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A REMARK ON THE GEOMETRY OF SPACES OF FUNCTIONS WITH PRIME FREQUENCIES

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Abstract. For any positive integer r, denote by $\mathcal{P}_r$ the set of all integers $\gamma \in \mathbb{Z}$ having at most r prime divisors. We show that $C_{\mathcal{P}_r}(\mathbb{T})$, the space of all continuous functions on the circle $\mathbb{T}$ whose Fourier spectrum lies in $\mathcal{P}_r$, contains a complemented copy of ℓ^1. In particular, $C_{\mathcal{P}_r}(\mathbb{T})$ is not isomorphic to $C(\mathbb{T})$, nor to the disc algebra $A(\mathbb{D})$. A similar result holds in the L^1 setting.

For any set of "frequencies" $\Lambda \subset \mathbb{Z}$, let us denote by $C_\Lambda(\mathbb{T})$ the space of all continuous functions f on the circle $\mathbb{T}$ whose Fourier spectrum lies in Λ, i.e. $\hat{f}(\gamma) = 0$ for every $\gamma \in \mathbb{Z} \setminus \Lambda$. A classical topic in harmonic analysis (see e.g. the monographs [8] or [15]) is to try to understand how various "thinness" properties of the set Λ are reflected in geometrical properties of the Banach space $C_\Lambda(\mathbb{T})$. For example, by a famous results of Varopoulos and Pisier, Λ is a Sidon set if and only of C_Λ is isomorphic to ℓ^1, if and only if C_Λ has cotype 2. More generally, one can start with any Banach space X naturally contained in $L^1(\mathbb{T})$ and consider the spaces X_Λ of all functions $f \in X$ whose Fourier spectrum lies in Λ.

We concentrate here on sets $\Lambda \subset \mathbb{Z}$ closely related to the set $\mathcal{P}$ of prime numbers. Considering the primes as a "thin" set is a question of point of view. On the one hand, $\mathcal{P}$ is a small set since it has zero density in $\mathbb{Z}$. On the other hand, the enveloping sieve philosophy as exposed in [13] and [12] says that the primes can be looked at as a subsequence of density of a linear combination of arithmetic sequences, a fact that is brilliantly illustrated in [3] by showing that $\mathcal{P}$ contains arbitrarily long arithmetic progressions.

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From the harmonic analysis standpoint, let us just quote two results going in opposite directions: it was shown by F. Lust-Piquard [9] that $C_{\mathcal{P}}(\mathbb{T})$ contains subspaces isomorphic to c_0, a property shared by $C(\mathbb{T})$ and the disc algebra $A(\mathbb{D}) = C_{\mathbb{Z}_+}(\mathbb{T})$, which indicates that $\mathcal{P}$ is not a very thin set; but on the other hand [6] any $f \in L^\infty_{\mathcal{P}}(\mathbb{T})$ is "totally ergodic" (i.e. fg has a unique invariant mean for any $g \in C(\mathbb{T})$), a property shared by very thin sets such as Sidon sets and not satisfied by $\mathbb{Z}$ and $\mathbb{Z}_+$.

In this note, we show that the geometry of $C_{\mathcal{P}}(\mathbb{T})$ is quite different from that of $C(\mathbb{T})$ or the disc algebra $A(\mathbb{D}) = C_{\mathbb{Z}_+}(\mathbb{T})$, and that $L^1_{\mathcal{P}}(\mathbb{T})$ is very different from $L^1(\mathbb{T})$. Hence, the primes definitely do not behave like an arithmetic progression. Further, this disruption occurs even with the set of integers having at most r prime factors, for some given r, though the sieve is known to behave rather well for such sequences as soon as $r \geqq 2$ ([2], [16]).

For any positive integer r, let us denote by $\mathcal{P}_r$ the set of all positive integers having at most r prime factors:

$$\mathcal{P}_r = \Big\{ \prod_{1 \leqq i \leqq r} p_i^{\alpha_i} \;\Big|\; p_i \in \mathcal{P},\ \alpha_i \in \mathbb{Z}_+ \Big\}.$$

Our main result reads as follows.

Theorem. *Let $\Lambda \subset \mathbb{Z}$ and assume that $\mathcal{P} \subset \Lambda \subset \mathcal{P}_r \cup (-\mathcal{P}_r)$. Then*
(1) *$C_\Lambda(\mathbb{T})$ contains a complemented copy of ℓ^1;*
(2) *$L^1_\Lambda(\mathbb{T})$ and $L^1_{\mathbb{Z}^- \cup \Lambda}(\mathbb{T})$ contain a complemented copy of ℓ^2.*

It is well known that neither $C(\mathbb{T})$ nor the disc algebra $A(\mathbb{D})$ contains a complemented copy of ℓ^1, and that $L^1(\mathbb{T})$ does not contain a complemented copy of ℓ^2 (see the remark below). Hence, we immediately get

