On Šnirel'man's constant[1]

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1. Introduction.

In a letter to Euler in 1742, Goldbach raised the problem of whether every even integer (other than 2) could be written as a sum of at most two primes. Euler noticed that this problem is equivalent to the fact that every integer larger than 5 can be expressed as a sum of three primes. Different approaches have since been developed to attack this still unsolved problem. Let us briefly mention here the sieve methods initiated by Brun in 1915, which were used by Chen Jing-run (1965) in proving that every sufficiently large even integer can be represented as a sum of a prime and an integer having at most two prime factors. The circle method of Hardy-Littlewood and Ramanujan eventually led Vinogradov to show in 1937 that every sufficiently large odd integer is a sum of three primes ; Chen Jing-run and Wang Tian-ze have shown in 1989 that this is indeed the case for odd integers larger than exp(100 000).

This work follows the approach initiated by Šnirel'man in 1930. By using an upper bound sieve to show that the set of sums of two primes has a positive density together with general results concerning addition of sequences (which he invented for that purpose), he proved that there exists an integer C such that every integer larger than 1 is a sum of at most C prime numbers. Klimov (1969) was the first to actually exhibit an explicit value for such an admissible C, namely 6.10^9. This value has subsequently been reduced by different authors, the latest being Riesel and Vaughan who showed in 1982 (cf [11]) that 19 is an admissible value. Here we prove the following result :

Theorem 1.
Every even integer is a sum of at most 6 primes.

This immediately leads to the following corollary.

Corollary.
Every integer larger than 1 is a sum of at most 7 primes.

The crucial step involved in Theorem 1 is an effective lower bound for the density of the sums of two primes.

Theorem 2. *For* $X \geq \exp(67)$, *we have*

$$Card\{N \in]X, 2X], \exists\, p_1, p_2 \text{ primes } : N = p_1 + p_2\} \geq \frac{X}{5}.$$

Let us recall that Montgomery & Vaughan have been able to show that the above cardinal is asymptotically equal to $X/2 + \mathcal{O}(X^{1-\delta})$ for some positive δ.

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The deduction of Theorem 1 from this result is performed in the last section. Thanks to an effective version of Ostman's additive Theorem due to Deshouillers one easily deduces that Theorem 1 holds true for all integers larger than 10^{30}. The remaining values are dealt with by an ascent process that combines known effective results on primes with a new one, which we proved in [9], concerning small effective intervals containing primes.

We outline the proof of Theorem 2 in the next section. Let us briefly mention here that it combines an enveloping sieve with effective results on primes in arithmetic progressions that we deduced from Rumely's computations [10] on zeroes of Dirichlet L-functions associated with characters to small moduli.

We present below the organisation of this article :

Notations :

ϕ is the Euler's totient function, μ is the Möbius' indicator, $\nu(n)$ is the number of prime factors of n, (n,m) denotes the gcd of n and m and $[n,m]$ their lcm. The arithmetical convolution of two functions f and g is noted $f \star g$ and defined by $(f \star g)(n) = \sum_{d|n} f(d)g(n/d)$. If f and g are two functions then $f(x) = \mathcal{O}^*(g(x))$ means $|f(x)| \leq g(x)$. We use the abbreviation $e(\alpha) = \exp(2i\pi\alpha)$, and if $S(\alpha) = \sum_n a_n e(n\alpha)$ is a trigonometric polynomial, we let $\|S\|_2^2 = \sum_n |a_n|^2$.

To denote a sum over all invertible residue classes a modulo q, we shall use the symbol

$$\sum_{a \bmod^* q}$$

We further need some special functions and constants :

ϕ_2 is defined by $\phi_2(d) = \prod_{p|d}(p-2)$,

ξ is defined by $\xi(d) = \prod_{p|d}(2p-3)/(p-1)$.

It will be numerically interesting to take a special care of the parity of our variables. To this end, we define the function $\kappa(a,.)$ by $\kappa(a,d) = a$ if $2|d$ and 1 otherwise. We denote by γ the Euler constant ($\gamma = 0.577\ 215\ 664\ 901\ 532\ ...$), $\mathfrak{S}_2 = 2\prod_{p\geq 3}(1-1/(p-1)^2)$ with $1.320\ 322 < \mathfrak{S}_2 < 1.320\ 323$, and $\mathcal{D} = \{d \geq 1, d \text{ odd, squarefree and } \phi(d) \leq 60\}$.

The notations concerning prime numbers are usual : the letter p always stands for a prime, $\theta(X) = \sum_{p\leq X} \operatorname{Log} p$, $\tilde{\theta}(X) = \sum_{\sqrt{X}<p\leq X} \operatorname{Log} p$, and $\theta(X; d, \ell) = \sum_{\substack{p\leq X \\ p\equiv \ell[d]}} \operatorname{Log} p$.

