(bn/M) \\
&= \sum_{d|M}\sum_{a \bmod^* d} \biggl(\frac{1}{M}\sum_{c\in\mathcal{K}_M} e(-ac/d)\biggr) e(an/d) \\
&= \sum_{d|M}\sum_{a \bmod^* d} \biggl(\frac{|\mathcal{K}_M|}{M|\mathcal{K}_d|}\sum_{c\in\mathcal{K}_d} e(-ac/d)\biggr) e(an/d).
\end{aligned}$$

†We define also

$$\psi_d^*(n) = \sum_{a \bmod^* d} \biggl(\frac{1}{|\mathcal{K}_d|}\sum_{c\in\mathcal{K}_d} e(-ac/d)\biggr) e(an/d). \tag{13.1}$$

---

*In fact $\mathrm{Log}\, M = \sum_{n\le N+2}\Lambda(n) = \psi(N+2)$, and this is the definition of the ψ-function that Riemann was using! Notation ψ here has nothing to do with (13.1) or (3.3).

†We have used the Johnsen-Gallagher condition, see (Gallagher, 1974), (Selberg, 1976), (Ramaré & Ruzsa, 2001) and (Ramaré, 2009).

Notice here that

$$\psi_q(n) = \sum_{d|q} \psi_d^*(n) = \frac{q}{|\mathcal{K}_q|} 1\!\!1_{\mathcal{K}_q}(n) \tag{13.2}$$

and this is what we shall call our *local model.* * We can rewrite our Fourier decomposition in a simpler manner:

$$1\!\!1_{\mathcal{K}_M}(n) = \frac{|\mathcal{K}_M|}{M} \sum_{d|M} \psi_d^*(n). \tag{13.3}$$

What can be done with such a writing? Since summation in d is too long, let us shorten it and write

$$\frac{|\mathcal{K}_M|}{M} \sum_{d\le D} \psi_d^*(n).$$

This is also the best L^2-approximation on the space of functions "that depend only their argument modulo d for all $d \le D$". We meet here one of the major difficulty of the theory: We have to deal with this space of functions; it has a good algebraical structure but where many geometrical concept have lost their meanings. Indeed, what is a point here? Furthermore an additionnal difficulty comes from the fact that we project on a space that is way smaller than the initial one.

Let us still try to continue. We first note that the function $\sum_{d\le D} \psi_d^*$ is constant over $\mathcal{K}_M$, since each ψ_d shares this property. More precisely we can invert (13.2) in

$$\psi_d^*(n) = \sum_{q|d} \mu(d/q)\psi_q(n) \tag{13.4}$$

which is thus equal to

$$h(d) = \sum_{q|d} \mu(d/q)\frac{|q|}{|\mathcal{K}_q|} = \prod_{p^\nu \| d} \Big(\frac{p^\nu}{|\mathcal{K}_{p^\nu}|} - \frac{p^{\nu-1}}{|\mathcal{K}_{p^{\nu-1}}|} \Big) \ge 0. \tag{13.5}$$

We define

$$G(D) = \sum_{d\le D} h(d). \tag{13.6}$$

The function $\sum_{d\le D} \psi_d^*/G(D)$ equals, over $\mathcal{K}_M$, to the characteristic function of $\mathcal{K}_M$†. We do not have any control of its value outside this set. We use here to the same technique that Selberg used, i.e. we consider

$$\beta(n) = \Big| \sum_{d\le D} \psi_d^*(n)/G(D) \Big|^2. \tag{13.7}$$

*See also (Kobayashi, 1973), (Jutila, 1977b), (Jutila, 1977a) and (Kowalski & Michel, 2002), as well as (Motohashi, 1978) and (Bombieri, 1987/1974b).

†This only means it takes the constant value 1 on this set!

This function is an upper bound of the characteristic function of $\mathcal{K}_M$. It turns out that this is the same upper boud that arises from the Selberg sieve ![*] This approach is the one developped in (Ramaré, 2009).

When we want to sieve the prime numbers, we check that (see (13.1) and (13.5))

$$\psi_d^*(n) = \mu(d)c_d(n)/\phi(d) \quad \text{and} \quad h(d) = \mu^2(d)/\phi(d).$$

The function G from (13.6) is indeed the one that arose in (2.3).

Let us conclude this general presentation by a lemma.

Lemma 13.1. *On writing*

$$\psi_d^*(n) = \sum_{a \bmod^* d} \hat{\psi}(a/d)e(na/d), \tag{13.8}$$

we have

$$\sum_{a \bmod^* d} |\hat{\psi}(a/d)|^2 = h(d).$$

Proof. Indeed, we have

$$\begin{aligned}
\sum_{a \bmod^* d} |\hat{\psi}(a/d)|^2 &= \sum_{a \bmod^* d} \left| \frac{1}{|\mathcal{K}_d|} \sum_{c \in \mathcal{K}_d} e(-ac/d) \right|^2 \\
&= \frac{1}{|\mathcal{K}_d|^2} \sum_{c,c' \in \mathcal{K}_d} \sum_{a \bmod^* q} e(-a(c-c')/d) \\
&= \frac{1}{|\mathcal{K}_d|^2} \sum_{c,c' \in \mathcal{K}_d} \sum_{\substack{q|d,\\ q|c-c'}} q\mu(d/q) = \frac{1}{|\mathcal{K}_d|^2} q\mu(d/q) \sum_{\substack{c,c' \in \mathcal{K}_d,\\ c \equiv c'[q]}} 1 \\
&= \frac{1}{|\mathcal{K}_d|^2} \sum_{q|d} q\mu(d/q) \sum_{b \in \mathcal{K}_q} \Bigl(\sum_{\substack{c \in \mathcal{K}_d,\\ c \equiv b[q]}} 1 \Bigr)^2 = \frac{1}{|\mathcal{K}_d|^2} \sum_{q|d} q\mu(d/q) \sum_{b \in \mathcal{K}_q} \frac{|\mathcal{K}_d|^2}{|\mathcal{K}_q|^2} = h(d)
\end{aligned}$$

by using (10.8) for the Ramanujan sums. □

13.2 Two inequalities involving local models

Here is the generalisation of (2.2) in this context.

