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MÖBIUS FUNCTION AND PRIMES: AN IDENTITY FACTORY WITH APPLICATIONS

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ABSTRACT. We study the sums $\sum_{n\le X,(n,q)=1}\frac{\mu(n)}{n^s}\log^k\left(\frac{X}{n}\right)$, where $k\in\{0,1\}$, $s\in\mathbb{C}$, $\Re s>0$ and give asymptotic estimations in an explicit manner. In order to do so, we produce a large family of arithmetical identities and derive several applications. Along similar ideas, we present an appendix showing the inequality $\sum_{n\le X}\Lambda(n)/n\le\log X$, valid for any $X\ge 1$.

1. Introduction and results

The Möbius function μ is a difficult object to study. From an explicit viewpoint, one would like to ask for estimates for averaging functions like $M(x)=\sum_{n\le X}\mu(n)$ and $m(X)=\sum_{n\le X}\frac{\mu(n)}{n}$, questions addressed for instance by R.A. MacLeod in [18], L. Schoenfeld in [32], N. Costa Pereira in [6], F. Dress and M. El Marraki in [11], [12] and [13], and with the help of H. Cohen in [5] and recently by K.A. Chalker in the memoir [4]. In this paper, we aim at evaluating explicitely the following two quantities

$$m_q(X;s)=\sum_{\substack{n\le X,\\(n,q)=1}}\frac{\mu(n)}{n^s},\quad \check{m}_q(X;s)=\sum_{\substack{n\le X,\\(n,q)=1}}\frac{\mu(n)}{n^s}\log\left(\frac{X}{n}\right),\tag{1}$$

where $s\in\mathbb{C}$ and $\Re s>0$. Here, $m_q(X;1)=m_q(X)$, $\check{m}_q(X;1)=\check{m}_q(X)$ and we shall omit the index when $q=1$.

A wide ranging estimate. Our first result is an easy but efficient estimate.

Theorem 1.1. *When $k\ge 1$ is an integer, $\sigma\ge 1$ and $X\ge 1$, we have*

$$0\le\sum_{\substack{n\le X\\(n,q)=1}}\frac{\mu(n)}{n^\sigma}\log^k\left(\frac{X}{n}\right)\le 1.00303\frac{q}{\varphi(q)}\big(k+(\sigma-1)\log X\big)(\log X)^{k-1}.$$

When $k\ne 1$, we may replace 1.00303 *by 1.*

The non-negativity is very useful in practice. The case $k=0$, $\sigma=1$ has been treated in [8, Lemma 1] by H. Davenport, and also in [14, Lemma 10.2] as in [34] by T. Tao. Its extension to $\sigma>1$ is dealt with in [25, Theorem 1.1]: we have $|\sum_{n\le X,(n,q)=1}\mu(n)/n^\sigma|\le\sigma$.

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Notation. Throughout the present work the variable p denotes a prime number. We also use the O^* *notation*: we write $f(X) = O^*(h(X))$, as $X \to a$ to indicate that $|f(X)| \le h(X)$ in a neighborhood of a, where, in absence of precision, a corresponds to ∞. Consider now $q, d \in \mathbb{Z}_{>0}$; we write $d|q^\infty$ to mean that d is in the set $\{d',\ p|d' \implies p|q\}$. Finally, we consider the *Euler* φ_s *function*: let s be any complex number, we define $\varphi_s : \mathbb{Z}_{>0} \to \mathbb{C}$ as $q \mapsto q^s \prod_{p|q}\left(1 - \frac{1}{p^s}\right)$.

Asymptotic results. Let us now turn to stronger results.

Theorem 1.2. *Let $q \in \mathbb{Z}_{>0}$. Let $\sigma = 1 + \varepsilon \in [1,2]$ and $X \ge 1$, we have the following estimation*

$$m_q(X;\sigma) = \frac{m_q(X)}{X^{\sigma-1}} + \frac{q^\sigma}{\varphi_\sigma(q)}\frac{1}{\zeta(\sigma)} + \frac{(\sigma-1)\Delta_q(X,\sigma-1)}{X^{\sigma-1}}$$

where

$$|\Delta_q(X,\varepsilon)| \le 0.0215\, g_1(q)\frac{q^\xi}{\varphi_\xi(q)}\frac{\mathbb{1}_{X\ge 10^{12}}}{\log(X)} + \left(4.1\, g_0(q) + \frac{(5+\varepsilon 2^\varepsilon)}{2}\right)\frac{\sqrt{q}}{\varphi_{\frac12}(q)}\frac{2^\varepsilon}{\sqrt{X}},$$

and

(2)

$$g_0(q) = \prod_{2|q}\frac{\sqrt{3}(\sqrt{2}-1)}{2}, \quad g_1(q) = \prod_{2|q} 2.06\left(1 - \frac{1}{2^\xi}\right), \quad \xi = 1 - \frac{1}{12\log 10}.$$

Moreover we have $m_q(X;\sigma) \ge m_q(X)/X^{\sigma-1}$ and

$$\begin{cases} \Delta_q(X,\varepsilon) \le 0 & \text{if } X < 10.85,\\ \Delta_q(X,\varepsilon) \le 0 & \text{if } X < 10.9,\ \varepsilon \le 4/25,\\ \Delta_q(X,\varepsilon)/X^\varepsilon \le 0 & \text{if } X < 41,\ \gcd\big(q, \prod_{p\le 37} p\big) \notin \{1,11,13\},\\ \Delta_q(X,\varepsilon)/X^\varepsilon \le 0 & \text{if } X < 47,\ \gcd\big(q, \prod_{p\le 43} p\big) \notin \{1,11,13,17\},\\ \Delta_q(X,\varepsilon)/X^\varepsilon \le 0.014 & \text{if } X \le 47,\ \gcd\big(q, \prod_{p\le 43} p\big) = 1,\\ \Delta_q(X,\varepsilon)/X^\varepsilon \le 0.00005 & \text{if } X < 47,\ \gcd\big(q, \prod_{p\le 43} p\big) \in \{11,13,17\}. \end{cases}$$

Finally, we also have $\Delta_q(X,\varepsilon)/X^\varepsilon \ge -q/\varphi(q)$. This lower bound may be refined to $-\frac{1}{\varepsilon\zeta(1+\varepsilon)}\frac{q^{1+\varepsilon}}{\varphi_{1+\varepsilon}(q)}$.

Observe that the function $\Delta_q(X,\varepsilon)$ is positive when $q = 1$, $X = 10.97$ or $X = 11$, and $\sigma \in [1,2]$. The condition $X < 47$ is set only to keep the running verification time within an acceptable bound, as we have to range over all the divisors of $\prod_{p<47} p$.

Theorem 1.3. *Let $\sigma = 1+\varepsilon \in [1, 11/10]$ and $X \ge 15$, we have the estimate*

$$\check{m}_q(X;\sigma) = \frac{q^\sigma}{\varphi_\sigma(q)}\left(\frac{\log(X)}{\zeta(\sigma)} - \frac{\zeta'(\sigma)}{\zeta^2(\sigma)} - \frac{1}{\zeta(\sigma)}\sum_{p|q}\frac{\log p}{p^\sigma - 1}\right) + \frac{\check{\Delta}_q(X,\sigma-1)}{X^{\sigma-1}}$$

where we have

$$\begin{aligned}|\check{\Delta}_q(X,\varepsilon)| \le & 0.0256 g_1(q) 2^\varepsilon \frac{q^\xi}{\varphi_\xi(q)}\frac{\mathbb{1}_{X\ge 10^{12}}}{\log(X)}\\ &+ (4.86 g_0(q) + 2.93 + 2.83\varepsilon\log(X) + 5.17\varepsilon)\frac{\sqrt{q}}{\varphi_{\frac12}(q)}\frac{2^\varepsilon}{\sqrt{X}},\end{aligned}$$

g_0 *and* g_1 *are defined in* (2).

Notice that, on letting σ go to 1, the obtained result is comparable to but weaker than [37, Lemma 3.3].

Extension of the above two results to complex s such that $\Re s > 0$ is given in Theorem 1.5 and Theorem 1.6, respectively..

The main engine. In order to prove the above two results, we use identities. The usage of identities in this context goes back at least to Meissel in [21]. In [2, 1], M. Balazard extended the set of available identities by elaborating on [19] by R.A. MacLeod; this extension also led to [25], [26], [27] and [30]. Further flexibility was attained by F. Daval in [7] and it is proved in [31] that, within a set of conditions, he has obtained *all* the (non-trivial) identities linking $M(X)$ to $m(X)$; this work also extends to $\check{m}_q(X)$. In between H. Helfgott in [17] found a much more efficient way to handle the coprimality conditions when $\varepsilon = 0$. It is worth remarking that the usage of identities affects the error term estimations with some rigidity, a problem for instance addressed in [25] and by M. Haye Betah in [16].

Although the previous methods above had some flexibility, it stubbornly led to expressions involving $\mu(n)/n$ and never $\mu(n)/n^{1+\varepsilon}$. We solve this problem here and provide a wider set of identities that extend those of Daval.

Theorem 1.4 (The Identity Factory)**.** *Let* f *and* g *be two arithmetic functions. We define* $S_f(t) = \sum_{n\le t} f(n)$ *and* $S_{f\star g}(t)$ *similarly. Let* $h : (0,1] \to \mathbb{C}$ *be Lebesgue-integrable over every segment* $\subset (0,1]$ *and let* H *be function over* $[1,\infty)$ *that is absolutely continuous on every finite interval of* $[1,\infty)$*. When* $X \ge 1$*, we have*

$$\sum_{n\le X} f(n) H\left(\frac{X}{n}\right) - H(1) S_f(X) = \int_1^X S_{f\star g}\Big(\frac{X}{t}\Big) h\left(\frac{1}{t}\right)\frac{dt}{t} + \int_1^X S_f\Big(\frac{X}{t}\Big)\bigg(H'(t) - \frac{1}{t}\sum_{n\le t} g(n) h\left(\frac{n}{t}\right)\bigg) dt.$$

In Section 2, we prove the above identity and describe several of its consequences.

Furthermore, we show below that Theorem 1.4 enables one to bound $\sum_{n\le X}\mu(n)/n^\sigma$, $\sigma > 0$, in terms of $\{\sum_{n\le t}\mu(n)/n\}_t$, which in turn may be expressed in terms of $\{\sum_{n\le t}\mu(n)\}_t$, if required.

As a surprising upshot, the important question of deriving efficient estimates for $\sum_{p\le X}\frac{\log p}{p}$ from estimates of $\sum_{p\le X}\log p$ can be settled with the choices $f = \Lambda$, $g = 1$, $H = \mathrm{Id}$, since $f\star g = \log$, and still have several possibilities for choosing h ($h = \mathbb{1}$ leads already to interesting results). Independently and recently, M. Balazard used a similar approach for this question, which we include in the Appendix A. For further references, let us note that this question has also been the subject of [24], [20] by R. Mawia and its extension to primes in arithmetic progression is treated in [23] (see also [10] by H. Diamond and Wen-Bin Zhang for a similar question on Beurling numbers). In addition, the general question, regardless of the explicit aspect, is treated in [29].

A methodological remark. The way error terms are handled in analytic number theory is of utmost importance. While the Perron summation formula exhibits the use of complex analysis and multiplicative characters, the exponential sum method in essence relies on the Fourier expansion of the sawtooth function $x \mapsto \{x\} - \frac{1}{2}$, and thus uses complex analysis in the additive world. In these series of articles, our approach is to use instead real analysis and to handle the arising error terms directly by absolute value bounds. In the language of Theorem 1.4, it corresponds to the terms $H'(t) - \frac{1}{t}\sum_{n\le t} g(n)h\left(\frac{n}{t}\right)$, where we are selecting either $g(n) = 1$ or $g(n) = (-1)^{n+1}$. Precisely, one of the novelties of this article is the usage of $g(n) = (-1)^{n+1}$ rather than solely of $g(n) = 1$; this will become clearer in Appendix A.

Extension to complex parameters.

Theorem 1.5. *Let $X \ge 1$, $q \in \mathbb{Z}_{>0}$ and $s \in \mathbb{C}$ such that $\zeta(s) \neq 0$ and $\Re s = \sigma \ge \sigma_0 > 0$. Then we have*

$$\left|\sum_{\substack{d\le X,\\(n,q)=1}} \frac{\mu(n)}{n^s} - \frac{m_q(X)}{X^{s-1}} - \frac{q^s}{\varphi_s(q)}\frac{1}{\zeta(s)}\right| \le \frac{\sigma+|s|}{\sigma|c(s)\zeta(s)|X^\sigma}\int_1^X |m_q(t)|dt + \frac{|c(s)| + 2^{\sigma_0}e(s)}{|c(s)\zeta(s)|X^{\sigma_0}}\frac{q^{\sigma-\sigma_0}}{\varphi_{\sigma-\sigma_0}(q)},$$

where $c(s) = \frac{1-2^{1-s}}{s-1}$ and $e(s) = 2^{1-\sigma}(1 + 2^{|s-1|-1}|s-1|\log 2)\log 2$. Notice that $c(1) = \log 2$. Furthermore, when $s = \sigma$ is real, the value $(\sigma + |s|)/\sigma$ may be replaced by 1.

Here is the counterpart concerning $\check{m}_q(X;s)$.