The knowledge which we will require about the distribution of primes in arithmetic progressions with a small modulus is contained in the following lemma (cf [10]) :

Lemma[3] **0.** *If $X \geq \exp(50)$, then for any d in $\mathcal{D}$ and ℓ prime to d, we have*

$$\max_{y\leq X} |\theta(y;d,\ell) - \frac{y}{\phi(d)}| \leq \epsilon_d \frac{X}{\phi(d)}$$

for ϵ_d given in table 1 at the end of this paper.

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2. The principle of the proof.

Here we present the proof of Theorem 2 (the key to Theorem 1) and at the end of the paper we show how to deduce Theorem 1 from it.

In the first part we detail Selberg upper sieve for the primes and consider the usual weights $\left(\sum_{d|y} \lambda_d\right)^2$ as a weighted sequence which is an upper bound for the characteristic function of the primes. Such a process has already been used by Hooley [5]and has been called "enveloping sieve" by Linnik ([6], chapter 1).

In the second part we use this enveloping sieve in order to build an upper bound for the number of representations of an even integer as the sum of two primes. We conjecture this upper bound to be twice larger than what we expect for the primes, a fact that we will be able only to establish on average.

We then state Propositions 1,2,3 and 4 and deduce Theorem 2 from them. Their proofs are given in the forthcoming sections.

[3]Since first writing this paper, the values of ϵ_d have been improved. However we only use the ones given here.

2.1 An enveloping sieve.

Here we fix a real number $X \geq 1$ and want to build an upper bound β for the characteristic function of the primes up to X with which we can work in a very explicit way.

First let us recall that Selberg's sieve provides such a function. We refer the reader to Halberstam & Richert [3] for a full account on this subject.

We choose a real parameter z between 1 and $X^{1/2}$ and define for every integer d and every real number t larger than 1 :

$$G_d(t) = \sum_{\substack{(k,d)=1 \\ k \leq t}} \frac{\mu^2(k)}{\phi(k)} \quad \text{and} \quad G(t) = G_1(t).$$

For fixed d and t going to infinity, we have (cf Lemma 3.4)

$$G_d(t) \sim \frac{\phi(d)}{d} \operatorname{Log} t. \tag{2.1}$$

Now we define λ_d, for all positive integers d, by

$$\lambda_d = \mu(d) \frac{\frac{d}{\phi(d)} G_d(z/d)}{G(z)} \tag{2.2}$$

so that it is, by (2.1), a weighted version of the usual Möbius $\mu(d)$. We see that $\lambda_1 = 1$ and $\lambda_d = 0$ if $d > z$. Finally we define $\beta(y)$ for all integers y, by

$$\beta(y) = \Big(\sum_{d|y} \lambda_d\Big)^2. \tag{2.3}$$

The weights $\beta(y)$ satisfy

(1) $\beta(y) \geq 0$ for every integer y and
(2) $\beta(p) = 1$ for every prime p larger than z

and therefore, the weighted sequence β is a good candidate for solving our problem, overlooking the anyway unimportant fact that we lose the primes less than z.

What are the advantages and drawbacks of this upper bound ?

(1) First of all, it can be easily checked that

$$\sum_{y \leq X} \beta(y) = \frac{X}{G(z)} + \mathcal{O}\Big(\frac{z^2}{\operatorname{Log}^2 z}\Big).$$

Hence if we choose z such that $\operatorname{Log} z \sim \frac{1}{2} \operatorname{Log} X$ as X goes to infinity, our sequence will be on average twice as large as the sequence of primes.

(2) But now we find that for every positive integer d the weighted sequence β is well-distributed in the progressions $\{a + kd, k \in \mathbb{Z}\}$ for all a coprime to d. To see this, define

$$w_d = \sum_{\substack{d_1, d_2 \\ d|[d_1,d_2]}} \frac{\lambda_{d_1} \lambda_{d_2}}{[d_1, d_2]}. \tag{2.4}$$

We can check that for any a coprime to d we have

$$w_d = \lim_{T\to\infty} \frac{1}{T} \sum_{y\leq T} \beta(y) e(ay/d),$$

which shows the equi-distribution since w_d does not depend on a. We also have

$$\sum_{y\leq X} \beta(y) e(ay/d) = X w_d + \mathcal{O}\big(d \frac{z^2}{\operatorname{Log}^2 z}\big). \tag{2.5}$$

In fact, we will see in the next section that only the w_d's and no error term appear in our problem. For these w_d's, we can prove in a very explicit way (see Lemma 4.3) that for all positive integers H :

$$w_d \sim \frac{1}{G(z)} \frac{\mu(d)}{\phi(d)} \quad \text{uniformly for} \quad d \leq (\operatorname{Log} z)^H, \tag{2.6}$$

which can be seen as proving a weak Brun-Titchmarsh inequality with the same sieve for all moduli. This will enable us to avoid the Theorem of Siegel-Walfisz.