Theorem 13.1. *For any $M \in \mathbb{R}$, we have*

$$\sum_{r \le R} h(r)^{-1} \Bigl| \sum_n u_n \psi_r^*(n+M) \Bigr|^2 dt \le \sum_n |u_n|^2 (N - 1 + R^2).$$

[*]The reader will find elements of this argument in the papers that show that the Selberg sieve is "dual" to the large sieve (in a vague sense), and for instance in (Kobayashi, 1973).

We should maybe add that the summation runs only over those r for which $h(r) \neq 0$. Indeed, using (13.8) together with (13.1), we see that $\psi_r^*(n) = 0$ when $h(r) = 0$. Let us further add that this summation runs over squarefree integers when the implied compact set is so.

Proof. We start from (13.1) (see also (Ramaré, 2009, (11.33))) :

$$\psi_r^*(m) = \sum_{a \bmod^* r} \hat{\psi}(a/r) e(ma/r),$$

and keep Lemma 13.1 in mind. There comes

$$\begin{aligned}\Big|\sum_n u_n \psi_r^*(n+M)\Big|^2 &= \Big|\sum_{a \bmod^* q} \hat{\psi}(a/r) e(Ma/r) \sum_n u_n e(na/r)\Big|^2 \\ &\leq h(r) \sum_{a \bmod^* q} \Big|\sum_n u_n e(na/r)\Big|^2\end{aligned}$$

and we only have to use the large sieve inequality (Theorem 2.1). □

This Theroem can be applied in the same way we used Theorem 2.2 to get a sieve*.
Here is a hybrid version that we can get with the help of Theorem 2.5 :

Theorem 13.2. *For every $M \in \mathbb{R}$ and every $T \geq \pi$, we have*

$$\sum_{r \leq R} h(r)^{-1} \int_{-T}^{T} \Big|\sum_{n \leq N} u_n n^{it} \psi_r^*(n+M)\Big|^2 dt \leq 500 \sum_{n \leq N} |u_n|^2 (N + R^2 T).$$

*We recover here the bound given by Montgomery sievem with a presentation which is very similar, though simplified, to (Motohashi, 1983, Theorem 3).

Chapter 14

Sieving multiplicative functions

We have seen in the first section how the notion of local model led to sieve results. One of the fundamental equation is (13.1). We have then exploited the local model corresponding the prime numbers (i.e. up to some factor, the Ramanujan sum) and the main lemma is Lemm 10.1. This one relies on expression (10.8). The question now arises to know id and how we could generalize the method to every local models. As it turns out, equation (10.8) is extremely linked with the primes and has no counterpart at the level of general local models.*

Let us analyse our use of local models:

1. We require to have a function of the shape $\mathbb{1} \star \lambda$ to have some equivalent of (10.8). The lemma in question has its equivalent (Kowalski & Michel, 2002, Lemma 16). The latter relies on the multiplicativity of their notion of pseudo-characters.

2. We need a large sieve kind of inequality, like Theorem 13.1 where the special shape of the pseudo-characters we are interested in in this time (13.8). This is (Kowalski & Michel, 2002, Proposition 13) where the authors do not rely on some more powerful inequality, like (Duke & Kowalski, 2000, Theorem 4), but establish directly a kind of quasi-orthogonality for the functions they consider. These two proofs use some (modified) Rankin-Selberg convolution products. As a matter of fact (Motohashi, 1978, Lemma 2) also avoids a writing of the shape (13.8).

3. We need an equivalent of Theorem 6.1 ; But this does not depend on pseudo-characters but on Barban & Vehov's weights, and (Kowalski & Michel, 2002, Lemma 15) is the variant that is required.

So, how are we going to generalize our proof? We look at sums of the type $\sum_{p\le N} f(p)$ for a non-negative function f. We can extend it by multiplicativity. Motohashi uses such a generalization to show the Gallagher prime number Theorem from (Gallagher, 1970) (see also Theorem 11.2) in case there is an exceptional character. He builds his generalization

*Here is a note for the reqders that have also read(Ramaré, 2009). Generalization of (10.8) is (Ramaré, 2009, (11.14)). The condition on n is thus $n \in \mathcal{L}_\ell$, which reduces to a single divisibility condition modulo ℓ only in the case of $\mathcal{U}$.

of the pseudo-characters on a minimisation problem and we show in this chapter how to build these as local models, that is to say that we *do not* try to minimize the functional

$$\sum_{n\le N} f(n)\bigl(\sum_{d|n}\lambda_d\bigr)^2$$

but strive to approximate* $p\mapsto f(p)$ by a function that looks like it "locally". We will reach the same result.

14.1 The hypothesis on f and some consequences

We start with a non-negative multiplicative function f, and we follow closely (Motohashi, 1978) and (Motohashi, 1983) concerning the hypothesis that we make on this function. The reader should keep in the mind the example $f(n)=1$! We define

$$F_p=\sum_{\nu\ge 0} f(p^\nu)/p^\nu$$

and we of course assume that this series converges. Note that $F_p=p/(p-1)$ when $f=1$.

(C_1) f is a non-negative multiplicative function such that $f(n)\ll_\varepsilon n^\varepsilon$ for all $\varepsilon>0$.