Corollary. *If $\Lambda \subset \mathbb{Z}$ satisfies $\mathcal{P} \subset \Lambda \subset \mathcal{P}_r \cup (-\mathcal{P}_r)$, then $C_\Lambda(\mathbb{T})$ is isomorphic neither to a complemented subspace of $C(\mathbb{T})$ nor to a complemented subspace of the disc algebra $A(\mathbb{D})$, and neither $L^1_\Lambda(\mathbb{T})$ nor $L^1_{\mathbb{Z}^- \cup \Lambda}(\mathbb{T})$ is isomorphic to a complemented subspace of $L^1(\mathbb{T})$.*

Remark. (i) One way of showing that neither $C(\mathbb{T})$ nor the disc algebra $A(\mathbb{D})$ contains a complemented copy of ℓ^1 is to use Pełczyński's property (V). A Banach space X has property (V) if every non weakly compact (bounded) operator $T : X \to Y$ into a Banach space Y "fixes a copy of c_0", i.e. there is a (closed) subspace $X_0 \subset X$ isomorphic to c_0 such that $T_{|X_0}$ is an isomorphism from X_0 onto $T(X_0)$. It is well known that $C(\mathbb{T})$ and $A(\mathbb{D})$ have property (V) (see [4]). Moreover, property (V) is clearly inherited by complemented subspaces. Since obviously ℓ^1 does not have property (V), the result follows.

(ii) To show that $L^1(\mathbb{T})$ does not contain a complemented copy of ℓ^2 one can use another well known Banach space property. Let us say that the

Banach space X *satisfies Grothendieck's theorem* if every operator from X into a Hilbert space is absolutely summing. A typical example is the space L^1: this is precisely "Grothendieck's theorem" (see e.g. [11]). Moreover, this property is again clearly inherited by complemented subspaces. Since obviously ℓ^2 does not satisfy it, the result follows.

Having introduced property (V) and GT, we can state a more general version of the above corollary:

COROLLARY'. *If* $\Lambda \subset \mathbb{Z}$ *satisfies* $\mathcal{P} \subset \Lambda \subset \mathcal{P}_r \cup (-\mathcal{P}_r)$, *then*
(i) $C_\Lambda(\mathbb{T})$ *does not have property* (*V*);
(ii) $L^1_\Lambda(\mathbb{T})$ *and* $L^1_{\mathbb{Z}^- \cup \Lambda}(\mathbb{T})$ *do not satisfy Grothendieck's theorem.*

The proof of the theorem relies on the following "gap" property of $\mathcal{P}_r$. Denote by $d(\lambda, A)$ the distance of an integer λ to a set $A \subset \mathbb{Z}$,

$$d(\lambda, A) = \inf \left\{ |\lambda - a|;\ a \in A \right\}.$$

LEMMA. *For any* $r \in \mathbb{N}$, *one can find an infinite set* $E_r \subset \mathcal{P}$ *such that*

$$\sup_{\lambda \in E_r} d\big(\lambda, \mathcal{P}_r \setminus \{\lambda\}\big) = +\infty.$$

In other words, there are *big symmetric holes* in any set Λ such that $\mathcal{P} \subset \Lambda \subset \mathcal{P}_r$. It is of course well known that there are big holes in $\mathcal{P}$. However, the standard argument is to consider the set $\{n! + 2, \dots, n! + n\}$ with n arbitrarily large (which is obviously disjoint from the set of prime numbers), but this provides only *one-sided* holes in $\mathcal{P}$. (One may consult [14] and [10] where the quantitative question of the sizes of these holes is studied.) Actually, the referee has kindly informed us that the case $\Lambda = \mathcal{P}$ was already known: see [1] and [5]. Nevertheless, our proof is both short and general. It would be interesting to estimate the (optimal) size of these symmetric holes.

PROOF OF THE LEMMA. For convenience, we put $\mathcal{P}_0 = \{1\}$. We prove by induction on $r \in \mathbb{Z}^+$ the following assertion, denoted by $(\mathcal{H}_r)$: *For every* $a \geqq 1$, *there exists a prime number* p *such that* $\{p - a, \dots, p - 1, p + 1, \dots, p + a\} \cap \mathcal{P}_r = \emptyset$.

Property $(\mathcal{H}_0)$ is obvious. Now, assume that $(\mathcal{H}_{r-1})$ has been proved for some $r \geqq 1$, and let us fix $a \geqq 1$.

We choose a prime number $A > a + 1$ such that

$$\{A - a, \dots, A - 1, A + 1, \dots, A + a\} \cap \mathcal{P}_{r-1} = \emptyset,$$

and put

$$Q = \prod_{\substack{1 \leqq k < 2A \\ k \neq A}} k^2 = \left(\frac{(2A - 1)!}{A} \right)^2.$$

Note that A and Q are coprime.