The idea of looking at the weights of the upper sieve as a weighted sequence, which had seemed new to the author, was in fact already used by Hooley (see [5] , where he used Brun's sieve instead of Selberg's). In the Lemma 11 of Chapter 5 of [5], Hooley proved a formula corresponding to (2.5) for his w_d's.

2.2 An upper bound for the number of representations of an even integer as the sum of two primes.

Let N be an even integer.

We are looking for an upper bound of

$$\rho(N) = \sum_{p_1+p_2=N} 1.$$

This quantity is conjectured to be asymptotic, when N goes to infinity, to $\mathfrak{S}_2(N) N/\operatorname{Log}^2 N$ where $\mathfrak{S}_2(N)$ is an arithmetical factor which takes care of local obstructions. We have

$$\mathfrak{S}_2(N) = \mathfrak{S}_2 \prod_{\substack{p|N \\ p\neq 2}} \frac{p-1}{p-2} \quad \text{with} \quad \mathfrak{S}_2 = 2 \prod_{p\geq 3} \big(1 - \frac{1}{(p-1)^2}\big). \tag{2.7}$$

In order to obtain an upper bound for $\rho(N)$ we replace the characteristic function of the prime p_1 by the weighted sequence β. For technical reasons we add some size conditions which will be helpful later and do not change the arithmetical nature of this quantity. Let us choose a real number $X \geq 1$ and assume that N lies in $]X, 2X]$. We consider

$$r_2(N) = \sum_{\substack{p_1+p_2=N \\ \sqrt{X}\leq p_1 \\ p_2 \leq X}} \operatorname{Log} p_2. \tag{2.8}$$

$r_2(N)$ is conjectured to be asymptotic to $\mathfrak{S}_2(N)X/\operatorname{Log} X$. Then we consider

$$R_2(N) = \sum_{\substack{y+p_2=N \\ p_2 \le X}} \beta(y) \operatorname{Log} p_2, \tag{2.9}$$

where β is defined by (2.3) and depends on a parameter z which will be choosen later. We have

$$r_2(N) \le R_2(N) \tag{2.10}$$

and conjecture that $R_2(N)$ is asymptotic to $\mathfrak{S}_2(N)X/G(z)$ as N goes to infinity. We will not be able to prove such a statement for every N but we show in the next section that this is true in a suitable average sense. It is worthwhile to mention that the individual upper bound is out of reach at present since even by using Bombieri-Vinogradov's Theorem the best we are able to get as an individual upper bound is

$$r_2(N) \le (4+o(1))\mathfrak{S}_2(N)\frac{X}{\operatorname{Log} X} \quad \text{for} \quad X < N \le 2X,$$

while the conjecture $R_2(N)$ equivalent to $\mathfrak{S}_2(N)X/G(z)$ for the given choice of z would permit us to replace the 4 by a 2 in the above bound.

Let us study $R_2(N)$ some more. We introduce

$$T(\alpha) = \sum_{p \le X} \operatorname{Log} p \; e(p\alpha) \tag{2.11}$$

and write (2.9) in a different way by expanding the square in $\beta(y)$, reversing the summations and using the identity :

$$\sum_{\substack{p \equiv N[q] \\ p \le X}} \operatorname{Log} p = \frac{1}{q} \sum_{d|q} \sum_{a \bmod^* d} T(a/d) e(-Na/d).$$

We arrive at the expression

$$R_2(N) = \sum_{d \le z^2} w_d \sum_{a \bmod^* d} T(a/d) e(-Na/d) \tag{2.12}$$

which must be considered as one of the main steps in our proof. Now two remarks have to be made :

(1) Our sieving process does not depend on N which is the most notable feature of the "enveloping sieve".
(2) According to (2.5), w_d is also related to the value of a trigonometric polynomial at a point a/d. That only w_d appears is very convenient but also shows that this method is not able to carry very precise information about the distribution of primes.