(C_2) There exist two constants $A>0$ and $\alpha\ge 1$ such that $F_p-1\ge A/p^\alpha$ for every prime number p.

(C_3) There exist constants $\beta\ge 0$, $\gamma\in[0,1]$, $\mathscr{F}>0$ and $C'\ge 1$ such that, for every $y\ge 1$ and every (non-necessarily primitive) Dirichlet character χ modulo q, we have

$$\sum_{n\le y}\chi(n)f(n)=\delta_{\chi=\chi_0}\mathscr{F}K(q)y+\mathcal{O}^*(C'q^\beta y^\gamma)$$

where

$$K(q)=\prod_{p|q}F_p^{-1}. \tag{14.1}$$

Note that when $f=1$, we have $K(q)=\phi(q)/q$ since $F_p=p/(p-1)$.

A first consequence of these hypotheses is that

$$\sum_{\substack{n\le y,\\ n\equiv a[q]}} f(n)=\mathscr{F}\frac{K(q)}{\phi(q)}y+\mathcal{O}^*(C'q^\beta y^\gamma) \tag{14.2}$$

as soon as a is prime† to q. Some more work‡ gives us the following estimate, valid for squarefree q, d a divisor of q and any a prime to q :

$$\sum_{\substack{n\le y,\\ n\equiv ad[q]}} f(n)=\frac{\mathscr{F}y}{\phi(q/d)}\prod_{\substack{p|d,\\ p\nmid q}}F_p^{-1}\prod_{p|q}(1-F_p^{-1})(1+o(1)).$$

*We will only be able to majorize.

†Simply by writing $\mathbb{1}_{n\equiv a[q]}=\sum_{b \bmod^* q}\overline{\chi(a)}\chi(n)/\phi(q)$.

‡The reader will find most of it in (Motohashi, 1978) and (Motohashi, 1983).

We continue our definitions and set as in (Motohashi, 1983)*,

$$g(r) = \prod_{p|r}(F_p - 1)^{-1}, \tag{14.3}$$

so that we may write the preceding estimate in a somewhat more compact form

$$\sum_{\substack{n\le y,\\ n\equiv ad[q]}} f(n) = \mathscr{F} y \frac{K(d)}{g(d)\phi(q/d)}(1+o(1)). \tag{14.4}$$

Note that, since q is squarefree, we have here $\phi(q/d) = \phi(q)/\phi(d)$. In case $f = 1$, the function g is the Euler ϕ function.

14.2 The local scalar products

Let us start as in chapter 13 by selecting a multiplicatively large modulus M, say

$$M = \mathrm{lcm}(r, r \le R).$$

We look at the sequence $(f(n))_{n\le N}$ from which we want to extract the $(f(p))_{p\le N}$. When compared to what we presented in chapter 13, the present situation is a weighted counterpart. This seemingly minor modification has however fairly important consequences and forces us to look at was being done until now with more depth.

The scalar product on sequences

This scalar product is simply

$$(h_1, h_2) \mapsto (1/N)\sum_{n\le N} f(n)h_1(n)\overline{h_2(n)}.$$

The scalar product modulo q

We want to define a local† scalar product that would be the "natural" counterpart of the one on sequences. The initial candidate is

$$(h_1, h_2) \mapsto \sum_{a \bmod q}(1/N)\sum_{\substack{n\le N,\\ n\equiv a[q]}} f(n)h_1(a)\overline{h_2(a)}$$

where, this time, the functions h_1 are h_2 are being defined on $\mathbb{Z}/q\mathbb{Z}$. By (14.4), the following expression is a approximation (up to the factor $\mathscr{F}$) to the summation over n in the above, with $d = (a, q)$:

$$W(q; d) = K(q)/\big(g(d)\phi(q/d)\big). \tag{14.5}$$

*__Careful!!__ The function $g(r)$ of (Motohashi, 1978, bottom of page 167) is here $1/g(r)$!

†That is to say "modulo q".

Hence, we define

$$[h_1|h_2]_q = \sum_{d|q} W(q;d) \sum_{\substack{a \bmod q,\\ (a,q)=d}} h_1(a)\overline{h_2(a)} \tag{14.6}$$

and have now the quadratic space $(\mathcal{F}(\mathbb{Z}/q\mathbb{Z}), [\cdot|\cdot]_q)$ at our disposal. We of course check that

$$\sum_{d|q} W(q;d)\phi(q/d) = 1 \tag{14.7}$$

which says that the norm of the function $\mathbb{1}$ remains 1 on all these spaces. Notice furthermore that these weights are multiplicative i.e. that they verify

$$W(q;d) = \prod_{p^\nu \| q} W(p^\nu; (p^\nu, d)).$$

In case $f = 1$, we have $W(q;d) = 1/q$.

Comparing different local products

Let us compare the quadratic spaces $(\mathcal{F}(\mathbb{Z}/d\mathbb{Z}), [\cdot|\cdot]_d)$ and $(\mathcal{F}(\mathbb{Z}/q\mathbb{Z}), [\cdot|\cdot]_q)$ when $d|q$. The lift is the first mapping that comes to mind and is indeed the first to be considered:

$$L_{\tilde{d}}^{\tilde{q}} h : x \in \mathbb{Z}/q\mathbb{Z} \mapsto h(x \text{ mod } d). \tag{14.8}$$

Its adjoined J_d^q is defined by

$$[h|L_{\tilde{d}}^{\tilde{q}} g]_q = [J_{\tilde{d}}^{\tilde{q}} h|g]_d. \tag{14.9}$$