Theorem 1.6. *Let $X \ge 1$, $q \in \mathbb{Z}_{>0}$ and $s \in \mathbb{C}$ such that $\zeta(s) \neq 0$ and $\Re s = \sigma > \sigma_0 > 0$. Then we have*

$$\left|\sum_{\substack{n\le X,\\(n,q)=1}} \frac{\mu(n)}{n^s}\log\left(\frac{X}{n}\right) - \frac{q^s}{\varphi_s(q)}\left(\frac{\log X}{\zeta(s)} - \frac{\zeta'(s)}{\zeta^2(s)} - \frac{1}{\zeta(s)}\sum_{p|q}\frac{\log(p)}{p^s-1}\right)\right| \le \frac{\Xi_1(s)}{X^\sigma}\int_1^X |m_q(t)|dt + \frac{\Xi_2(X;s,\sigma_0)}{X^{\sigma_0}}\frac{q^{\sigma-\sigma_0}}{\varphi_{\sigma-\sigma_0}(q)},$$

where

$$\Xi_1(s) = \frac{(\sigma+|s|)((\sigma+|s-1|)|C(s)| + \sigma|s-1||C'(s)|)}{\sigma^2 C(s)^2}, \tag{3}$$

$$\Xi_2(X,\sigma_0,s) = \frac{2^{\sigma_0}\left(\log X + \delta\left(\frac{X}{2},\sigma_0\right)\max\left\{\log\left(\frac{X}{2}\right),\frac{1}{\sigma_0}\right\}\right)}{|\zeta(s)|} + \frac{2^{\sigma_0}\log 2}{|c(s)\zeta(s)|} + \frac{2^{\sigma_0}}{|\zeta(s)|}\left|\frac{C'(s)}{C(s)} - \frac{1}{(s-1)} - \frac{\zeta'(s)}{\zeta(s)}\right|$$

$$+\frac{2^{\sigma_0}}{|\zeta(s)|}\left|\frac{C'(s)}{C(s)} - \frac{1}{(s-1)}\right| + 2^{\sigma_0}\frac{e(s)}{|C(s)|}\left|\frac{1}{C(s)} + \frac{(s-1)C'(s)}{C(s)}\right| \tag{4}$$

Here, we have $C(s) = (1-2^{1-s})\zeta(s)$, $\delta(\frac{X}{2},\sigma_0) = 1 + \mathbb{1}_{\log(X/2)<1/\sigma_0}$, *and* $c(s)$, $e(s)$ *are defined as in Theorem 1.5.*

The integral of $|m_q(t)|$ appearing in the above two results is treated in Lemma 7.2. Moreover, the case $q = 1$ and $s = \sigma > 1$ is particularly significant.

Theorem 1.7. *For any* $\sigma \in [1, 1.04]$, *we have the following estimation*

$$\left|\sum_{n\leq X}\frac{\mu(n)}{n^\sigma}\log\left(\frac{X}{n}\right) - \frac{\log X}{\zeta(\sigma)} + \frac{\zeta'(\sigma)}{\zeta^2(\sigma)}\right|$$
$$\leq \frac{15.5 + 3.11\varepsilon\log X}{X^{\sigma-\frac{1}{2}}}, \qquad \text{if } 15 \leq X \leq 10^{14},$$
$$\leq \frac{0.043}{X^{\sigma-1}\log X}, \qquad \text{if } X \geq 10^{14}.$$

2. The Identity Factory, proof of Theorem 1.4

Proof of Theorem 1.4. On the one hand, by the local absolute continuity of H, H has a derivate almost everywhere, which is Lebesgue-integrable. Thus, by integration by parts, we obtain

$$\int_1^X S_f\Big(\frac{X}{t}\Big)H'(t)dt = \sum_{n\leq X} f(n)H\left(\frac{X}{n}\right) - H(1)S_f(X).$$

On the other hand, we have

$$\int_1^X S_f\Big(\frac{X}{t}\Big)\frac{1}{t}\sum_{n\leq t} g(n)h\left(\frac{n}{t}\right)dt = \sum_{n\leq X} g(n)\int_n^X S_f\Big(\frac{x}{t}\Big)h\left(\frac{n}{t}\right)\frac{dt}{t}$$
$$= \sum_{n\leq X} g(n)\int_1^{\frac{X}{n}} S_f\left(\frac{\frac{X}{n}}{t}\right)h\left(\frac{1}{t}\right)\frac{dt}{t}$$
$$= \int_1^X \sum_{n\leq X/t} g(n)S_f\left(\frac{\frac{X}{t}}{n}\right)h\left(\frac{1}{t}\right)\frac{dt}{t},$$

where we have used summation by parts, a change of variables and then Fubini's theorem, respectively. The proof follows on noticing the identity

$$\sum_{n\leq X/t} g(n)\sum_{m\leq \frac{X}{tn}} f(m) = \sum_{\ell\leq X/t}(f\star g)(\ell)$$

valid for any real number $X \geq 1$. □

The case $H = \mathrm{Id}$, $g = \mathbb{1}$ and $f = \mu$ yields the following statement, which is the initial result in [7].

Corollary 2.1. *Let* $h : (0,1] \to \mathbb{C}$ *be any Lebesgue-integrable function over every segment of* $(0,1]$. *When* $X \geq 1$, *we have*

$$m(X) - \frac{M(X)}{X} = \frac{1}{X}\int_{\frac{1}{X}}^1 \frac{h(t)}{t}dt - \frac{1}{X}\int_1^X M\left(\frac{X}{t}\right)\left(1 - \frac{1}{t}\sum_{n\leq t} h\left(\frac{n}{t}\right)\right)dt.$$

Although not required, it is better to normalize h by imposing the condition $\int_0^1 h(t)dt = 1$.

In [30, Theorem 7.4], it is proven that one recovers all the (regular enough) identities linking $m(X)$ and $M(X)$ with the result of Corollary 2.1, so that the above is not only a curiosity that is included in a further stream of identities.

For example, on selecting $h = \mathbb{1}$, we recover the Meissel identity

$$\sum_{n\le X}\mu(n)\left\{\frac{X}{n}\right\} = -1 + Xm(X),$$

and on selecting $h = 2\cdot\mathrm{Id}$, we obtain the identity of MacLeod

$$\sum_{n\le X}\mu(n)\frac{\left\{\frac{X}{n}\right\}^2 - \left\{\frac{X}{n}\right\}}{\frac{X}{n}} = Xm(X) - M(X) - 2 + \frac{2}{X},$$

both valid for any $X \ge 1$.

The functional transform that, to a function h, associates the function $X > 0 \mapsto \int_0^1 h(t)dt - \frac{1}{X}\sum_{n\le X} h(\frac{n}{X})$ is closely related to a transform introduced by Ch. Müntz in [22]. This is also discussed by E. Titchmarsh in [35, Section 2.11] and more information can be found in [36] by S. Yakubovich.

Corollary 2.2. *With the same notation as the one of Corollary 2.1, we have, for $X \ge 1$,*

$$\sum_{n\le X}\frac{\mu(n)}{n}\log\left(\frac{X}{n}\right) + \gamma\Big(\sum_{n\le X}\frac{\mu(n)}{n} - \frac{M(X)}{X}\Big) \\ = 1 - \frac{1}{X} + \frac{1}{X}\int_1^X M\left(\frac{X}{t}\right)\Big(\log t + \gamma + \frac{1}{t} - \sum_{n\le t}\frac{1}{n}\Big)dt,$$

where γ is Euler's constant.

Proof. We select $H = \mathrm{Id}\cdot\log - \log + \gamma\cdot\mathrm{Id}$, $h = \mathrm{Id}^{-1}$, $f = \mu$ and $g = \mathbb{1}$ in Theorem 1.4. We readily derive

$$\sum_{n\le X}\frac{\mu(n)}{n}\log\left(\frac{X}{n}\right) - \sum_{n\le X}\frac{\mu(n)}{X}\log\left(\frac{X}{n}\right) + \gamma\Big(\sum_{n\le X}\frac{\mu(n)}{n} - \frac{M(X)}{X}\Big) \\ = 1 - \frac{1}{X} + \frac{1}{X}\int_1^X M\left(\frac{X}{t}\right)\Big(\log t + \gamma - \sum_{n\le t}\frac{1}{n}\Big)dt.$$

The result is obtained by observing that, by summation by parts, we have

$$\sum_{n\le X}\mu(n)\log\left(\frac{X}{n}\right) = -\int_1^X M\left(\frac{X}{t}\right)\frac{dt}{t}.$$

□

Such identities and several others have been put to use in [27]. Here is a novel corollary.

Corollary 2.3. *For every $X \geq 1$, we have*

$$\sum_{n\leq X}\frac{\lambda(n)}{n} - \frac{1}{X}\sum_{n\leq X}\lambda(n) = \frac{2}{\sqrt{X}} - \frac{1}{X} - \frac{1}{X}\int_1^X \left\{\frac{X}{t}\right\}\frac{dt}{t} + \frac{1}{X}\int_1^X \sum_{n\leq X/t}\lambda(n)\{t\}\frac{dt}{t}.$$

Proof. Apply Theorem 1.4 with $H = \mathrm{Id}$, $h = \mathbb{1}$, $f = \lambda$ and $g = \mathbb{1}$. With this choice, observe that

$$S_{f\star g}\left(\frac{X}{t}\right) = \left[\frac{X}{t}\right].$$

□

3. Proof of Theorem 1.1

In [27, Corollary 1.10], we find the next lemma.

Lemma 3.1. *For any $q \in \mathbb{Z}_{>0}$ and any $X > 0$, we have*

$$0 \leq \sum_{\substack{n\leq X\\(n,q)=1}}\frac{\mu(n)}{n}\log\left(\frac{X}{n}\right) \leq 1.00303\frac{q}{\varphi(q)}\log X.$$

In [33, Prop. A.4, p. 126] by P. Srivasta, we find the following.

Lemma 3.2. *For any $q \in \mathbb{Z}_{>0}$, any $X > 0$, and any integer $k \geq 2$, we have*

$$0 \leq \sum_{\substack{n\leq X\\(n,q)=1}}\frac{\mu(n)}{n}\log^k\left(\frac{X}{n}\right) \leq k\frac{q}{\varphi(q)}\log^{k-1}(X)$$

Proof. The case $k = 2$ is proved in [27, Corollary 1.11]. The case $k \geq 3$ is readily deduced from this one by summation by parts. □

Proof of Theorem 1.1. On using the expansion

$$\left(\frac{X}{n}\right)^{\varepsilon} = \exp(\varepsilon)\log\left(\frac{X}{n}\right) = \sum_{\ell\geq 0}\frac{\varepsilon^\ell}{\ell!}\log^\ell\left(\frac{X}{n}\right),$$

we deduce that

$$X^\varepsilon\sum_{\substack{n\leq X\\(n,q)=1}}\frac{\mu(n)}{n^{1+\varepsilon}}\log^k\left(\frac{X}{n}\right) = \sum_{\ell\geq 0}\frac{\varepsilon^\ell}{\ell!}\sum_{\substack{n\leq X\\(n,q)=1}}\frac{\mu(n)}{n}\log^{k+\ell}\left(\frac{X}{n}\right).$$

When $k \geq 1$, by Lemma 3.2, this implies that

$$0 \leq X^\varepsilon\sum_{\substack{n\leq X\\(n,q)=1}}\frac{\mu(n)}{n^{1+\varepsilon}}\log^k\left(\frac{X}{n}\right) \leq 1.00303\frac{q}{\varphi(q)}\sum_{\ell\geq 0}\frac{\varepsilon^\ell\log^{k+\ell-1}(X)(k+\ell)}{\ell!}$$

$$\leq 1.00303\frac{q}{\varphi(q)}\Big(k\log^{k-1}(X)\exp(\varepsilon\log X) + \varepsilon\log^k(X)\exp(\varepsilon\log X)\Big),$$

from which the theorem follows. □

4. Evaluating $m_q(X;s)$, proof of Theorem 1.5

Lemma 4.1. *Let $X>0$ and $q\in\mathbb{Z}_{>0}$. Consider the arithmetic function $g_1 : n\in\mathbb{Z}_{>0}\mapsto(-1)^{n+1}$. We have the following identities*

$$(i)\qquad \sum_{n\le X} g_1(n) = \mathbb{1}_{\{([X],2)=1\}}(X),$$

$$(ii)\qquad G_1(n)=\sum_{\substack{d_1d_2=n\\(d_1,q)=1}}\mu(d_1)g_1(d_2)=\mathbb{1}_{\{n|q^\infty\}}(n)-2\cdot\mathbb{1}_{\{2|n,\ \frac{n}{2}|q^\infty\}}(n),$$

$$(iii)\qquad \sum_{n\le X}\frac{G_1(n)}{n}=\sum_{\substack{\frac{X}{2}<\ell\le X\\ \ell|q^\infty}}\frac{1}{\ell}.$$

Proof. The definition of g_1 gives (i). On the other hand, the Dirichlet series of g_1 is $(1-2^{1-s})\zeta(s)$ while the one of the $\mathbb{1}_{\{(n,q)=1\}}(n)\mu(n)$ is $\prod_{p\nmid q}(1-p^{-s})$. Their product satisfies

$$(1-2^{1-s})\prod_{p|q}(1-p^{-s})^{-1}=(1-2^{1-s})\sum_{\ell|q^\infty}\frac{1}{\ell^s}=\sum_{\ell|q^\infty}\frac{1}{\ell^s}-\sum_{\substack{2|\ell\\ \frac{\ell}{2}|q^\infty}}\frac{2}{\ell^s},$$

which gives the convolution identity (ii). We readily obtain (iii) from (ii). □

Lemma 4.2. *When $\Re s=\sigma>0$ and $X>0$, we have*

$$\sum_{n\le X}\frac{g_1(n)}{n^s}=\sum_{n\le X}\frac{(-1)^{n+1}}{n^s}=C(s)+O^*\Big(\frac{\sigma+|s|}{\sigma X^\sigma}\Big).$$

with $C(s)=(1-2^{1-s})\zeta(s)$, $C(1)=\log 2$. When $s=\sigma$ is real, the error term is non-positive and reduces to $O^*(1/X^\sigma)$.