Conjectural behaviour of $R_2(N)$:

Let us see how the summands in (2.12) behave in the ideal case, that is to say, when we replace w_d by $\frac{\mu(d)}{\phi(d)G(z)}$ and $T(a/d)$ by $\frac{\mu(d)}{\phi(d)}\theta(X)$. Then we get the expression

$$\frac{\theta(X)}{G(z)}\sum_{d\le z^2}\frac{\mu^2(d)}{\phi^2(d)}c_d(N) \tag{2.13}$$

where $c_d(N)$ is Ramanujan's sum. This expression is equivalent to $\mathfrak{S}_2(N)X/G(z)$ if z is large enough, but our series does not converge uniformly in N. Hence we cannot take only, say, the first 60 terms and claim it is a good approximation to $R_2(N)$. However, it is easily seen that this convergence is "almost everywhere" uniform (for instance,

$$\max_{X>1}\frac{1}{X}\sum_{X<N\le 2X}|\sum_{d\le t}\frac{\mu^2(d)}{\phi^2(d)}c_d(N)-\mathfrak{S}_2(N)|=\mathcal{O}(t^{-1/2})\quad\text{as}\quad t\to\infty).$$

We will retain from this discussion that on average over N the first terms of (2.12) give a good approximation of $R_2(N)$.

2.3 Structure of the proof of Theorem 2.

Let us make a comment on the content of Theorem 2. The number of even integers in $]X,2X]$ is approximatively $X/2$, and according to Goldbach's conjecture, our cardinal should be $X/2$. By comparison, Riesel & Vaughan's result corresponds to the lower bound $X/(2\times 9)$.

Now let us look at the principle of the proof. Following Shapiro & Warga [17], we study

$$\mathcal{R}=\sum_{N\in]X,2X]}\mathfrak{S}_2^{-1}(N)r_2(N). \tag{2.14}$$

By using effective numerical results on the distribution of the primes in arithmetic progression we easily get the following lower bound (cf section 5) :

Proposition 1. *For $X\ge\exp(67)$, we have*

$$\sum_{N\in]X,2X]}\mathfrak{S}_2^{-1}(N)r_2(N)\ge 0.478\frac{X^2}{\operatorname{Log}X}.$$

Note : We can prove that $\mathcal{R}\sim X^2/(2\operatorname{Log}X)$ as $X\to\infty$, and therefore our result does not lose much.

Shapiro & Warga's proof continues as follows :

$$\mathcal{R}\le\max_{X<N\le 2X}\{\mathfrak{S}_2^{-1}(N)r_2(N)\}\sum_{\substack{N\in]X,2X]\\ r_2(N)\ne 0}}1, \tag{2.15}$$

and it should be noted that the above inequality is conjectured to be an asymptotic equality since $r_2(N)$ is expected to be asymptotic to $\mathfrak{S}_2(N)X/\operatorname{Log}X$; therefore all the summands in $\mathcal{R}$ should be equal. Their proof continues by taking an individual sieve upper bound for $r_2(N)$.

The previous section provides us with an upper bound for $r_2(N)$, but we are unable to compute it. However, we have seen that the first terms of (2.12) must give, on average over N, the main contribution to $R_2(N)$. This is enough for our purpose since in (2.15) we need only the maximum out of a set of small measure , up to a neglegible error term.

More precisely and following the previous section, we have

$$\mathcal{R} \le \mathcal{R}^* = \sum_{\substack{N\in]X,2X] \\ r_2(N)\neq 0}} \mathfrak{S}_2^{-1}(N) R_2(N), \tag{2.16}$$

where $\mathcal{R}^*$ is expected to be

$$\frac{X}{G(z)} \mathrm{Card}\{N \in]X, 2X], \exists\ (p_1, p_2) \in \mathcal{P}^2\ :\ N = p_1 + p_2\}.$$

Putting

$$U(\alpha) = \sum_{\substack{N\in]X,2X] \\ r_2(N)\neq 0}} \mathfrak{S}_2^{-1}(N) e(N\alpha), \tag{2.17}$$

we get by using (2.12)

$$\mathcal{R}^* = \sum_{d\le z^2} w_d \sum_{a\ mod^* d} T(a/d)\bar{U}(a/d). \tag{2.18}$$

From this expression we see that we are dealing more with a ternary problem than with a binary one, but we do not know much about $U(\alpha)$. For d not in $\mathcal{D}$, we shall control $U(a/d)$ by using the large sieve inequality. A way of evaluating $\|U\|_2$ is given at the end of section 3.

Let us recall the notation

$$\tilde{\theta}(X) = \theta(X) - \theta(\sqrt{X}), \tag{2.19}$$

which is asymptotic to X (even numerically, we can replace this function by X).