It is here easier to use the notation $\tilde{W}(q;a) = W(q,(a,q))$. We get

$$\sum_{a \bmod q} \tilde{W}(q;a)h(a)\overline{g(a \text{ mod } d)} = \sum_{b \bmod d} \tilde{W}(d;b) \sum_{\substack{a \bmod q,\\ a\equiv b[d]}} \frac{\tilde{W}(q;a)}{\tilde{W}(d;b)} h(a)\overline{g(b)}$$

which gives us

$$J_{\tilde{d}}^{\tilde{q}} h(b) = \sum_{\substack{a \bmod q,\\ a\equiv b[d]}} \frac{\tilde{W}(q;a)}{\tilde{W}(d;b)} h(a). \tag{14.10}$$

What about $J_d^q \mathbb{1}_{\mathcal{U}_q}$? We have

$$J_{\tilde{d}}^{\tilde{q}} \mathbb{1}_{\mathcal{U}_q}(b) = \sum_{\substack{a \bmod^* q,\\ a\equiv b[d]}} \frac{\tilde{W}(q;a)}{\tilde{W}(d;b)} = \mathbb{1}_{(b,d)=1} \frac{\phi(q)}{\phi(d)} \frac{W(q;1)}{W(d;1)}$$

hence, finally,

$$J_{\tilde{d}}^{\tilde{q}} \mathbb{1}_{\mathcal{U}_q} = \frac{\phi(q)}{\phi(d)} \frac{W(q;1)}{W(d;1)} \mathbb{1}_{\mathcal{U}_d}. \tag{14.11}$$

Lemma 14.1. *When $d|q$, we have $J^{\tilde{q}}_{\tilde{d}} L^{\tilde{d}}_{\tilde{q}} = \mathrm{Id}$.*

This Lemma is very important, because it is the one that tells us that, if h_1 are h_2 are orthogonal modulo d, then their lift modulo q stays so. Indeed

$$\Big[L^{\tilde{d}}_{\tilde{q}} h_1 | L^{\tilde{d}}_{\tilde{q}} h_2\Big]_q = \Big[J^{\tilde{q}}_{\tilde{d}} L^{\tilde{d}}_{\tilde{q}} h_1 | h_2\Big]_d = \Big[h_1|h_2\Big]_d.$$

This proof even shows that the lift of $\mathcal{F}(\mathbb{Z}/d\mathbb{Z})$ modulo q equipped with $[|]_q$ is isomorphic asa quadratic space to $(\mathcal{F}(\mathbb{Z}/d\mathbb{Z}), [\cdot|\cdot]_d)$.

Proof. We look at (14.10) and see that we only have to show that

$$\sum_{\substack{a \bmod q,\\ a\equiv b[d]}} \tilde{W}(q;a) = \tilde{W}(d;b)$$

which is obvious if one remembers that

$$\tilde{W}(q;a) = \lim_{N\to infty} \sum_{\substack{n\le N,\\ n\equiv a[q]}} f(n).$$

It is also readily verified from the formal definition of $W(q;d)$. □

14.3 An orthogonal basis

Let us start by looking at what happens modulo a prime moduli p. We have the function $\mathbb{1}$ that comes from $\mathbb{Z}/\mathbb{Z}$ and which we lift to $\mathbb{Z}/p\mathbb{Z}$. We can then choose, when a is coprime to q, an orthonormal family $(\gamma(p;a))_{a \bmod {}^*p}$ in $(\mathcal{F}(\mathbb{Z}/p\mathbb{Z}), [\cdot|\cdot]_p)$ which is furthermore orthogonal to $\mathbb{1}$. And then further the construction by multiplicativity $(\gamma(q;a))_{a \bmod {}^*q}$.

Such a general way of doing is absolutely alright, and may even shed some light on what is going on, but we shall require later a basis having a very explicit form.

Here is the basis we choose:

$$\{\mathbb{1}\} \cup \{\chi_p^\sharp\} \cup \{\chi \text{ mod } p, \chi \text{ primitive}\} \tag{14.12}$$

where

$$\chi_d^\sharp = \prod_{p|d} (\mathbb{1} - K(p)^{-1}\chi_{0,p}) \tag{14.13}$$

and $\chi_{0,p}$ is the principal character modulo p. In (14.15), we should understand $\mathbb{1}$ as $L^{1}_{\tilde{p}}\mathbb{1}$ since it is the function over $\mathbb{Z}/p\mathbb{Z}$ that we consider and not the function over $\mathbb{Z}/\mathbb{Z}$.

Lemma 14.2. *The basis given by* (14.15) *is orthogonal for the scalar product defined in* (14.6)*.*

Proof. Indeed, the fact that

$$\{1\!\!1\} \cup \{\chi \bmod p, \chi \text{ primitive}\} \tag{14.14}$$

is an orthogonal system is clear. It however contains only $1 + p - 2 = p - 1$ functions and one is missing to form a complete system. The culprit is the principal character $\chi_{0,p}$ which is orthogonal to each χ when χ is primitive modulo p, but not to $1\!\!1$. So we first look at $g = \chi_{0,p} - \frac{(p-1)W(p;1)}{W(p;p)} 1\!\!1_{p|n}$ which has all the good properties, save that we prefer another expression! We modify it as follows

$$g = \chi_{0,p} - \frac{(p-1)W(p;1)}{W(p;p)}(1\!\!1 - \chi_{0,p}) = -\frac{(p-1)W(p;1)}{W(p;p)}\left(1\!\!1 - \left(1 + \frac{W(p;p)}{(p-1)W(p;1)}\right)\chi_{0,p}\right).$$

We remark then that

$$1 + \frac{W(p;p)}{(p-1)W(p;1)} = 1 + \frac{1 - F_p^{-1}}{F_p^{-1}} = F_p = K(p)^{-1}$$

and thus we may replace g by $\chi_p^\sharp$. □

Building a basis modulo higher powers of p is easy by Lemma 14.1. Modulo p^2, we take

$$\{1\!\!1\} \cup \{\chi_p^\sharp\} \cup \{\chi \bmod p, \chi \text{ primitive}\} \cup \{\chi \bmod p^2, \chi \text{ primitive}\} \tag{14.15}$$

and continue like that for higher powers.