Proof. When s is real non-negative, the sum is alternating hence the error bound in this case. Otherwise, observe that

$$\sum_{n\le X}\frac{(-1)^{n+1}}{n^s}=\sum_{n\le X}(-1)^{n+1}\bigg(\frac{1}{X^s}+s\int_n^X\frac{dt}{t^{s+1}}\bigg)$$

$$=\frac{\mathbb{1}_{\{([X],2)=1\}}(X)}{X^s}+s\int_1^X\frac{\mathbb{1}_{\{([t],2)=1\}}(t)}{t^{s+1}}dt,\tag{5}$$

where we used Lemma 4.1 (i) and then Fubini's theorem. By letting $X\to\infty$, we can write (5) as $C(s)+E(s)$, where

$$C(s)=\int_1^\infty\frac{\mathbb{1}_{\{([t],2)=1\}}(t)}{t^{s+1}}dt=\sum_n\frac{(-1)^{n+1}}{n^s}=(1-2^{1-s})\zeta(s),$$

$$|E(s)|=\bigg|\frac{\mathbb{1}_{\{([X],2)=1\}}(X)}{X^s}-s\int_X^\infty\frac{\mathbb{1}_{\{([t],2)=1\}}(t)}{t^{s+1}}dt\bigg|\le\frac{\sigma+|s|}{\sigma X^\sigma}.$$

Finally, observe that, for any $z\ge1$, the function $C_z : s\in\mathbb{C}\mapsto(1-z^{1-s})\zeta(s)$ is entire, satisfying $C_z(1)=\log z$. □

Lemma 4.3. *Let $X > 0$ and $s \in \mathbb{C}$. If $\Re s = \sigma \geq \sigma_0 > 0$, then*

$$(i) \left|\sum_{\substack{\ell > X,\\ \ell | q^\infty}} \frac{1}{\ell^s}\right| \leq \frac{1}{X^{\sigma_0}} \frac{q^{\sigma-\sigma_0}}{\varphi_{\sigma-\sigma_0}(q)}$$

$$(ii) \left|\sum_{\substack{\frac{X}{2} < \ell \leq X\\ \ell | q^\infty}} \frac{2^{1-s} - \left(\frac{X}{\ell}\right)^{1-s}}{s-1} \frac{1}{\ell^s}\right| \leq \frac{2^{\sigma_0} e(s)}{X^{\sigma_0}} \frac{q^{\sigma-\sigma_0}}{\varphi_{\sigma-\sigma_0}(q)},$$

where $e(s) = 2^{1-\sigma}(1 + 2^{|s-1|-1}|s-1|\log 2)\log 2$.

Proof. Observe that $\sum_{\ell | q^\infty} \frac{1}{\ell^\omega}$ converges to $\frac{q^\omega}{\varphi_\omega(q)}$ for any $\omega \in \mathbb{C}$ such that $\Re\omega > 0$. Thus, as $\sigma - \sigma_0 > 0$,

$$\left|\sum_{\substack{\ell > X,\\ \ell | q^\infty}} \frac{1}{\ell^s}\right| \leq \frac{1}{X^{\sigma_0}} \sum_{\ell | q^\infty} \frac{1}{\ell^{\sigma-\sigma_0}} = \frac{1}{X^{\sigma_0}} \frac{q^{\sigma-\sigma_0}}{\varphi_{\sigma-\sigma_0}(q)},$$

whence (i).

On the other hand, we use $2^{1-s} - (X/\ell)^{1-s} = 2^{1-s}(1 - (X/(2\ell))^{1-s}$. Recall that, for any $z \in [0,1]$, we have $(z^{1-s} - 1)(1-s)^{-1} = \log z + O^*(2^{-1}|s-1|z^{-|s-1|}\log^2 z)$. Therefore, by taking $z = \frac{X}{2\ell} \leq 1$ and using that $\sigma_0 > 0$, we derive (ii). □

Proof of Theorem 1.5. We use Theorem 1.4 with $h : t \in (0,1] \mapsto (s-1)t^{1-s}C(s)^{-1}$, $H : t \in [1,\infty) \mapsto t^{s-1}$, $g = \frac{g_1}{\mathrm{Id}} : n \in \mathbb{Z}_{>0} \mapsto \frac{(-1)^{n+1}}{n}$ and $f = \mathbb{1}_{\{(\cdot,q)=1\}} \frac{\mu}{\mathrm{Id}}$, where $C(s) = (1-2^{s-1})\zeta(s)$. As $\zeta(s) \neq 0$, h is well-defined.

Moreover, by Lemma 4.1 (ii), we have $f \star g = \frac{G_1}{\mathrm{Id}}$. Therefore, by Lemma 4.1 (iii) and Lemma 4.2, we may express

$$(6) \quad \sum_{\substack{n \leq X,\\ (n,q)=1}} \frac{\mu(n)}{n^s} - \frac{m_q(X)}{X^{s-1}} = \frac{(s-1)}{C(s)} M_1(X;q,s,\sigma_0) + O^*\left(\frac{R_1(X;q,s)}{X^\sigma}\right),$$

where, by Fubini's theorem, considering the holomorphic function $c : s \in \mathbb{C} \mapsto \frac{1-2^{1-s}}{s-1}$, $c(1) = \log 2$, and recalling Lemma 4.3, we have

$$\begin{aligned}
M_1(X; q, s, \sigma_0) &= \int_1^X \Bigg(\sum_{\substack{\frac{t}{2} < \ell \le t \\ \ell | q^\infty}} \frac{1}{\ell} \Bigg) \frac{dt}{t^s} = \sum_{\substack{\ell \le X, \\ \ell | q^\infty}} \frac{1}{\ell} \int_\ell^{\min(2\ell, X)} \frac{dt}{t^s} \\
&= \frac{1}{s-1} \sum_{\substack{\ell \le X, \\ \ell | q^\infty}} \frac{1}{\ell^s} \left(1 - \min\left(2, \frac{X}{\ell}\right)^{1-s} \right) \\
&= \frac{1}{s-1} \sum_{\substack{\ell \le \frac{X}{2}, \\ \ell | q^\infty}} \frac{1 - 2^{1-s}}{\ell^s} + \frac{1}{s-1} \sum_{\substack{\frac{X}{2} < \ell \le X, \\ \ell | q^\infty}} \frac{1}{\ell^s} \left(1 - \left(\frac{X}{\ell}\right)^{1-s} \right) \\
&= \frac{1}{s-1} \sum_{\substack{\ell \le X, \\ \ell | q^\infty}} \frac{1 - 2^{1-s}}{\ell^s} + \frac{1}{s-1} \sum_{\substack{\frac{X}{2} < \ell \le X, \\ \ell | q^\infty}} \frac{1}{\ell^s} \left(2^{1-s} - \left(\frac{X}{\ell}\right)^{1-s} \right) \\
&= c(s) \frac{q^s}{\varphi_s(q)} + O^*\left(\frac{|c(s)| + 2^{\sigma_0} e(s)}{X^{\sigma_0}} \frac{q^{\sigma - \sigma_0}}{\varphi_{\sigma - \sigma_0}(q)} \right), \qquad (7)
\end{aligned}$$

and where, by using Lemma 4.2,

$$\begin{aligned}
R_1(X; q, s) &= X \int_1^X m_q\left(\frac{X}{t}\right) (s-1) t^{s-2} \left(1 - \frac{1}{C(s)} \sum_{n \le t} \frac{g_1(n)}{n} \right) dt \\
&= X \int_1^X m_q\left(\frac{X}{t}\right) O^*\Big(\frac{(\sigma + |s|)|s-1|}{\sigma |C(s)|} \Big) \frac{dt}{t^2} \\
&= O^*\left(\frac{(\sigma + |s|)|s-1|}{\sigma |C(s)|} \int_1^X |m_q(t)| dt \right). \qquad (8)
\end{aligned}$$

Finally, by estimations (7), (8) and by observing that $\frac{s-1}{C(s)} = \frac{1}{c(s)\zeta(s)}$, we deduce from (6) that

$$\left| \sum_{\substack{n \le X, \\ (n,q)=1}} \frac{\mu(n)}{n^s} - \frac{m_q(X)}{X^{s-1}} - \frac{q^s}{\varphi_s(q)} \frac{1}{\zeta(s)} \right| \le \frac{R_1(X; q, s)}{X^\sigma} + \frac{R_2(X; s, \sigma_0)}{X^{\sigma_0}} \frac{q^{\sigma - \sigma_0}}{\varphi_{\sigma - \sigma_0}(q)},$$

where

$$R_2(X; s, \sigma_0) = \frac{|c(s)| + 2^{\sigma_0} e(s)}{|c(s)\zeta(s)|}. \qquad (9)$$

□

5. Evaluating $\check{m}_q(X; s)$, proof of Theorem 1.6

Lemma 5.1. *When $\Re s = \sigma > 0$ and $X > 0$, we have*

$$\sum_{n \le X} \frac{(-1)^{n+1}}{n^s} \log\left(\frac{X}{n}\right) = C(s) \log X + C'(s) + O^*\Big(\frac{\sigma + |s|}{\sigma^2 X^\sigma} \Big).$$

where $C'(s)$ is the derivative of $C(s) = (1-2^{1-s})\zeta(s)$ with respect to s. When $s=\sigma$ is real, the error term is $O^*(1/(e\sigma X^\sigma))$.

Proof. By recalling Lemma 4.1 (i) and using summation by parts, we observe that for any $Y>0$,

$$\sum_{n\le Y}\frac{(-1)^{n+1}}{n^s}\log\left(\frac{X}{n}\right) = \sum_{n\le Y}(-1)^{n+1}\left(\frac{\log\left(\frac{X}{Y}\right)}{Y^s} + \int_n^Y \frac{1+s\log\left(\frac{X}{t}\right)}{t^{s+1}}dt\right)$$

(10)

$$= \frac{\mathbb{1}_{([Y],2)=1}\log\left(\frac{X}{Y}\right)}{Y^s} + \int_{1^-}^Y \frac{\mathbb{1}_{([t],2)=1}\left(1+s\log\left(\frac{X}{t}\right)\right)}{t^{s+1}}dt$$

(11)

$$= \frac{\mathbb{1}_{([Y],2)=1}\log\left(\frac{X}{Y}\right)}{Y^s} + B(X,s) - \int_Y^\infty \frac{\mathbb{1}_{([t],2)=1}\left(1+s\log\left(\frac{X}{t}\right)\right)}{t^{s+1}}dt$$

where

$$B(X,s) = \int_{1^-}^\infty \frac{\mathbb{1}_{([t],2)=1}\left(1+s\log\left(\frac{X}{t}\right)\right)}{t^{s+1}}dt. \tag{12}$$

Suppose that $\Re s>1$, then we immediately see that

$$\sum_{n=1}^\infty \frac{(-1)^{n+1}}{n^s} = C(s), \quad -\sum_{n=1}^\infty \frac{(-1)^{n+1}\log n}{n^s} = C'(s).$$

Therefore, by letting $Y\to\infty$ in (10), we obtain

$$B(X,s) = C(s)\log X + C'(s). \tag{13}$$

As $s\mapsto B(X,s)$ is holomorphic for $\Re s>0$, by analytic continuation, (13) is valid for $\Re s>0$. Thereupon, by selecting $Y=X$ in (11), we obtain

$$\sum_{n\le X}\frac{(-1)^{n+1}}{n^s}\log\left(\frac{X}{n}\right) = C(s)\log X + C'(s) + O^*\left(\frac{\sigma+|s|}{\sigma^2X^\sigma}\right),$$

valid for $\Re s=\sigma>0$. Indeed, the error term is bounded by noticing that

$$\left|\int_X^\infty \frac{\mathbb{1}_{([t],2)=1}\left(1+s\log\left(\frac{X}{t}\right)\right)}{t^{s+1}}dt\right| \le \int_X^\infty \frac{1+|s|\log\left(\frac{t}{X}\right)}{t^{\sigma+1}}dt$$

$$= \left[-\frac{1}{\sigma t^\sigma} - \frac{|s|}{\sigma}\left(\frac{\log\left(\frac{t}{X}\right)}{t^\sigma} + \frac{1}{\sigma t^\sigma}\right)\right]_X^\infty = \frac{\sigma+|s|}{\sigma^2X^\sigma}.$$

When $s=\sigma$ is a real number, we see that

$$\sum_{n\le X}\frac{(-1)^{n+1}}{n^s}\log\left(\frac{X}{n}\right) - (C(s)\log X + C'(s)) = -\sum_{n>X}\frac{(-1)^{n+1}\log(n/X)}{n^\sigma}.$$

By computing the derivative of $n\mapsto \log(n/X)/n^\sigma$, we see that these terms increase up to a maximal value at $n/X = e^{1/\sigma}$ that is at most $1/(e\sigma X^\sigma)$ and decrease afterwards. As this sum is alternating, the error is of size at most this term. □

Lemma 5.2. *Let $X > 0$ and $s \in \mathbb{C}$. If $\Re s = \sigma > \sigma_0 > 0$, then*

$$\sum_{\substack{\ell \le X \\ \ell | q^\infty}} \frac{\log \ell}{\ell^s} = \frac{q^s}{\varphi_s(q)} \sum_{p|q} \frac{\log p}{p^s - 1} + O^*\left(\frac{\delta(X,\sigma_0)\max\{\log X, 1/\sigma_0\}}{X^{\sigma_0}} \frac{q^{\sigma-\sigma_0}}{\varphi_{\sigma-\sigma_0}(q)} \right),$$

where $\delta(X,\sigma_0) = 1 + \mathbb{1}_{\{\log X < \frac{1}{\sigma_0}\}}(X)$.