By Dirichlet's theorem on arithmetic progressions, one can find a prime number $p > A$ such that $p \equiv A$ [mod Q]. We claim that this prime number p satisfies

$$\{p-a,\dots,p-1,p+1,\dots,p+a\} \cap \mathcal{P}_r = \emptyset,$$

which will give $(\mathcal{H}_r)$.

To prove the claim, let us fix $j \in \{\pm 1,\dots,\pm a\}$ and, towards a contradiction, assume that $p+j \in \mathcal{P}_r$, i.e.

$$p+j = \prod_{1 \leqq i \leqq r} q_i^{\alpha_i},$$

where $q_i \in \mathcal{P}$ and $\alpha_i \in \mathbb{Z}_+$. Since $A+j$ divides Q and $p+j \equiv A+j$ [mod Q], the number $A+j$ divides $p+j$. Hence we can write

$$A+j = \prod_{1 \leqq i \leqq r} q_i^{\beta_i},$$

where $\beta_i \leqq \alpha_i$. If at least one of the β_i vanished, then $A+j$ would belong to $\mathcal{P}_{r-1}$, but this is false. Hence for every $1 \leqq i \leqq r$, we have $\beta_i \geqq 1$. On the other hand, since $A < p$, there exists some $i_0 \in \{1\dots,r\}$ such that $\beta_{i_0} < \alpha_{i_0}$. Now, consider $N = q_{i_0}(A+j)$. Since $\beta_{i_0} \geqq 1$, we have that q_{i_0} divides $A+j$, hence N divides $(A+j)^2$; but $(A+j)^2$ divides Q so that N divides Q. On the other hand N divides $p+j$ since $\beta_{i_0}+1 \leqq \alpha_{i_0}$. Hence N must divide the gcd of Q and $p+j$. The latter is equal to $A+j$ since $A+j$ divides Q and $p+j \equiv A+j$ [mod Q]. This is the required contradiction.

Thus, $(\mathcal{H}_r)$ is true for all $r \in \mathbb{Z}^+$, and the lemma follows immediately. □

Proof of the Theorem. It would suffice to apply the lemma and [6, Proposition 2]. However, we repeat quickly the argument in our framework.

From the lemma, we get by induction a sequence of prime numbers $H = (p_n)_{n \geqq 1}$ (that we identify with the set $\{p_1, p_2, \dots,\}$) with the following properties:

(a) H is a Hadamard sequence with ratio at least 3;

(b) the "mesh" $[H] = \left\{ \sum_{1 \leqq n \leqq m} \varepsilon_n p_n \;\middle|\; \varepsilon_n \in \{-1,0,1\};\ m \geqq 1 \right\}$ intersects the set $\mathcal{P}_r \cup (-\mathcal{P}_r)$ only at the $\pm p_n$'s.

Then we consider the Riesz products

$$R_N(x) = 2 \prod_{n=1}^{N} \left[1 + \cos(2\pi p_n x)\right].$$

Identifying R_N with the positive measure $R_N(x)dx$ (where dx is the normalized Lebesgue measure on $\mathbb{T}$), we have $\|R_N\| = \int_{\mathbb{T}} R_N(x)dx = 2$. If μ is any limit point of the sequence (R_N) in $M(\mathbb{T})$, then μ is a positive measure on $\mathbb{T}$ whose Fourier coefficients are 1 on H and vanish on $\Lambda \setminus [H]$.

Since $H \subset \Lambda \subset \mathcal{P}_r \cup (-\mathcal{P}_r)$ and $[H] \cap \big(\mathcal{P}_r \cup (-\mathcal{P}_r)\big) = H \cup (-H)$ by (b), it follows that the convolution operator associated with μ is then a translation-invariant projection from $C_\Lambda(\mathbb{T})$ onto $C_E(\mathbb{T})$, where $E = H \cup \big(\Lambda \cap (-H)\big)$. Since E is a Sidon set by (a), $C_E(\mathbb{T})$ is isomorphic to ℓ^1, and this proves part (1) of the theorem.

The same convolution operator also defines a projection from $L^1_\Lambda(\mathbb{T})$ onto $L^1_E(\mathbb{T})$, which is isomorphic to ℓ^2 because E is a $\Lambda(2)$ set by (a). Hence, $L^1_\Lambda(\mathbb{T})$ contains a complemented copy of ℓ^2. Finally, using [7, Proposition 1.4] this holds for $L^1_{\mathbb{Z}^- \cup \Lambda}(\mathbb{T})$ as well (see [6] for some details). This proves part (2) of the theorem. □

Remark. If we knew that there are infinitely many primorial numbers (resp. factorial numbers), then the above proposition would not be needed: we could apply directly [6, Proposition 2] to prove the theorem.

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