From there onwards, it is easy to build a basis modulo q by mutiplicativity:

$$\bigcup_{\substack{t|q,\\ \mu^2(t)=1}} \bigcup_{\substack{\mathfrak{f}|d,\\ (\mathfrak{f},t)=1}} \{\chi_t^\sharp \chi, \chi \text{ primitive modulo } \mathfrak{f}\} \tag{14.16}$$

This basis splits according to $t\mathfrak{f}$ and each subspace generated by

$$\bigcup_{\substack{s|t\mathfrak{f},\\ \mu^2(s)=1,\\ (s,t\mathfrak{f}/s)=1}} \{\chi_s^\sharp \chi, \chi \text{ primitive modulo } t\mathfrak{f}/s\}$$

is in fact an adapted space of primitive functions modulo $t\mathfrak{f}$. Note here the following expression for $\chi_t^\sharp(n)$:

$$\chi_t^\sharp(n) = \sum_{\ell|t} \mu(\ell) K(\ell)^{-1} \chi_{0,\ell}(n) \tag{14.17}$$

Note further that, with $d = t\mathfrak{f}$,

$$\begin{aligned}
[\chi_t^\sharp \chi | \chi_t^\sharp \chi]_d &= \prod_{p|\mathfrak{f}} (p-1)W(p;1) \prod_{p|t} ((p-1)W(p;1)(1 - K(p)^{-1})^2 + W(p;p)) \\
&= \prod_{p|\mathfrak{f}} F_p^{-1} \prod_{p|t} (F_p^{-1}(1 - F_p)^2 + 1 - F_p^{-1}) \\
&= \prod_{p|\mathfrak{f}} F_p^{-1} \prod_{p|t} (F_p - 1) = \mathcal{K}(\mathfrak{f})/g(t).
\end{aligned}$$

14.4 The local models

Let us select a multiplicatively large modulus M and consider the decomposition of $\mathbb{1}_{\mathcal{U}_M}$ in a the basis we have exhibited. We proceed in two steps. First we find the orthornormal projection of $\mathbb{1}_{\mathcal{U}_M}$ on each primitive space and then express this projection in terms of our orthogonal basis. We can reduce the work to a minimum by exhibiting the good candidate:

$$\vartheta_d(n) = \prod_{p|d}(F_p - 1)\prod_{\substack{p|d,\\ p|n}}(1 - F_p)^{-1} = \mu((d,n))\frac{g((d,n))}{g(d)}. \tag{14.18}$$

This expression is, upto a multiplicative factor that depends only on d the same as (Motohashi, 1983, (1.4.10))* When $f = 1$, the function $g(d)\vartheta_d$ is simply the Ramanujan sum modulo d.

Lemma 14.3. *We have*

$$\mathbb{1}_{\mathcal{U}_M} = K(M)\sum_{d|M}\vartheta_d(n).$$

Proof. By multiplicativity, the RHS is simply

$$K(M)\prod_{p|M}\Big(1 + \mu((p,n))\frac{g((p,n))}{g(p)}\Big) = \begin{cases} 0 & \text{when } \exists p|M/p|n,\\ K(p)\big(1 + g(p)^{-1}\big) = 1 & \text{otherwise.}\end{cases}$$

as required. □

A more conceptual proof. We have exhibited the function ϑ_d, and we show here how we have found it. Write

$$\mathbb{1}_{\mathcal{U}_M} = \sum_{q|M}\sum_{a \bmod^* q}[\mathbb{1}_{\mathcal{U}_M}|\gamma(q;a)]_M\gamma(q;a).$$

Firts note that, via (14.11),

$$[\mathbb{1}_{\mathcal{U}_M}|\gamma(q;a)]_M = [J_q^M\mathbb{1}_{\mathcal{U}_M}|\gamma(q;a)]_q = \frac{\phi(M)}{\phi(q)}\frac{W(M;1)}{W(q;1)}[\mathbb{1}_{\mathcal{U}_q}|\gamma(q;a)]_q.$$

We thus want to define

$$\vartheta_q = \frac{1}{\phi(q)W(q;1)}\sum_{a \bmod^* q}[\mathbb{1}_{\mathcal{U}_q}|\gamma(q;a)]_q\gamma(q;a).$$

We remark at this level that

$$\begin{aligned}\sum_{d|q}\vartheta_d\phi(q)W(q;1) &= \sum_{d|q}\sum_{a \bmod^* d}\frac{\phi(q)}{\phi(d)}\frac{W(q;1)}{W(d;1)}[\mathbb{1}_{\mathcal{U}_d}|\gamma(d;a)]_d\gamma(d;a)\\ &= \sum_{d|q}\sum_{a \bmod^* d}[\mathbb{1}_{\mathcal{U}_q}|\gamma(d;a)]_q\gamma(d;a) = \mathbb{1}_{\mathcal{U}_q}\end{aligned}$$

*The pseudo-character Motohashi considers is $g(d)\vartheta_d$. It is also the expression found in (Motohashi, 1978, top of page 168) but the g used there is the inverse of ours, as already mentionned.

so that, by using a Moebius inversion, we reach

$$\vartheta_d = \sum_{q|d} \mu(d/q) \frac{1\!\!1_{\mathcal{U}_q}}{\phi(q)W(q;1)}.$$