Proof. As $\sum_{\ell|q^\infty} \ell^{-\omega} = \frac{q^\omega}{\varphi_\omega(q)}$ for any $\omega \in \mathbb{C}$ such that $\Re\omega > 0$, we may differentiate that equality with respect to ω. Thus,

$$-\sum_{\ell|q^\infty} \frac{\log \ell}{\ell^\omega} = \frac{\mathrm{d}}{\mathrm{d}\omega}\left(\prod_{p|q}\left(1 - \frac{1}{p^\omega}\right)^{-1} \right) = -\frac{q^\omega}{\varphi_\omega(q)} \sum_{p|q} \frac{\log p}{p^\omega - 1}. \tag{14}$$

Therefore

$$\sum_{\substack{\ell \le X \\ \ell | q^\infty}} \frac{\log \ell}{\ell^s} = \frac{q^s}{\varphi_s(q)} \sum_{p|q} \frac{\log p}{p^s - 1} - \sum_{\substack{\ell > X, \\ \ell | q^\infty}} \frac{\log \ell}{\ell^s}.$$

Furthermore, note that $t > 0 \mapsto (\log t)t^{-\sigma_0}$ is decreasing for $t \ge e^{\frac{1}{\sigma_0}}$. Thus, if $X \ge e^{\frac{1}{\sigma_0}}$, as $\sigma - \sigma_0 > 0$, we have

$$\left| \sum_{\substack{\ell > X \\ \ell | q^\infty}} \frac{\log \ell}{\ell^s} \right| \le \frac{\log X}{X^{\sigma_0}} \sum_{\ell|q^\infty} \frac{1}{\ell^{\sigma-\sigma_0}} = \frac{\max\{\log X, \frac{1}{\sigma_0}\}}{X^{\sigma_0}} \frac{q^{\sigma-\sigma_0}}{\varphi_{\sigma-\sigma_0}(q)};$$

whereas, if $X < e^{\frac{1}{\sigma_0}}$, we have

$$\begin{aligned} \left| \sum_{\substack{\ell > X \\ \ell | q^\infty}} \frac{\log \ell}{\ell^s} \right| &\le \sum_{\substack{\ell > e^{\frac{1}{\sigma_0}} \\ \ell | q^\infty}} \frac{\log \ell}{\ell^\sigma} + \sum_{\substack{X < \ell \le e^{\frac{1}{\sigma_0}} \\ \ell | q^\infty}} \frac{\log \ell}{\ell^\sigma} \le \left(\frac{1}{\sigma_0 e} + \frac{1}{\sigma_0 X^{\sigma_0}} \right) \sum_{\ell|q^\infty} \frac{1}{\ell^{\sigma-\sigma_0}} \\ &\le \frac{2}{\sigma_0 X^{\sigma_0}} \frac{q^{\sigma-\sigma_0}}{\varphi_{\sigma-\sigma_0}(q)} = \frac{2\max\{\log X, \frac{1}{\sigma_0}\}}{X^{\sigma_0}} \frac{q^{\sigma-\sigma_0}}{\varphi_{\sigma-\sigma_0}(q)}, \end{aligned}$$

whence the result. □

Lemma 5.3. *Let $q \in \mathbb{Z}_{>0}$. For any $X \ge 1$ we have the following estimation*

$$\int_1^X \left(\sum_{\substack{\frac{t}{2} < \ell \le t \\ \ell | q^\infty}} \frac{1}{\ell} \right) \log\left(\frac{X}{t}\right) \frac{dt}{t^s} =$$

$$\frac{q^s}{\varphi_s(q)}\left(c(s)\log X + c'(s) - c(s)\sum_{p|q} \frac{\log(p)}{p^s - 1} \right) + O^*\left(\frac{R_3(X; s, \sigma_0)}{X^{\sigma_0}} \frac{q^{\sigma-\sigma_0}}{\varphi_{\sigma-\sigma_0}(q)} \right),$$

where $c'(s)$ is the derivate of $c(s) = \frac{1-2^{1-s}}{s-1}$ with respect to s and

$$\begin{aligned} R_3(X; s, \sigma_0) = {} & 2^{\sigma_0}\log 2 \\ & + 2^{\sigma_0}\left(|c(s)\log X + c'(s)| + |c(s)|\delta\Big(\frac{X}{2}, \sigma_0\Big) \max\left\{ \log\Big(\frac{X}{2}\Big), \frac{1}{\sigma_0} \right\} \right). \end{aligned} \tag{15}$$

Proof. Suppose first that $s \neq 1$. By Fubini's theorem, we derive

$$\int_1^X \Bigg(\sum_{\substack{\frac{t}{2}<\ell\leq t\\ \ell|q^\infty}} \frac{1}{\ell}\Bigg) \log\left(\frac{X}{t}\right)\frac{dt}{t^s} = \sum_{\substack{\ell\leq X\\ \ell|q^\infty}} \frac{1}{\ell}\int_\ell^{\min\{2\ell,X\}} \log\left(\frac{X}{t}\right)\frac{dt}{t^s}$$

$$= \sum_{\substack{\ell\leq \frac{X}{2}\\ \ell|q^\infty}} \frac{1}{\ell}\int_\ell^{2\ell} \log\left(\frac{X}{t}\right)\frac{dt}{t^s} + \sum_{\substack{\frac{X}{2}<\ell\leq X\\ \ell|q^\infty}} \frac{1}{\ell}\int_\ell^{X} \log\left(\frac{X}{t}\right)\frac{dt}{t^s}. \tag{16}$$

By Lemma 4.3 (i), we have

$$\left|\sum_{\substack{\frac{X}{2}<\ell\leq X\\ \ell|q^\infty}} \frac{1}{\ell}\int_\ell^{X} \log\left(\frac{X}{t}\right)\frac{dt}{t^s}\right| \leq \sum_{\substack{\frac{X}{2}<\ell\leq X\\ \ell|q^\infty}} \frac{1}{\ell}\int_\ell^{2\ell} \log(2)\frac{dt}{t^\sigma}$$

$$\leq \sum_{\substack{\frac{X}{2}<\ell\\ \ell|q^\infty}} \frac{\log 2}{\ell^\sigma} \leq \frac{\log(2)2^{\sigma_0}}{X^{\sigma_0}}\frac{q^{\sigma-\sigma_0}}{\varphi_{\sigma-\sigma_0}(q)}. \tag{17}$$

On the other hand, by observing that $c'(s) = \frac{1}{s-1}(-c(s)+\log(2)2^{1-s})$ and using Lemma 5.2, we have

$$\sum_{\substack{\ell\leq \frac{X}{2}\\ \ell|q^\infty}} \frac{1}{\ell}\int_\ell^{2\ell} \log\left(\frac{X}{t}\right)\frac{dt}{t^s} = \sum_{\substack{\ell\leq \frac{X}{2}\\ \ell|q^\infty}} \frac{1}{\ell}\left(\frac{-c(s)+\log(2)2^{1-s}}{(s-1)}\frac{1}{\ell^{s-1}} + c(s)\frac{\log\left(\frac{X}{\ell}\right)}{\ell^{s-1}}\right)$$

$$= \sum_{\substack{\ell|q^\infty,\\ \ell\leq \frac{X}{2}}} \frac{c'(s)+c(s)\log\left(\frac{X}{\ell}\right)}{\ell^s} = \frac{q^s}{\varphi_s(q)}\left(c(s)\log X + c'(s) - c(s)\sum_{p|q}\frac{\log(p)}{p^s-1}\right)$$

$$+ O^*\left(\frac{|c(s)\log X + c'(s)| + |c(s)|\delta(\frac{X}{2},\sigma_0)\max\{\log(\frac{X}{2}),\frac{1}{\sigma_0}\}}{\left(\frac{X}{2}\right)^{\sigma_0}}\frac{q^{\sigma-\sigma_0}}{\varphi_{\sigma-\sigma_0}(q)}\right). \tag{18}$$

Finally, if $s=1$, as $c(1)=\log 2$, $c'(1) = -\frac{\log^2(2)}{2}$, we derive

$$\sum_{\substack{\ell\leq \frac{X}{2}\\ \ell|q^\infty}} \frac{1}{\ell}\int_\ell^{2\ell} \log\left(\frac{X}{t}\right)\frac{dt}{t} = \sum_{\substack{\ell|q^\infty,\\ \ell\leq \frac{X}{2}}} \frac{1}{\ell}\left(\log(2)\log\left(\frac{X}{\ell}\right) - \frac{\log^2(2)}{2}\right)$$

$$= \sum_{\substack{\ell|q^\infty,\\ \ell\leq \frac{X}{2}}} \frac{c'(1)+c(1)\log\left(\frac{X}{\ell}\right)}{\ell},$$

so that the estimation (18) holds for any s with $\Re s = \sigma > \sigma_0 > 0$. The result is concluded by adding (17) to (18). □

Proof of Theorem 1.6. Let $K_1, K_2 : \mathbb{C} \to \mathbb{C}$ be any two functions. Let $h = K_1(s)\mathrm{Id}^{1-s}\cdot\log + K_2(s)\mathrm{Id}^{1-s}$, $H = \mathrm{Id}^{s-1}\cdot\log$, $g : n \in \mathbb{Z} \mapsto \frac{(-1)^{n+1}}{n}$ and

$f = \mathbb{1}_{(\cdot,q)=1}\frac{\mu}{\mathrm{Id}}$. Then, by Lemma 4.2 and Lemma 5.1,

$$\begin{aligned}
&H'(t) - \frac{1}{t}\sum_{n\le t} g_1(n)h\left(\frac{n}{t}\right)\\
&= \frac{1+(s-1)\log t}{t^{2-s}} + \frac{K_1(s)}{t^{2-s}}\sum_{n\le t}\frac{(-1)^{n+1}}{n^s}\log\left(\frac{t}{n}\right) - \frac{K_2(s)}{t^{2-s}}\sum_{n\le t}\frac{(-1)^{n+1}}{n^s}\\
&= \frac{1+(s-1)\log t}{t^{2-s}} + \frac{K_1(s)}{t^{2-s}}\big(C(s)\log(t)+C'(s)\big) - \frac{K_2(s)}{t^{2-s}}C(s)\\
&\qquad + O^*\left(\frac{(|K_1(s)|+\sigma|K_2(s)|)(\sigma+|s|)}{\sigma^2t^2}\right).
\end{aligned}\tag{19}$$

By selecting

$$K_1(s) = -\frac{s-1}{C(s)},\quad K_2(s) = \frac{C(s)+(s-1)C'(s)}{C(s)^2},\tag{20}$$

the main term in (19) vanishes.

Moreover, by Lemma 4.1 (ii), we have $f\star g = \frac{G_1}{\mathrm{Id}}$. Therefore, by Lemma 4.1 (iii) and Theorem 1.4, we obtain

$$\begin{aligned}
\sum_{\substack{n\le X,\\(n,q)=1}}\frac{\mu(n)}{n^s}\log\left(\frac{X}{n}\right) &= \int_1^X\Bigg(\sum_{\substack{\frac{t}{2}<\ell\le t\\ \ell|q^\infty}}\frac{1}{\ell}\Bigg)\left(-K_1(s)\log\left(\frac{X}{t}\right)+K_2(s)\right)\frac{dt}{t^s}\\
&\quad + X^{1-s}\int_1^X m_q\left(\frac{X}{t}\right)O^*\left(\frac{(|K_1(s)|+\sigma|K_2(s)|)(\sigma+|s|)}{\sigma^2t^2}\right)dt.
\end{aligned}\tag{21}$$

The first integral above can be handled with the help of Lemma 5.3. Likewise, we can handle the second integral by recalling estimation (7). Hence

$$\begin{aligned}
\sum_{\substack{n\le X,\\(n,q)=1}}\frac{\mu(n)}{n^s}\log\left(\frac{X}{n}\right) &= -\frac{q^s}{\varphi_s(q)}K_1(s)\Bigg(c(s)\log X + c'(s) - c(s)\sum_{p|q}\frac{\log p}{p^s-1}\Bigg)\\
&\quad + \frac{q^s}{\varphi_s(q)}K_2(s)c(s) + O^*\left(\frac{R_4(X;q,s)}{X^\sigma} + \frac{R_5(X;s,\sigma_0)}{X^{\sigma_0}}\frac{q^{\sigma-\sigma_0}}{\varphi_{\sigma-\sigma_0}(q)}\right),
\end{aligned}\tag{22}$$

where

$$\begin{aligned}
R_4(X;q,s) &= X\int_1^X m_q\left(\frac{X}{t}\right)O^*\left(\frac{(|K_1(s)|+\sigma|K_2(s)|)(\sigma+|s|)}{\sigma^2t^2}\right)dt\\
&= O^*\left(\frac{(|K_1(s)|+\sigma|K_2(s)|)(\sigma+|s|)}{\sigma^2}\int_1^X|m_q(t)|dt\right)\\
&= O^*\left(\frac{(\sigma+|s|)((\sigma+|s-1|)|C(s)|+\sigma|s-1||C'(s)|)}{\sigma^2C(s)^2}\int_1^X|m_q(t)|dt\right)
\end{aligned}\tag{23}$$

Note that (23) allows us to define $\Xi_1(s)$ as in the statement.