We infer from the above that

$$\begin{aligned}
\vartheta_d(n) &= \sum_{\substack{q|d,\\(n,q)=1}} \frac{\mu(d/q)}{\phi(q)W(q;1)} = \sum_{q|d} \sum_{\substack{\delta|q,\\ \delta|n}} \mu(\delta) \frac{\mu(d/q)}{\phi(q)W(q;1)} \\
&= \sum_{\substack{\delta|d,\\ \delta|n}} \mu(\delta) \sum_{\delta|q|d} \frac{\mu(d/q)}{\phi(q)W(q;1)} = \sum_{\substack{\delta|d,\\ \delta|n}} \frac{\mu(\delta)}{\phi(\delta)W(\delta;1)} \prod_{p|d/\delta} \Big(-1 + \frac{1}{\phi(p)W(p;1)}\Big) \\
&= \mu(d) \prod_{p|d} \Big(1 - \frac{1}{\phi(p)W(p;1)}\Big) \sum_{\substack{\delta|d,\\ \delta|n}} \mu(\delta) \prod_{p|\delta} \frac{1}{1-\phi(p)W(p;1)}.
\end{aligned}$$

All that simplifies into

$$\vartheta_d(n) = \mu(d) \prod_{p|d} (1 - (\phi(p)W(p;1))^{-1}) \prod_{\substack{p|d,\\ p|n}} \frac{\phi(p)W(p;1)}{\phi(p)W(p;1) - 1}.$$

On using $W(p;1) = K(p)/\phi(p) = F_p^{-1}/\phi(p)$ (see (14.5)), we get the announced expression. □

Expressing the local model in terms of our chosen orthogonal basis

We need to express our local model in terms of the basis we have exhibited. We again give two proofs.

Lemma 14.4. *We have* $\vartheta_d = \mu(d)\chi_d^\sharp$.

Proof. Just note that

$$\vartheta_p(n) = \begin{cases} F_p - 1 & \text{when } (n,p) = 1, \\ -1 & \text{when } p|n, \end{cases} \qquad \chi_p^\sharp(n) = \begin{cases} 1 - F_p & \text{when } (n,p) = 1, \\ 1 & \text{when } p|n. \end{cases}$$

which proves our assertion when d is a prime number. We conclude by multiplicativity. □

A more conceptual proof. Let us write

$$\vartheta_d = \sum_{\mathfrak{f}|d} \sum_{\chi \bmod^* \mathfrak{f}} \frac{[\vartheta_d|\chi_{d/\mathfrak{f}}^\sharp \chi]_d}{[\chi_{d/\mathfrak{f}}^\sharp \chi|\chi_{d/\mathfrak{f}}^\sharp \chi]_d} \chi_{d/\mathfrak{f}}^\sharp \chi.$$

We should compute $[\vartheta_d|\chi^\sharp_{d/\mathfrak{f}}\chi]_d$. We proceed by multiplicativity. There comes

$$[\vartheta_d|\chi^\sharp_{d/\mathfrak{f}}\chi]_d = \prod_{p|\mathfrak{f}}[\vartheta_p|\chi]_p \prod_{p|d/\mathfrak{f}}[\vartheta_p|\chi^\sharp_p]_p.$$

We check that the scalar product $[\vartheta_p|\chi]_p$ simply vanished! Which means that only $\mathfrak{f} = 1$ contributes, i.e. that ϑ_d is colinear to $\chi^\sharp_d$. Moreover

$$\begin{aligned}[\vartheta_p|\chi^\sharp_p]_p &= (1 - K(p)^{-1})(F_p - 1)W(p;1)(p-1) - W(p;p)\\ &= -(F_p-1)^2F_p^{-1} - (F_p-1)F_p^{-1} = 1 - F_p\end{aligned}$$

and, since $[\chi^\sharp_d|\chi^\sharp_d]_d = 1/g(d)$, we get the lemma. □

14.5 A large sieve inequality adapted to f

We want to majorize

$$\sum_{\substack{t\mathfrak{f}\le D,\\ (t,\mathfrak{f})=1}} \mu^2(t) \sum_{\chi \bmod^* \mathfrak{f}} \Bigl| \sum_{N_0<n\le N_0+N} f(n)u_n\chi(n)\chi^\sharp_t(n)\Bigr|^2 / \|\chi\chi^\sharp_t\|^2_d$$

in terms of $\sum_{M<n\le M+N} f(n)|u_n|^2$. In order to do so, we use a Lemma due to Selberg (see (Bombieri, 1987/1974a, Lemma 1.1)*). The reader will find trouvera des precursory results in (Rényi, 1959). Let us mention here that $\chi\chi^\sharp_{d/\mathfrak{f}}$ is seen in the numerator as a function over $\mathbb{Z}$ and in the denominator as a function over $\mathbb{Z}/d\mathbb{Z}$.

Lemma 14.5. *In any prehilbertian space, we have*

$$\sum_i M_i^{-1}|[h|\varphi_i]|^2 \le \|h\|^2$$

with $M_i = \sum_j |[\varphi_i|\varphi_j]|$.

We apply this lemma to the family $(\chi\chi^\sharp_t)$. We have to evaluate

$$M(\chi\chi^\sharp_t) = \sum_{\substack{\mathfrak{g},t'\le D,\\ (\mathfrak{g},t')=1}} \mu^2(t') \sum_{\psi \bmod^* \mathfrak{g}} \Bigl| \sum_{N_0<n\le N_0+N} f(n)\chi(n)\chi^\sharp_s(n)\overline{\psi(n)}\chi^\sharp_t(n)\Bigr|.$$

This evaluation relies on a preliminary computation of

$$S = \sum_{N_0<n\le N_0+N} f(n)\chi(n)\chi^\sharp_t(n)\overline{\psi(n)}\chi^\sharp_{t'}(n).$$

*See also (Bombieri, 1971) and (Ramaré, 2009, Lemma 1.2).

We can write the product $\chi\overline{\psi}$ as $\theta\chi_{0,[\mathfrak{g},\mathfrak{f}]/\mathfrak{h}}$ where θ is a primitive character modulo $\mathfrak{h}$ and $([\mathfrak{g},\mathfrak{f}]/\mathfrak{h},\mathfrak{h})=1$. We use (14.17) to get*

$$S=\sum_{r|t}\mu(r)K(r)^{-1}\sum_{s|t'}\mu(s)K(s)^{-1}\sum_{N_0<n\le N_0+N}f(n)\theta(n)\chi_{0,rs[\mathfrak{g},\mathfrak{f}]/\mathfrak{h}}(n).$$

Hypothesis (C_3) applies and gives us

$$\begin{aligned}S&=\sum_{r|t}\mu(r)K(r)^{-1}\sum_{s|t'}\mu(s)K(s)^{-1}\Big(\delta_{\mathfrak{h}=1}\mathscr{F}NK([r,s,[\mathfrak{f},\mathfrak{g}]/\mathfrak{h}])+\mathcal{O}^*(C'[r,s,[\mathfrak{f},\mathfrak{g}]/\mathfrak{h}]^{\beta}(N_0+N)^{\gamma})\Big)\\&=\mathscr{F}N\delta_{\mathfrak{h}=1}\sum_{r|t}\mu(r)K(r)^{-1}\sum_{s|t'}\mu(s)K(s)^{-1}K([r,s,[\mathfrak{f},\mathfrak{g}]/\mathfrak{h}])\\&\qquad+\mathcal{O}^*\big(C'\sum_{r|t}K(r)^{-1}(r\mathfrak{f})^{\beta}\sum_{s|t'}K(s)^{-1}(s\mathfrak{g})^{\beta}(N_0+N)^{\gamma}\big).\end{aligned}$$

What about the main term? It vanishes when $\mathfrak{h}\neq 1$, i.e. when $\chi\neq\psi$, which implies that $\mathfrak{f}=\mathfrak{g}$. Moreover, when $\mathfrak{f}=\mathfrak{g}$ and $\mathfrak{h}=1$,

$$\begin{aligned}\sum_{r|t}\mu(r)\sum_{s|t'}\mu(s)\frac{K([r,s,\mathfrak{f}])}{K(r)K(s)}&=K(\mathfrak{f})\sum_{r|t}\mu(r)\sum_{s|t'}\mu(s)\frac{K([r,s])}{K(r)K(s)}\\&=K(\mathfrak{f})\sum_{r|t}\mu(r)\sum_{s|t'}\mu(s)K((r,s))^{-1}.\end{aligned}$$

We readily check that this expression vanishes as soon as† $t\neq t'$. Moreover, when $t=t'$ we have

$$\begin{aligned}K(\mathfrak{f})\sum_{r|t}\mu(r)\sum_{s|t}\mu(s)\frac{K([r,s])}{K(r)K(s)}&=K(\mathfrak{f})\prod_{p|t}\Big(1-1-1+\frac{1}{K(p)}\Big)\\&=K(\mathfrak{f})\prod_{p|t}(F_p-1)=\|\chi\chi_t^{\sharp}\|_{t\mathfrak{f}}^2.\end{aligned}$$

We let the reader meditate on this equality which shows how and why we have selected our local scalar products. Notice that this norm is at least $\mathcal{O}(D^{-\alpha})$ by (C_2). Let us now turn our attention to the error term. It is at most $(N_0+N)^{\gamma}$ times

$$\begin{aligned}&C'\sum_{r|t}K(r)^{-1}(r\mathfrak{f})^{\beta}\sum_{t'}\sum_{\substack{\mathfrak{g}/(\mathfrak{g},t')=1,\\ \mathfrak{g}t'\le D}}\phi(\mathfrak{g})\sum_{s|t'}K(s)^{-1}(s\mathfrak{g})^{\beta}\\&\qquad\le C'\sum_{r|t}K(r)^{-1}(t\mathfrak{f})^{\beta}\sum_{\mathfrak{g}\le D}\sum_{s\le D/\mathfrak{g}}\phi(\mathfrak{g})K(s)^{-1}(s\mathfrak{g})^{\beta}\frac{D}{s\mathfrak{g}}\\&\qquad\qquad\ll_{\varepsilon}(t\mathfrak{f})^{\varepsilon+\beta}\sum_{\mathfrak{g}\le D}D^{1+\beta}\ll_{\varepsilon}(t\mathfrak{f})^{\varepsilon+\beta}D^{2+\beta}\ll_{\varepsilon}D^{2+2\beta+\varepsilon}\end{aligned}$$

*We have to note that $\chi_{0,\ell}$ depends only, as a function over $\mathbb{Z}$, on the prime factors that divide ℓ. In particular here, $\chi_{0,[r,s]}=\chi_{0,r}\chi_{0,s}$.