Moreover, on recalling the definition of $R_2(X;q,s)$ and $R_3(X;q,s)$,

$$R_5(X;s,\sigma_0) = |K_2(s)c(s)\zeta(s)|R_2(X;s,\sigma_0) + |K_1(s)|R_3(X;s,\sigma_0).\tag{24}$$

Now, by Eq. (20), we immediately check that $-K_1(s)c(s) = \zeta(s)^{-1}$. Furthermore, by writing $c(s) = \frac{C(s)}{(s-1)\zeta(s)}$, we observe that

$$-K_1(s)c'(s) + K_2(s)c(s) = \frac{1}{C(s)\zeta(s)}\left(C'(s) - \frac{C(s)}{s-1} - \frac{C(s)\zeta'(s)}{\zeta(s)}\right)$$
$$+\frac{C(s)-(s-1)C'(s)}{C(s)}\frac{1}{(s-1)\zeta(s)} = -\frac{\zeta'(s)}{\zeta^2(s)}, \tag{25}$$

and that $R_5(X;s,\sigma_0) \leq \Xi_2(X;s,\sigma_0)$, where, by recalling (9) and (15),

$$\Xi_2(X;s,\sigma_0) = \frac{2^{\sigma_0}\left(\log X + \delta\left(\frac{X}{2},\sigma_0\right)\max\left\{\log\left(\frac{X}{2}\right),\frac{1}{\sigma_0}\right\}\right)}{|\zeta(s)|}$$
$$+\frac{2^{\sigma_0}\log(2)|s-1|}{|(1-2^{1-s})\zeta(s)|} + 2^{\sigma_0}\left|\frac{C'(s)}{C(s)\zeta(s)} - \frac{1}{(s-1)\zeta(s)} - \frac{\zeta'(s)}{\zeta^2(s)}\right|$$
$$+2^{\sigma_0}\left|\frac{C'(s)}{C(s)\zeta(s)} - \frac{1}{(s-1)\zeta(s)}\right| + 2^{\sigma_0}|e(s)|\left|\frac{C(s)+(s-1)C'(s)}{C(s)^2}\right|.$$

The result is concluded by noticing (25) and bounding $R_5(X;s,\sigma_0)$ by $\Xi_2(X;s,\sigma_0)$ in (22).

□

6. Auxiliaries on m_q

By [17, Prop. 5.15], we have, for any $X \geq 1$ and $q \in \mathbb{Z}_{q>0}$,

$$\left|\sum_{\substack{n\leq X\\(n,q)=1}}\frac{\mu(n)}{n}\right| \leq \frac{\sqrt{2}\sqrt{q}}{\varphi_{\frac{1}{2}}(q)}\frac{1}{\sqrt{X}} + \frac{0.0144\ q^{\theta}}{\varphi_\theta(q)}\frac{\mathbb{1}_{X\geq 10^{14}}}{\log X}, \tag{26}$$

where $\theta = 1 - \frac{1}{14\log 10}$. This is based on [17, Lemma 5.10]. When $q = 2$, we have [17, Eq. (5.79) and (5.89)] at our disposal, namely

$$|m_2(X)| \leq \sqrt{\frac{3}{X}}, \quad \text{if } 0 < X \leq 10^{12},$$
$$|m_2(X)| \leq \frac{0.0296}{\log X}, \quad \text{if } X \geq 5379, \tag{27}$$

Following the proof of [17, Prop. 5.15], we readily deduce from the above that, when $(q,2) = 1$ and with $\xi = 1 - \frac{1}{12\log 10}$, we have, for any $X > 0$,

$$\left|\sum_{\substack{n\leq X\\(n,2q)=1}}\frac{\mu(n)}{n}\right| \leq \frac{\sqrt{3}\sqrt{q}}{\varphi_{\frac{1}{2}}(q)}\frac{1}{\sqrt{X}} + \frac{0.0296q^{\xi}}{\varphi_\xi(q)}\frac{\mathbb{1}_{X\geq 10^{12}}}{\log X}. \tag{28}$$

We now extend the second one to every q.

Lemma 6.1. *For any $q \in \mathbb{Z}_{>0}$ and $X > 0$, we have*

$$\left|\sum_{\substack{n\leq X\\(n,2q)=1}}\frac{\mu(n)}{n}\right| \leq \frac{g_0(q)\sqrt{q}}{\varphi_{\frac{1}{2}}(q)}\frac{\sqrt{2}}{\sqrt{X}} + \frac{0.0144\ g_1(q)q^{\xi}}{\varphi_\xi(q)}\frac{\mathbb{1}_{X\geq 10^{12}}}{\log X},$$

the multiplicative functions g_0 and g_1 being defined in (2).

The value at small values of the parameter X are often of crucial impact while the slight worsening of the second term has much less effect.

Proof. We may assume q to be squarefree. When q is odd, this is a slight degrading of (26). When q is even, this is a consequence of (28) since $0.0144\, g_1(2) \geq 0.0296$. □

7. The integral of $|m_q|$

Let us first recall [27, Lemma 7.1].

Lemma 7.1. *Let $A > e$ be a given parameter. The function*

$$T : y \mapsto \frac{\log y}{y} \int_A^y \frac{dt}{\log t}$$

is first increasing and then decreasing. It reaches its maximum at $y_0(A)$ where $y_0(A)$ is the unique solution of $y = (\log y - 1) \int_A^y dt/\log t$. Moreover we have $T(y_0(A)) = (\log y_0(A))/(\log y_0(A) - 1)$.

With the results of Section 6 at hand, we derive the following result.

Lemma 7.2. *For any $X \geq 1$ and $q \in \mathbb{Z}_{q>0}$, we have*

$$\int_1^X |m_q(t)| dt \leq \frac{0.0149\ g_1(q) q^\xi}{\varphi_\xi(q)} \frac{X \mathbb{1}_{X \geq 10^{12}}}{\log X} + \frac{g_0(q)\sqrt{q}}{\varphi_{\frac{1}{2}}(q)} \sqrt{8X},$$

where the multiplicative functions g_0 and g_1 are defined in (2).

Proof. We distinguish two cases. If $X < 10^{12}$, then by Lemma 6.1, we have

$$\int_1^X |m_q(t)| dt \leq \frac{g_0(q)\sqrt{q}}{\varphi_{\frac{1}{2}}(q)} \int_1^X \frac{\sqrt{2}}{\sqrt{t}} dt \leq \frac{g_0(q)\sqrt{q}}{\varphi_{\frac{1}{2}}(q)} \sqrt{8X}.$$

On the other hand, we first use the following Pari/GP script

```
g(y)=(log(y)-1)*intnum(t=10^(12), y, 1/log(t));
solve(y=10^12, 10^16, g(y)-y),
```

which tells us that the value $y(10^{12})$ defined in Lemma 7.1 corresponds to $y_0(10^{12}) = 1365396548134370.8\cdots$. We have $T(y_0(10^{12}) \leq 1.03$. Therefore

$$\begin{aligned}\int_1^X |m_q(t)| dt &\leq \frac{g_0(q)\sqrt{q}}{\varphi_{\frac{1}{2}}(q)} \int_1^X \frac{\sqrt{2}}{\sqrt{t}} dt + \frac{g_1(q) q^\xi}{\varphi_\xi(q)} \int_{10^{12}}^X \frac{0.0144}{\log t} dt \\ &\leq \frac{g_0(q)\sqrt{q}}{\varphi_{\frac{1}{2}}(q)} \sqrt{8X} + \frac{0.0149\ g_1(q) q^\xi}{\varphi_\xi(q)} \frac{X}{\log X},\end{aligned}$$

whence the result. □

8. Estimates for $\Re s = \sigma \geq 1$

In this section, we specialize Theorem 1.5 and Theorem 1.6 to the case $\Re s = \sigma = 1 + \varepsilon \geq 0$ and we derive explicit bounds. It may be necessary for some bounds in this section to first assume $\sigma > 1$ and then let σ tend to 1^+. In order to do that, we need first a series of analytic estimations.

Analytic estimates.

Lemma 8.1. *Let* $\varepsilon > 0$ *and* $c(1+\varepsilon) = \frac{1-2^{-\varepsilon}}{\varepsilon}$. *Then, we have*

$$(i) \qquad \frac{1}{\varepsilon} < \zeta(1+\varepsilon) \leq \frac{e^{\gamma\varepsilon}}{\varepsilon}$$

$$(ii) \qquad \frac{1}{\log 2} < \frac{1}{c(1+\varepsilon)} < \frac{2^{\varepsilon}}{\log 2},$$

$$(iii) \qquad -\log 2 + \frac{1}{\varepsilon} < \frac{\log 2}{2^{\varepsilon}-1} < \frac{1}{\varepsilon}$$

Proof. (i). The upper bound is found in [28, Lemma 5.4]. With respect to the lower bound, for $\sigma = 1+\varepsilon > 1$, we have

$$\zeta(\sigma) = \sigma \int_1^{\infty} \frac{[t]}{t^{\sigma+1}} dt = \frac{\sigma}{\sigma-1} - \sigma \int_1^{\infty} \frac{\{t\}}{t^{\sigma+1}} dt = \frac{\sigma}{\sigma-1} - 1 + \sigma \int_1^{\infty} \frac{1-\{t\}}{t^{\sigma+1}} dt > \frac{1}{\sigma-1}. \tag{29}$$

In order to prove (ii), observe that

$$\frac{\varepsilon}{2^{\varepsilon}} < \int_0^{\varepsilon} 2^{-t} dt = \frac{1-2^{-\varepsilon}}{\log 2} < \varepsilon.$$

Thereupon, we derive (iii) by observing that

$$\frac{1}{\varepsilon} - \log 2 < \frac{1}{\varepsilon} - \frac{1-2^{-\varepsilon}}{\varepsilon} = \frac{1}{\varepsilon 2^{\varepsilon}} < \frac{\log 2}{2^{\varepsilon}-1} < \frac{1}{\varepsilon}.$$

□

Lemma 8.2. *Let* $\varepsilon > 0$ *and* $C(1+\varepsilon) = (1-2^{-\varepsilon})\zeta(1+\varepsilon)$. *Then, we have*

$$(i) \qquad -\frac{1}{\varepsilon} + \frac{1}{2(1+\varepsilon)^2} < \frac{\zeta'(1+\varepsilon)}{\zeta(1+\varepsilon)} < -\frac{1}{\varepsilon} + 2 - \frac{1}{1+\varepsilon},$$

$$(ii) \qquad \mathbb{1}_{\varepsilon < \frac{1}{\log 2}} \left(\frac{1}{\log 2} - \varepsilon\right)\left(\frac{2}{e^{\gamma}}\right)^{\varepsilon} < \frac{1}{C(1+\varepsilon)} < \frac{2^{\varepsilon}}{\log 2},$$

$$(iii) \qquad -\log 2 + \frac{1}{2(1+\varepsilon)^2} < \frac{C'(1+\varepsilon)}{C(1+\varepsilon)} < 2 - \frac{1}{1+\varepsilon}.$$

Proof. Let $\sigma > 1$. By [9], we have

$$\frac{\zeta'(\sigma)}{\zeta(\sigma)} > -\frac{1}{\sigma-1} + \frac{1}{2\sigma^2}. \tag{30}$$

On the other hand, upon multypling by $(\sigma-1)$, we may differentiate (29) with respect to σ and obtain

$$\zeta(\sigma) + (\sigma-1)\zeta'(\sigma) = 1 - (2\sigma-1)\int_1^{\infty} \frac{\{t\}}{t^{\sigma+1}} dt + \sigma(\sigma-1)\int_1^{\infty} \frac{\{t\}\log t}{t^{\sigma+1}} dt.$$

Therefore, as $2\sigma - 1 > 0$,

$$\zeta(\sigma) + (\sigma-1)\zeta'(\sigma) < 1 + \sigma(\sigma-1)\int_1^{\infty} \frac{\log t}{t^{\sigma+1}} dt = 1 + \frac{\sigma-1}{\sigma},$$

so that, by Lemma 8.1 (i),

$$\frac{\zeta'(\sigma)}{\zeta(\sigma)} < -\frac{1}{\sigma-1} + \frac{2\sigma-1}{\sigma(\sigma-1)\zeta(\sigma)} < -\frac{1}{\sigma-1} + 2 - \frac{1}{\sigma},$$

whence (i). With respect to (ii), observe that, by definition and with the help of Lemma 8.1 (ii), (iii), we have

$$\mathbb{1}_{\sigma<1+\frac{1}{\log 2}}\left(-1+\frac{1}{(\sigma-1)\log 2}\right)\frac{2^{\sigma-1}}{\zeta(\sigma)}<\frac{1}{C(\sigma)}=\frac{2^{\sigma-1}}{(2^{\sigma-1}-1)\zeta(\sigma)},$$
$$\frac{1}{C(\sigma)}=\frac{2^{\sigma-1}}{(2^{\sigma-1}-1)\zeta(\sigma)}<\frac{2^{\sigma-1}}{\log(2)(\sigma-1)\zeta(\sigma)}.$$