†Remember that they are both squarefree; them being distinct means that there exists a prime that divides one and not the other one.

for every $\varepsilon > 0$. Consequently

$$M(\chi\chi_t^\sharp) = \|\chi\chi_t^\sharp\|_d^2 \big(\mathscr{F}N + \mathcal{O}_\varepsilon(D^{2+2\beta+\alpha+\varepsilon}(N_0+N)^\gamma)\big).$$

We have reached

$$\sum_{t\le D}\mu^2(t)\sum_{\substack{\mathfrak{f}\le D/t,\\ (\mathfrak{f},t)=1}}\sum_{\chi \bmod^* \mathfrak{f}}\Big|\sum_{N_0<n\le N_0+N} f(n)u_n\chi(n)\chi_t^\sharp(n)\Big|^2/\|\chi\chi_t^\sharp\|_d^2$$
$$\le \sum_n f(n)|u_n|^2\big(\mathscr{F}N + \mathcal{O}_\varepsilon(D^{2+2\beta+\alpha+\varepsilon}(N_0+N)^\gamma)\big).$$

We finally note that $\|\chi\chi_t^\sharp\|_d^2 = K(\mathfrak{f})/g(t)$ and $\chi_t^\sharp(n) = \mu(t)\vartheta_t(n)$. Here is the result we have established:

Theorem 14.1.

$$\sum_{t\le D}\sum_{\substack{\mathfrak{f}\le D/t,\\ (\mathfrak{f},t)=1}}\frac{\mu^2(t)}{K(\mathfrak{f})g(t)}\Big|\sum_{N_0<n\le N_0+N} f(n)u_n\chi(n)g(t)\vartheta_t(n)\Big|^2$$
$$\le \sum_n f(n)|u_n|^2\big(\mathscr{F}N + \mathcal{O}_\varepsilon(D^{2+2\beta+\alpha+\varepsilon}(N_0+N)^\gamma)\big).$$

It is essentially (Motohashi, 1978, Lemma 2) or (Motohashi, 1983, Theorem 5). Once this result is established, we can easily get a hybrid form by using Theorem 2.4 as in the proof of Theorem 2.5.

Chapter 15

While reading Motohashi...

Remarks

1. Comparing (Ramaré, 2009, (11.30)) (recall last equation of (Ramaré, 2009, (11.5))) with (Motohashi, 1978, 4) hints at

$$\phi_{r^*} = h(r)\psi_r \tag{15.1}$$

(and ψ_r will be called Ψ_r later in this paper, when corresponding to a multiplicative function).

2. **Careful!!** The function $g(r)$ of (Motohashi, 1978, bottom of page 167) is here $1/g(r)$ according to the notations of (Motohashi, 1983)! We follow (Motohashi, 1983).

Notations

Notations used throughout these notes are standard ... in one way or the other! Here is a guideline:

— $e(y) = \exp(2i\pi y)$.

— The use of the letter p for a variable always implies this variable is a prime number.

— $[d, d']$ stands for the lcm and (d, d') for the gcd of d and d'.

— $|\mathcal{A}|$ stands for the cardinality of the set $\mathcal{A}$ while $\mathbb{1}_{\mathcal{A}}$ stands for its characteristic function.

— $q\|d$ means that q divides d in such a way that q and d/q are coprime. In words we shall say that q *divides* d *exactly*.

— The squarefree kernel of the integer $d = \prod_i p_i^{\alpha_i}$ is $\prod_i p_i$, the product of all prime factors of d.

— $\omega(d)$ is the number of prime factors of d, counted without multiplicity.

— $\phi(d)$ is the Euler totient, i.e. the cardinality of the multiplicative group of $\mathbb{Z}/d\mathbb{Z}$.

— $\tau(d)$ is the number of positive divisors of d.

— $\tau_k(d)$ is the number of k-tuples of (positive) integers $(d_1, \cdots, d_k)$ such that $d_1 \cdots d_k = d$, so that $\tau_2 = \tau$.

— $\mu(d)$ is the Moebius function, that is 0 when d is divisible by a square > 1 and otherwise $(-1)^r$ otherwise, where r is the number of prime factors of d.

— $c_q(n)$ is the Ramanujan sum. It is the sum of $e(an/q)$ over all a modulo q that are prime to q.

— $\Lambda(n)$ is van Mangoldt function: which is $\operatorname{Log} p$ is n is a power of the prime p and 0 otherwise.

— The notation $f = \mathcal{O}_A(g)$ means that there exists a constant B such that $|f| \leq Bg$ but that this constant may depend on A. When we put in several parameters as subscripts, it simply means the implied constant depends on all of them.

- The notation $f = \mathcal{O}^*(g)$ means that $|f| \le g$, that is a $\mathcal{O}$-like notation, but with an implied constant equal to 1.
- The notation $f \star g$ denotes the arithmetic convolution of f and g, that is to say the function h on positive integers such that $h(d) = \sum_{q|d} f(q)g(d/q)$. exists for every real number x.
- $\mathcal{U}$ is the compact set $(\mathcal{U}_d)_d$ where, for each d, $\mathcal{U}_d$ is the set of invertible elements modulo d.
- The letter ψ is used in two different context: either to denote the summatory function of the van Mangoldt function, that is to say $\psi(x) = \sum_{n \le x} \Lambda(n)$, with the variation $\psi(x, \chi) = \sum_{n \le x} \chi(n)\Lambda(n)$. Or for local model as in chapter 13, see (13.1) and (3.3).
- We used the Chebyshev functions ϑ and ψ as well as their variations $\vartheta(x; \chi)$, $\vartheta(x; q, a)$, $\psi(x, \chi)$ and $\psi(x; q, a)$. See chapter 11 for details.
- $1\!\!1$ denotes a characteristic function in one way or another. For instance, $1\!\!1_{\mathcal{K}_d}$ is 1 if $n \in \mathcal{K}_d$ and 0 otherwise, but we could also write it as $1\!\!1_{n \in \mathcal{K}_d}$, closer to what is often called the Dirac δ-symbol. We also use $1\!\!1_{(n,d)=1}$ and $1\!\!1_{q=q'}$.

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