The estimation is the derived by using Lemma 8.1 (i). Finally, again by definition,

$$\frac{C'(\sigma)}{C(\sigma)}=\frac{\log 2}{2^{\sigma-1}-1}+\frac{\zeta'(\sigma)}{\zeta(\sigma)}.$$

Thus, by (i) and Lemma 8.1 (ii), (iii), we derive (iii). □

Proof of Theorem 1.2. Part 1. Theorem 1.5 with $\sigma=1+\varepsilon$, $\sigma_0=\frac{1}{2}+\varepsilon$ gives us

$$\left|\sum_{\substack{n\le X,\\(n,q)=1}}\frac{\mu(n)}{n^{1+\varepsilon}}-\frac{m_q(X)}{X^\varepsilon}-\frac{q^{1+\varepsilon}}{\varphi_{1+\varepsilon}(q)}\frac{1}{\zeta(1+\varepsilon)}\right|$$
$$\le\frac{1}{|c(1+\varepsilon)\zeta(1+\varepsilon)|X^{1+\varepsilon}}\int_1^X|m_q(t)|dt+\frac{|c(1+\varepsilon)|+2^{1/2+\varepsilon}e(1+\varepsilon)}{|c(1+\varepsilon)\zeta(1+\varepsilon)|X^{\frac{1}{2}+\varepsilon}}\frac{\sqrt{q}}{\varphi_{\frac{1}{2}}(q)},$$

where $e(1+\varepsilon)=2^{-\varepsilon}(1+2^{\varepsilon-1}\varepsilon\log 2)\log 2$. Further, by Lemma 8.1, we have

$$\frac{1}{|c(1+\varepsilon)\zeta(1+\varepsilon)|}\le\frac{\varepsilon 2^\varepsilon}{\log 2}\tag{31}$$
$$\frac{|c(1+\varepsilon)|+2^{\frac{1}{2}+\varepsilon}e(1+\varepsilon)}{|c(1+\varepsilon)\zeta(1+\varepsilon)|}\le\varepsilon(1+2^{\frac{1}{2}+\varepsilon}(1+\varepsilon 2^{\varepsilon-1}\log 2))$$

Now, by using Lemma 7.2, we conclude that

$$\sum_{\substack{n\le X,\\(n,q)=1}}\frac{\mu(n)}{n^{1+\varepsilon}}=\frac{m_q(X)}{X^\varepsilon}+\frac{q^{1+\varepsilon}}{\varphi_{1+\varepsilon}(q)}\frac{1}{\zeta(1+\varepsilon)}+\frac{\varepsilon\Delta_q(X,\varepsilon)}{X^\varepsilon},$$

where

$$|\Delta_q(X,\varepsilon)|\le\frac{0.0149\ 2^\varepsilon}{\log 2}\frac{g_1(q)\ q^\xi\ \mathbb{1}_{X\ge 10^{12}}}{\varphi_\xi(q)\log X}$$
$$+\left(1+2^{\frac{1}{2}+\varepsilon}(1+\varepsilon 2^{\varepsilon-1}\log 2)+\frac{2^\varepsilon\sqrt{8}g_0(q)}{\log 2}\right)\frac{\sqrt{q}}{\varphi_{\frac{1}{2}}(q)\sqrt{X}}.$$

We further simplify this bound into

$$|\Delta_q(X,\varepsilon)|\le 0.0215\frac{g_1(q)\ q^\xi\ \mathbb{1}_{X\ge 10^{12}}}{\varphi_\xi(q)\log X}$$
$$+\left(\frac{\sqrt{8}}{\log 2}g_0(q)+1+2^{\frac{1}{2}}(1+\varepsilon 2^{\varepsilon-1}\log 2)\right)\frac{\sqrt{q}}{\varphi_{\frac{1}{2}}(q)}\frac{2^\varepsilon}{\sqrt{X}},$$

that is,

$$|\Delta_q(X,\varepsilon)| \le 0.0215\frac{g_1(q)\ q^{\xi}\ \mathbb{1}_{X\ge 10^{12}}}{\varphi_\xi(q)\log X} + (4.09\,g_0(q) + 2.42 + 0.50\,\varepsilon 2^{\varepsilon})\,\frac{\sqrt{q}}{\varphi_{\frac12}(q)}\frac{2^{\varepsilon}}{\sqrt{X}} \tag{32}$$

from which the statement of the theorem follows.

The inequality $X^{\sigma-1}m_q(X,\sigma) \ge m_q(X)$ follows by expanding $X^{\sigma-1}m_q(X,\sigma)$ in Taylor series as in the proof of Theorem 1.1 and in using inequality (32). This readily implies that $\Delta_q(X,\varepsilon)/X^{\varepsilon} \ge -q/\varphi(q)$.

The bounds for $\Delta_q(X,\varepsilon)$ are dealt with in Section 9. □

Proof of Theorem 1.3. By using Theorem 1.6 with $\sigma = 1+\varepsilon$, $\sigma_0 = \frac12+\varepsilon$ and writing $\Xi_2(X;\varepsilon) = \Xi_2(X;1+\varepsilon,\frac12+\varepsilon)$, we obtain

$$\begin{aligned}\check{m}_q(X;1+\varepsilon) =& \frac{q^{1+\varepsilon}}{\varphi_{1+\varepsilon}(q)}\left(\frac{\log X}{\zeta(1+\varepsilon)} - \frac{\zeta'(1+\varepsilon)}{\zeta^2(1+\varepsilon)} - \frac{1}{\zeta(1+\varepsilon)}\sum_{p|q}\frac{\log p}{p^{1+\varepsilon}-1}\right)\\ &+\ O^*\left(\frac{\Xi_1(\varepsilon)}{X^{1+\varepsilon}}\int_1^X |m_q(t)|dt + \frac{\Xi_2(X;\varepsilon)}{X^{\frac12+\varepsilon}}\frac{\sqrt{q}}{\varphi_{\frac12}(q)}\right).\end{aligned} \tag{33}$$

Concerning $\Xi_1(\varepsilon)$, we may reduce it to

$$\Xi_1^0(\varepsilon) = \frac{(1+2\varepsilon)C(1+\varepsilon) + \varepsilon e^{-1}|C'(1+\varepsilon)|}{(1+\varepsilon)C(1+\varepsilon)^2}.$$

Therefore Lemma 7.2 gives us

$$\begin{aligned}\check{m}_q(X;1+\varepsilon) =& \frac{q^{1+\varepsilon}}{\varphi_{1+\varepsilon}(q)}\left(\frac{\log X}{\zeta(1+\varepsilon)} - \frac{\zeta'(1+\varepsilon)}{\zeta^2(1+\varepsilon)} - \frac{1}{\zeta(1+\varepsilon)}\sum_{p|q}\frac{\log p}{p^{1+\varepsilon}-1}\right)\\ &+ O^*\left(\frac{0.0149\,g_1(q)q^{\xi}\mathbb{1}_{X\ge 10^{12}}}{\varphi_\xi(q)X^{\varepsilon}\log X}\Xi_1^0(\varepsilon) + \frac{(\sqrt{8}g_0(q)\Xi_1^0(\varepsilon) + \Xi_2(X;\varepsilon)\sqrt{q}}{\varphi_{\frac12}(q)X^{\frac12+\varepsilon}}\right).\end{aligned}$$

As $\varepsilon < \frac{1}{\log 2}$, we may use Lemma 8.2 (ii), (iii) and obtain

$$\begin{aligned}|\Xi_1^0(\varepsilon)| \le& \frac{(1+2\varepsilon)}{(1+\varepsilon)|C(1+\varepsilon)|} + \frac{\varepsilon|C'(1+\varepsilon)|}{(1+\varepsilon)C(1+\varepsilon)^2}\\ \le& \frac{2^{\varepsilon}(1+2\varepsilon)}{(1+\varepsilon)\log 2} + \frac{2^{\varepsilon}\varepsilon}{(1+\varepsilon)\log 2}\max\left\{2-\frac{1}{1+\varepsilon}, \log 2 - \frac{1}{2(1+\varepsilon)^2}\right\}\\ =& \frac{2^{\varepsilon}}{\log 2}\left(2-\frac{1}{1+\varepsilon}\right)^2,\end{aligned} \tag{34}$$

where we have used that $2-\log 2 > 1 > \frac{1}{1+\varepsilon} - \frac{1}{2(1+\varepsilon)^2}$.

On the other hand, we can bound $\Xi_2(X;1+\varepsilon,\frac12+\varepsilon) = \Xi_2(X;\varepsilon)$ by noticing that, as $X \ge 15$ and $\varepsilon \in (0,\frac12]$, $\max\left\{\log\left(\frac{X}{2}\right), \frac{1}{\frac12+\varepsilon}\right\} = \log\left(\frac{X}{2}\right) < \log X$.

Thus

$$
\begin{aligned}
|\Xi_2(X;\varepsilon)| \leq &\frac{2^{\frac{3}{2}+\varepsilon}\log X}{|\zeta(1+\varepsilon)|} + \frac{2^{\frac{1}{2}+\varepsilon}}{|\zeta(1+\varepsilon)|}\left|\frac{C'(1+\varepsilon)}{C(1+\varepsilon)} - \frac{1}{\varepsilon} - \frac{\zeta'(1+\varepsilon)}{\zeta(1+\varepsilon)}\right| \\
&+ \frac{2^{\frac{1}{2}+\varepsilon}\log 2}{|c(1+\varepsilon)\zeta(1+\varepsilon)|} + \frac{2^{\frac{1}{2}+\varepsilon}}{|\zeta(1+\varepsilon)|}\left|\frac{C'(1+\varepsilon)}{C(1+\varepsilon)} - \frac{1}{\varepsilon}\right| \\
&+ 2^{\frac{1}{2}+\varepsilon}\frac{e(1+\varepsilon)}{|C(1+\varepsilon)|}\left|1 + \frac{\varepsilon C'(1+\varepsilon)}{C(1+\varepsilon)}\right|, \qquad (35)
\end{aligned}
$$

where $e(1+\varepsilon) = \log 2 + 2^{\varepsilon-1}\log^2(2)\varepsilon$. In order to further estimate (35), by recalling the definition of $C(1+\varepsilon)$ and on using Lemma 8.1 (iii), we have

$$
\left|\frac{C'(1+\varepsilon)}{C(1+\varepsilon)} - \frac{1}{\varepsilon} - \frac{\zeta'(1+\varepsilon)}{\zeta(1+\varepsilon)}\right| = \left|\frac{\log 2}{2^\varepsilon - 1} - \frac{1}{\varepsilon}\right| < \log 2. \qquad (36)
$$

Also, by (36) and Lemma 8.2 (i), we have

$$
\begin{aligned}
\left|\frac{C'(1+\varepsilon)}{C(1+\varepsilon)} - \frac{1}{\varepsilon}\right| = \left|\frac{\log 2}{2^\varepsilon - 1} - \frac{1}{\varepsilon} + \frac{\zeta'(1+\varepsilon)}{\zeta(1+\varepsilon)}\right| &< \left|\frac{\log 2}{2^\varepsilon - 1} - \frac{1}{\varepsilon}\right| + \left|\frac{\zeta'(1+\varepsilon)}{\zeta(1+\varepsilon)}\right| \\
&< \frac{1}{\varepsilon} + \log 2 - \frac{1}{2(1+\varepsilon)^2} \qquad (37)
\end{aligned}
$$

where we have used that $2 < \frac{1}{\varepsilon} + \frac{1}{1+\varepsilon}$. So, by Lemma 8.2, (iii), we have

$$
\begin{aligned}
\left|1 + \frac{\varepsilon C'(1+\varepsilon)}{C(1+\varepsilon)}\right| &< \left(1 + \varepsilon\max\left\{2 - \frac{1}{1+\varepsilon}, \log 2 - \frac{1}{2(1+\varepsilon)^2}\right\}\right) \\
&= 1 + 2\varepsilon - \frac{\varepsilon}{1+\varepsilon} \qquad (38)
\end{aligned}
$$

where we have used that $2 - \log 2 > \frac{1}{1+\varepsilon}$. Subsequently, by using Lemma 8.1 (i) and Lemma 8.2 (ii) and putting (36), (31), (37) and (38) together with (35), we obtain

$$
\begin{aligned}
|\Xi_2(X;\varepsilon)|2^{-\varepsilon} \leq &\varepsilon 2^{\frac{3}{2}}\log X + \varepsilon 2^{\frac{1}{2}}\log 2 + 2^{\frac{1}{2}}\left(1 + \varepsilon\log 2 - \frac{\varepsilon}{2(1+\varepsilon)^2}\right) \qquad (39) \\
&+ \varepsilon 2^{\frac{1}{2}+\varepsilon} + 2^{\frac{1}{2}+\varepsilon}\left(1 + \frac{\varepsilon 2^\varepsilon\log 2}{2}\right)\left(1 + 2\varepsilon - \frac{\varepsilon}{1+\varepsilon}\right) \qquad (40) \\
\leq &2.93 + 2.83\varepsilon\log X + 5.17\varepsilon \qquad (41)
\end{aligned}
$$

where, in the last line, we have used $\varepsilon \leq 1/10$. □

9. Bounding $\Delta_q(X,\varepsilon)$ from above

Let us first notice that

$$
\Delta_q(X,\varepsilon) = \sum_{\substack{n\leq X \\ (n,q)=1}} \frac{\mu(n)}{n}\frac{(X/n)^\varepsilon - 1}{\varepsilon} - \frac{q^{1+\varepsilon}}{\varphi_{1+\varepsilon}(q)}\frac{X^\varepsilon}{\varepsilon\zeta(1+\varepsilon)}. \qquad (42)
$$

Our aim in this section is to examine this quantity algorithmically for small values of the parameters X and q and varying $\varepsilon \in [0,1]$. Let us notice that when q is squarefree, if $q'|q$ and the only prime factors of q/q' are (strictly) larger than X, then $\Delta_q(X,\varepsilon) \leq \Delta_{q'}(X,\varepsilon)$, so that we may restrict our attention, while bounding $\Delta_q(X,\varepsilon)$ from above, to values q that only have their prime factors below X.

Discretising in ε. Let us start with a (rough) bound for the derivative of $\Delta_q(X,\varepsilon)$ with respect to ε.

Lemma 9.1. *For any $X \geq 1$, $\varepsilon \in [0,1]$ and $q \in \mathbb{Z}_{>0}$, we have the following inequalities*

$$\frac{\varphi(q)}{q}\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\frac{\Delta_q(X,\varepsilon)}{X^\varepsilon} \leq \log X + \sum_{p|q}\frac{\log p}{p-1} - \frac{1}{2\varepsilon(1+\varepsilon)^2\zeta(1+\varepsilon)},$$

$$-\log X - \frac{1+2\varepsilon}{\varepsilon(1+\varepsilon)\zeta(1+\varepsilon)} \leq \frac{\varphi(q)}{q}\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\frac{\Delta_q(X,\varepsilon)}{X^\varepsilon}.$$

Proof. By definition,

$$\frac{\Delta_q(X,\varepsilon)}{X^\varepsilon} = \frac{X^\varepsilon m_q(X,1+\varepsilon) - m_q(X)}{\varepsilon\ X^\varepsilon} - \frac{q^{1+\varepsilon}}{\varphi_{1+\varepsilon}(q)}\frac{1}{\varepsilon\ \zeta(1+\varepsilon)}. \tag{43}$$

Notice that

$$\begin{aligned}\frac{X^\varepsilon m_q(X,1+\varepsilon) - m_q(X)}{\varepsilon} &= \sum_{\substack{n\leq X\\(n,q)=1}}\frac{\mu(n)}{n}\frac{(X/n)^\varepsilon - 1}{\varepsilon}\\ &= \sum_{k\geq 1}\frac{\varepsilon^{k-1}}{k!}\sum_{\substack{n\leq X\\(n,q)=1}}\frac{\mu(n)}{n}\log^k\left(\frac{X}{n}\right). \qquad (44)\end{aligned}$$

By Lemma 3.2, (44) is non-negative and upper bounded by

$$\sum_{k\geq 1}\frac{\varepsilon^{k-1}}{k!}k\frac{q}{\varphi(q)}\log^{k-1}(X) = \frac{q}{\varphi(q)}X^\varepsilon, \tag{45}$$

Moreover, the derivative of the expression (44) with respect to ε is

$$\sum_{k\geq 2}\frac{(k-1)\varepsilon^{k-2}}{k!}\sum_{\substack{d\leq X\\(d,q)=1}}\frac{\mu(d)}{d}\log^k\left(\frac{X}{d}\right).$$

which, again by Lemma 3.2, is non-negative and bounded from above by

$$\sum_{k\geq 2}\frac{(k-1)\varepsilon^{k-2}}{(k-1)!}\frac{q}{\varphi(q)}\log^{k-1}(X) = \frac{q}{\varphi(q)}X^\varepsilon\log X.$$

We then conclude that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\left(\frac{X^\varepsilon m_q(X,1+\varepsilon) - m_q(X)}{\varepsilon\ X^\varepsilon}\right) &\in \left[0,\frac{q}{\varphi(q)}\log X\right] - \log(X)\left[0,\frac{q}{\varphi(q)}\right]\\ &\in \left[-\frac{q}{\varphi(q)}\log X, \frac{q}{\varphi(q)}\log X\right]. \qquad (46)\end{aligned}$$

On the other hand, by recalling (14), and using the chain rule, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\frac{q^{1+\varepsilon}}{\varphi_{1+\varepsilon}(q)} = -\frac{q^{1+\varepsilon}}{\varphi_{1+\varepsilon}(q)}\sum_{p|q}\frac{\log p}{p^{1+\varepsilon}-1}.$$

Further, thanks to Lemma 8.2 (i), we compute that

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\left(\frac{-1}{\varepsilon\zeta(1+\varepsilon)}\right)=\frac{\frac{1}{\varepsilon}+\frac{\zeta'(1+\varepsilon)}{\zeta(1+\varepsilon)}}{\varepsilon\zeta(1+\varepsilon)}\in\left[\frac{1}{2\varepsilon(1+\varepsilon)^2\zeta(1+\varepsilon)},\frac{1+2\varepsilon}{\varepsilon(1+\varepsilon)\zeta(1+\varepsilon)}\right].$$

Therefore, we have

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\left(-\frac{q^{1+\varepsilon}}{\varphi_{1+\varepsilon}(q)}\frac{1}{\varepsilon\zeta(1+\varepsilon)}\right)=\frac{q^{1+\varepsilon}}{\varphi_{1+\varepsilon}(q)}\frac{\sum_{p|q}\frac{\log p}{p^{1+\varepsilon}-1}-\left(\frac{1}{\varepsilon}+\frac{\zeta'(1+\varepsilon)}{\zeta(1+\varepsilon)}\right)}{\varepsilon\zeta(1+\varepsilon)}$$

$$\in\frac{q^{1+\varepsilon}}{\varphi_{1+\varepsilon}(q)}\left[-\frac{1+2\varepsilon}{\varepsilon(1+\varepsilon)\zeta(1+\varepsilon)},\sum_{p|q}\frac{\log p}{p-1}-\frac{1}{2\varepsilon(1+\varepsilon)^2\zeta(1+\varepsilon)}\right] \tag{47}$$

where we have used that $1\le\varepsilon\zeta(1+\varepsilon)$, by Lemma 8.1 (i). Finally, by using that $\frac{q^{1+\varepsilon}}{\varphi_{1+\varepsilon}(q)}\le\frac{q}{\varphi(q)}$ and putting (46) and (47) together, we conclude the result. □

Proof of Theorem 1.2. Part 2.

Discretising in X. For any positive integer N, we have

$$\max_{N\le X<N+1}\frac{\Delta_q(X,\varepsilon)}{X^\varepsilon}=\frac{m_q(N,1+\varepsilon)}{\varepsilon}+\frac{1}{\varepsilon}\max\left(\frac{-m_q(N)}{N^\varepsilon},\frac{-m_q(N)}{(N+1)^\varepsilon}\right)-\frac{q^{1+\varepsilon}}{\varphi_{1+\varepsilon}(q)}\frac{1}{\varepsilon\zeta(1+\varepsilon)}, \tag{48}$$

the maximum depending or whether or not $m_q(N)\ge 0$.

Further, at $\varepsilon=0$, we have

$$\begin{aligned}\Delta_q(X,0)=\lim_{\varepsilon\to0^+}\Delta_q(X,\varepsilon)&=\frac{\mathrm{d}}{\mathrm{d}\varepsilon}(X^\varepsilon m_q(X,1+\varepsilon))\Big|_{\varepsilon=0}-\lim_{\varepsilon\to0^+}\frac{q^{1+\varepsilon}}{\varphi_{1+\varepsilon}(q)}\frac{X^\varepsilon}{\varepsilon\,\zeta(1+\varepsilon)}\\&=[X^\varepsilon\check{m}_q(X,1+\varepsilon)|_{\varepsilon=0}-\frac{q}{\varphi(q)}\\&=\sum_{\substack{d\le X\\(d,q)=1}}\frac{\mu(d)}{d}\log\left(\frac{X}{d}\right)-\frac{q}{\varphi(q)}\\&=\check{m}_q([X])-\frac{q}{\varphi(q)}+m_q([X])\log\left(\frac{X}{[X]}\right).\end{aligned}$$

Algorithm. The points N enable us to build a Pari/GP script to determine whether $\Delta_q(X,\varepsilon)\le 0$ for all $\varepsilon\in[0,1]$ and $X\le X_0$ for some $X_0>0$. Let $\varepsilon_0=0$ and $N=1$.

(1) Suppose initially that $X\in[N,N+1)$.
(2) Determine a uniform upper bound M for the derivative with respect to ε of $X^{-\varepsilon}\Delta_q(X,\varepsilon)$ via Lemma 9.1, for $\varepsilon\in[0,1]$.
(3) Compute $\check{m}_q(N)$ and $m_q(N)$.
(4) Compute the maximum t_0 of $\check{m}_q(N)-\frac{q}{\varphi(q)}$ and $\check{m}_q(N)-\frac{q}{\varphi(q)}+m_q(N)\log\left(\frac{N+1}{N}\right)$, depending on whether or not $m_q(N)\ge 0$.
(5) It $t_0\ge 0$, exit with value `FAIL`.

(6) If $t_0 < 0$, set $\varepsilon_1 = \varepsilon_0 - t_0/M = -t_0/M$. Indeed, by the mean value theorem, for any $\varepsilon^* \in [\varepsilon_0, \varepsilon_1]$, $\Delta_q(X, \varepsilon^*)X^{-\varepsilon^*} \leq M\varepsilon_1 + t_0 \leq 0$, so $\Delta_q(X, [\varepsilon_0, \varepsilon_1]) \leq 0$.
(7) Continue until $\varepsilon_k \geq 1$:
   (a) Compute $t_k = \max_{N \leq X < N+1} \Delta_q(X, \varepsilon_k)X^{-\varepsilon_k}$ using (48).
   (b) If $t_k \geq 0$, exit with value `FAIL`.
   (c) If $t_k < 0$, set $\varepsilon_{k+1} = \varepsilon_k - t_k/M$.
(8) Replace N by $N + 1$.

This algorithm works when the values of $\Delta_q(X, \varepsilon)$ denoted by t_k are negative and far enough from 0. Nonetheless, it fails at $q = 1$ several times because, in fact, $\Delta_1(X, 0) \geq 0$ often. Moreover, for such values of X, while trying to bootstrap the algorithm at a latter value of ε, we see that $\Delta_1(X, \varepsilon)$ remains non-negative whenever $\varepsilon \in [0, 1]$. □

Appendix A. A classical inequality for the primes

We shall prove Lemma A.1, which follows thanks to Lemma A.2 by using a similar approach to that of Theorem 1.4.

Lemma A.1. *Pour $X \geq 1$, on a* $\displaystyle\sum_{n \leq X} \frac{\Lambda(n)}{n} \leq \log X$.

An integral identity. Define

$$\alpha(t) = \frac{1 - 2\{t\}}{t} - \frac{\{t\} - \{t\}^2}{t^2}, \qquad t > 0.$$

On one hand, we have

$$\frac{2}{X^2} \sum_{n \leq X} n = 1 + \alpha(X), \qquad X > 0. \tag{49}$$

On the other hand, α is precisely the right derivate of the function

$$\beta : t \mapsto \frac{\{t\} - \{t\}^2}{t}$$

over $(0, \infty)$.

Let $\varphi :]0, \infty[\to \mathbb{R}$ be a locally integrable function that vanishes in a neighbourhood of 0. Then the function $S_{\mathbf{1}}\varphi$, defined over $(0, \infty)$ as

$$S_{\mathbf{1}}\varphi(x) = \sum_n \varphi(x/n) \quad (x > 0),$$

satisfies the same conditions as φ.

Lemma A.2. *On considering the above conditions, we have*

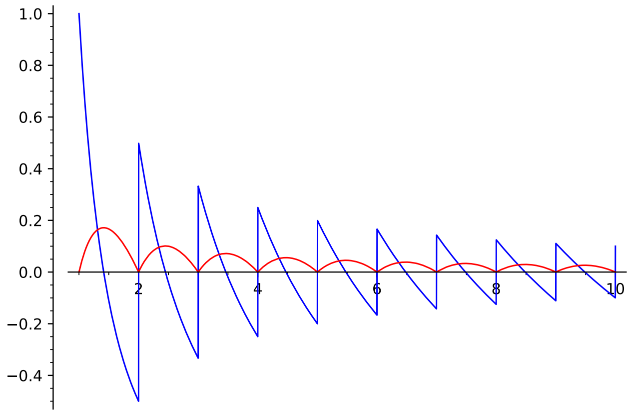
$$\int_0^X \varphi(t) \frac{dt}{t^2} = \frac{2}{X^2} \int_0^X S_{\mathbf{1}}\varphi(t)\,dt - \int_0^X \varphi(t)\,\alpha\left(\frac{X}{t}\right) \frac{dt}{t^2}, \qquad X > 0.$$

Proof. Thanks to (49) and by Fubini's theorem, we have

$$\begin{aligned}\int_0^X S_{\mathbf{1}}\varphi(t)\,dt &= \int_0^X \Big(\sum_n \varphi\Big(\frac{t}{n}\Big)\Big)\,dt = \sum_n \int_0^X \varphi\Big(\frac{t}{n}\Big)\,dt \\ &= \sum_n n\int_0^{\frac{X}{n}} \varphi(t)\,dt = \int_0^X \varphi(t)\Big(\sum_{n\le \frac{X}{t}} n\Big)\,dt \\ &= \frac{X^2}{2}\int_0^X \varphi(t)\Big(1+\alpha\Big(\frac{X}{t}\Big)\Big)\frac{dt}{t^2}.\end{aligned}$$

□

Analysis on two functions. Recall the definitions of α and β. We may visualize as follows: the function β is red colored, and its right derivate, the function α is blue colored.



By setting $t = \{x\} \in [0, 1)$, we have over any interval $[k, k+1)$, $k \in \mathbb{Z}_{>0}$, that

$$\beta(x) = \frac{\{x\}-\{x\}^2}{x} = \frac{t-t^2}{t+k},$$
$$\alpha(x) = \frac{1-2\{x\}}{x} - \frac{\{x\}-\{x\}^2}{x^2} = \frac{1-2t}{t+k} - \frac{t-t^2}{(t+k)^2}.$$

Hence

$$x^2\alpha(x) = (t+k)(1-2t) + t^2 - t = k^2 + k - (t+k)^2.$$

Moreover, by defining $t_k = \sqrt{k^2+k} - k$, which is a real number between 0 and 1/2, we derive

$$\alpha(x) > 0, \quad \text{if } 0 \le t < t_k \quad ; \quad \alpha(x) < 0, \quad \text{if } t_k < t < 1.$$

On the other hand, observe that

$$\begin{aligned}\int_{k+t_k}^{k+1} |\alpha(x)| \, \frac{dx}{x} &= -\int_{t_k}^{1} \alpha(t+k)\,\frac{dt}{t+k} \\ &= -\left[\frac{\beta(t+k)}{t+k}\right]_{t_k}^{1} - \int_{t_k}^{1} \beta(t+k)\frac{dt}{(t+k)^2} \\ &= \frac{t_k - t_k^2}{k(k+1)} - \int_{t_k}^{1} (t-t^2)\frac{dt}{(t+k)^3},\end{aligned}$$

where

$$\begin{aligned}\int_{t_k}^{1} (t-t^2)\frac{dt}{(t+k)^3} &= -\left[\frac{t-t^2}{2(t+k)^2}\right]_{t_k}^{1} + \int_{t_k}^{1} (1-2t)\frac{dt}{2(t+k)^2} \\ &= \frac{t_k - t_k^2}{2k(k+1)} - \left[\frac{1-2t}{2(t+k)}\right]_{t_k}^{1} - \int_{t_k}^{1} \frac{dt}{t+k} \\ &= \frac{t_k - t_k^2}{2k(k+1)} + \frac{1}{2(k+1)} + \frac{1-2t_k}{2\sqrt{k(k+1)}} - \frac{1}{2}\log\left(1+\frac{1}{k}\right).\end{aligned}$$

Thus,

$$\int_{k+t_k}^{k+1} |\alpha(x)| \, \frac{dx}{x} = \frac{t_k - t_k^2}{2k(k+1)} - \frac{1-2t_k}{2\sqrt{k(k+1)}} + \frac{1}{2}\log\left(1+\frac{1}{k}\right) - \frac{1}{2(k+1)}.$$

Now,

$$\begin{aligned}\frac{t_k - t_k^2}{2k(k+1)} - \frac{1-2t_k}{2\sqrt{k(k+1)}} &= \frac{\sqrt{k(k+1)} - k - k(k+1) - k^2 + 2k\sqrt{k(k+1)}}{2k(k+1)} \\ &\quad - \frac{1 - 2\sqrt{k(k+1)} + 2k}{2\sqrt{k(k+1)}} \\ &= \frac{(2k+1)\sqrt{k(k+1)}}{2k(k+1)} - 1 - \frac{2k+1}{2\sqrt{k(k+1)}} + 1 = 0.\end{aligned}$$

Therefore,

$$\int_{k+t_k}^{k+1} |\alpha(x)| \, \frac{dx}{x} = \frac{1}{2}\log\left(1+\frac{1}{k}\right) - \frac{1}{2(k+1)},$$

and we deduce that

$$\begin{aligned}2\int_{1}^{\infty} |\alpha(x)| \, \mathbb{1}_{\alpha(x)<0} \, \frac{dx}{x} &= \sum_{k=1}^{\infty} \left(\log\left(1+\frac{1}{k}\right) - \frac{1}{k+1}\right) \\ &= \lim_{K\to\infty} \left(\sum_{k=1}^{K} \log\left(1+\frac{1}{k}\right) - \sum_{k=1}^{K} \frac{1}{k+1}\right) \\ &= 1-\gamma.\end{aligned}$$

We have just proved the following lemma.

Lemma A.3. *We have* $\displaystyle\int_{1}^{\infty} |\alpha(x)| \, \mathbb{1}_{\alpha(x)<0} \, \frac{dx}{x} = \frac{1-\gamma}{2}$.

The reader should compare the above equality with the following

$$\int_1^\infty \alpha(x)\,\frac{dx}{x} = \left[\frac{\beta(x)}{x}\right|_1^\infty + \int_1^\infty \beta(x)\frac{dx}{x^2} = \int_1^\infty (\{x\}-\{x\}^2)\frac{dx}{x^3}$$
$$= -\left[\frac{\{x\}-\{x\}^2}{2x^2}\right|_1^\infty + \int_1^\infty (1-2\{x\})\frac{dx}{2x^2} = \gamma - \frac{1}{2},$$

which for instance follows from de [3, Eq. (10)].

Integrating the Stirling formula. Consider the Stirling formula in the version given in [3, p.17]

$$\sum_{n\le t}\log n = t\log t - t + \left(\frac{1}{2}-\{t\}\right)\log t + \gamma_{0,1} + \varepsilon_{0,1}(t), \quad t>0,$$

where

$$\gamma_{0,1} = 1 + \int_1^\infty (\{u\}-1/2)\,\frac{du}{u} = \frac{\log 2\pi}{2},$$
$$\varepsilon_{0,1}(t) = \int_t^\infty (1/2-\{u\})\,\frac{du}{u}.$$

For any $X>0$, we then have

$$\int_0^X \Big(\sum_{n\le t}\log n\Big)\,dt =$$
$$\int_0^X (t\log t - t)\,dt + \int_0^X \left(\frac{1}{2}-\{t\}\right)\log t\,dt + \gamma_{0,1}X + \int_0^X \varepsilon_{0,1}(t)\,dt. \tag{50}$$

We shall calculate the above three integrals. The first integral satisifies

$$\int_0^X (t\log t - t)\,dt = \frac{X^2}{2}\left(\log X - \frac{3}{2}\right), \quad X>0.$$

As for the second integral in (50), it may be expressed as

$$\int_0^X \left(\frac{1}{2}-\{t\}\right)\log t\,dt = \int_0^1 \left(\frac{1}{2}-t\right)\log t\,dt + \int_1^X \left(\frac{1}{2}-\{t\}\right)\log t\,dt,$$

where

$$\int_0^1 \left(\frac{1}{2}-t\right)\log t\,dt = \left[\frac{t-t^2}{2}\log t\right|_0^1 - \int_0^1 \frac{1-t}{2}\,dt = -\frac{1}{4}.$$

Finally, the third integral in (50) may be written as

$$\int_0^X \varepsilon_{0,1}(t)\,dt = X\varepsilon_{0,1}(X) + \int_0^X \left(\frac{1}{2}-\{t\}\right)dt = X\varepsilon_{0,1}(X) + \frac{\{X\}-\{X\}^2}{2}.$$

Subsequently,

$$\int_0^X \Big(\sum_{n\le t}\log n\Big)\,dt = \frac{X^2}{2}\left(\log X - \frac{3}{2}\right) \tag{51}$$
$$+\gamma_{0,1}X + \int_1^X \left(\frac{1}{2}-\{t\}\right)\log t\,dt - \frac{1}{4} + X\varepsilon_{0,1}(X) + \frac{\{X\}-\{X\}^2}{2}.$$

Proof of Lemma A.1. Apply Lemma A.2 to the following function

$$\varphi(X) = \psi(X) = \sum_{n \le X} \Lambda(n), \quad X > 0.$$

Thus, for any $X > 0$,

$$\int_0^X \psi(t)\,\frac{dt}{t^2} = \frac{2}{X^2}\int_0^X S_{\mathbf{1}}\psi(t)\,dt - \int_0^X \psi(t)\,\alpha\Big(\frac{X}{t}\Big)\,\frac{dt}{t^2},$$

so that

$$\sum_{n\le X}\Lambda(n)\Big(\frac{1}{n}-\frac{1}{X}\Big) = \frac{2}{X^2}\int_0^X \Big(\sum_{n\le t}\log n\Big)\,dt - \int_0^X \psi(t)\,\alpha\Big(\frac{x}{t}\Big)\,\frac{dt}{t^2}. \tag{52}$$

Hence, by putting (51) into (52), we obtain

$$\sum_{n\le X}\frac{\Lambda(n)}{n} = \log X + f(X),$$

where

$$\begin{aligned} f(X) =&\frac{\psi(X)}{X} - \frac{3}{2} - \int_0^X \psi(t)\,\alpha\Big(\frac{X}{t}\Big)\,\frac{dt}{t^2} + \frac{2\gamma_{0,1}}{X}\\ &+ \frac{2}{X^2}\int_1^X\Big(\frac{1}{2}-\{t\}\Big)\log t\,dt - \frac{1}{2X^2} + \frac{2\varepsilon_{0,1}(X)}{X} + \frac{\{X\}-\{X\}^2}{X^2}. \end{aligned} \tag{53}$$

In order to conclude the result, we just need to show that $f(X) \le 0$ for any $X \ge 1$. Observe first that

$$|\varepsilon_{0,1}(X)| \le \frac{1}{8X} \qquad [3, \text{p.17}],$$

$$\int_1^X\Big(\frac{1}{2}-\{t\}\Big)\log t\,dt \le \frac{\log X}{8}, \qquad \text{from the second mean value theorem.}$$

Thus, for any $X \ge 1$,

$$\begin{aligned} \frac{2\gamma_{0,1}}{X} + \frac{2}{X^2}\int_1^X\Big(\frac{1}{2}-\{t\}\Big)\log t\,dt - \frac{1}{2X^2} + \frac{2\varepsilon_{0,1}(X)}{X} + \frac{\{X\}-\{X\}^2}{X^2}&\\ \le \frac{2\gamma_{0,1}}{X} + \frac{\log X}{4X^2}&. \end{aligned} \tag{54}$$

On the other hand, for the terms appearing in (53) that involve the function ψ, we are going to use Hanson's inequality [15], namely

$$\psi(X) \le X\log 3, \quad X \ge 1. \tag{55}$$

Thereupon, we observe that

$$\begin{aligned} -\int_0^X \psi(t)\,\alpha\Big(\frac{X}{t}\Big)\,\frac{dt}{t^2} &\le \int_0^X \psi(t)\,\Big|\alpha\Big(\frac{X}{t}\Big)\Big|\,\mathbb{1}_{\alpha(X/t)<0}\,\frac{dt}{t^2}\\ &\le (\log 3)\int_0^x \Big|\alpha\Big(\frac{X}{t}\Big)\Big|\,\mathbb{1}_{\alpha(X/t)<0}\,\frac{dt}{t}\\ &= (\log 3)\int_1^\infty |\alpha(u)|\,\mathbb{1}_{\alpha(u)<0}\,\frac{du}{u} = \frac{(1-\gamma)\log 3}{2}, \end{aligned} \tag{56}$$

thanks to Lemma A.3. All in all, by putting (54), (55) and (56) together, and recalling the definition (53), we deduce the following inequality

$$f(X) \leq \log 3 - \frac{3}{2} + \frac{(1-\gamma)\log 3}{2} + \frac{2\gamma_{0,1}}{X} + \frac{\log X}{4X^2} = g(X), \quad X \geq 1.$$

Moreover, we have

$$g'(X) = \frac{1}{4X^3} - \frac{\log(2\pi)}{X^2} - \frac{\log X}{4X^3} < 0, \quad X \geq 1,$$

so that g is strictly decreasing in $[1, \infty)$. Since $g(12) = -0.011679\ldots$, we derive

$$f(X) \leq g(X) < 0, X \geq 12.$$

Finally, it is sufficient to see that

$$\sum_{n \leq X} \frac{\Lambda(n)}{n} \leq \log X, \qquad 1 \leq X \leq 12,$$

which is indeed true for $X = 2, 3, 4, 5, 7, 8, 9, 11$. □

If we used an inequality of the form $\psi(X) \leq aX$ instead of Hanson's, we would have obtained Lemma A.1 for $X \geq X_0(a)$, for some $X_0(a) > 0$, provided that

$$a < \frac{3}{3-\gamma} = 1.23824\ldots\ .$$

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