We split the summation over d into two parts, according to whether d is larger or smaller than $\lambda = X^{0.3}$ and we evaluate the main term via a computation of dispersion.

$$\mathcal{R}^* = \mathcal{R}_1^* + \mathcal{R}_2^* + \mathcal{R}_3^*, \tag{2.20}$$

with

$$\mathcal{R}_1^* = \sum_{\substack{N\in]X,2X] \\ r_2(N)\neq 0}} \mathfrak{S}_2^{-1}(N) \Big\{ \sum_{d\le\lambda} w_d \sum_{a\,\mathrm{mod}^*\,d} \frac{\mu(d)}{\phi(d)} \tilde{\theta}(X) e(-Na/d) \Big\}, \tag{2.21}$$

$$\mathcal{R}_2^* = \sum_{d\le\lambda} \sum_{a\,\mathrm{mod}^*\,d} \bar{U}(a/d) \Big\{ w_d \Big(T(a/d) - \frac{\mu(d)}{\phi(d)} \tilde{\theta}(X) \Big) \Big\}, \tag{2.22}$$

$$(2.23)\qquad \mathcal{R}_3^* = \sum_{\lambda<d\le z^2} w_d \sum_{a \bmod^* d} T(a/d)\bar{U}(a/d).$$

We now study each of these three terms. From now on we have

$$(Hyp.)\qquad X \ge \exp(67)\ \ ,\ \ \lambda = X^{0.3}\ \text{ and }\ z^4 = \frac{X^2}{15000\,\mathrm{Log}\,X}.$$

We also define

$$(2.24)\qquad \delta = \delta(X) = \frac{1}{X}\mathrm{Card}\{N \in]X,2X], \exists\,(p_1,p_2)\in\mathcal{P}^2/N = p_1+p_2\}.$$

Study of $\mathcal{R}_1^*$:

By using an asymptotic expansion (with respect to z) of the w_d's we get the following proposition (cf section 6) :

Proposition 2. *Under (Hyp.), we have*

$$\mathcal{R}_1^* \le \frac{X^2}{G(z)}\{\delta + \frac{0.232}{G(z)}\delta^{36/37} + 0.0008\ \delta^{9/19} + 0.0008\}.$$

Study of $\mathcal{R}_2^*$:

We apply Cauchy's inequality to separate $\bar{U}(a/d)$ and $w_d(T(a/d) - \frac{\mu(d)}{\phi(d)}\tilde{\theta}(X))$. For the summation corresponding to the U's we use the large sieve inequality, and for the other one which is a dispersion, we express $T(a/d)$ in terms of multiplicative characters ; this part of the proof is similar to the proof of Barban-Davenport-Halberstam's Theorem except that the weights w_d enable us to avoid any appeal to Siegel-Walfisz's Theorem. We will get (cf section 7) :

Proposition 3. *Under (Hyp.), we have*

$$|\mathcal{R}_2^*| \le \frac{X^2}{G(z)}\{0.0309\ \delta^{9/19}\}.$$

Study of $\mathcal{R}_3^*$:

We apply Cauchy's inequality to separate $\bar{U}(a/d)$ and $T(a/d)$ and use a weighted version of the large sieve inequality for both summations. This weighted version (our Theorem 8.1) requires some careful treatment of the w_d's (cf section 8) .

Proposition 4. *Under (Hyp.), we have*

$$|\mathcal{R}_3^*| \le \frac{X^2}{G(z)}\{0.0102\ \delta^{9/19}\}.$$

Remark : At this stage, we choose both z and λ.

Proof of Theorem 2.

We collect Propositions 1, 2, 3, 4, (2.14), (2.16) and (2.20) to obtain

$$0.478\frac{X^2}{\mathrm{Log}\,X} \le \frac{X^2}{G(z)}\{\delta + \frac{0.232}{G(z)}\delta^{36/37} + 0.0419\ \delta^{9/19} + 0.0008\}.$$

Now, we have $0.4683\,\mathrm{Log}\,X \le G(z)$ and $31.37 \le G(z)$ (cf Lemma 3.4), thus

$$0 \le \delta + 0.00740\ \delta^{36/37} + 0.0419\ \delta^{9/19} - 0.2230.$$

This function of δ is increasing as a sum of increasing functions and is negative for $\delta = 1/(2\times 2.48)$, hence the result. $\diamond\diamond\diamond$

About the numerical orders of magnitude :

We will work with $\operatorname{Log} X \geq 67$ *and if possible with a smaller lower bound. For* z, *(Hyp.) ensures that* $\operatorname{Log} z \geq 30$ *but we will try as long as it is possible and not too cumbersome to work with* $\operatorname{Log} z \geq 18$. *In fact the bounds* 67 *and* 30 *could be lowered a little and we would get an analog of Theorem 2 with a lower bound of the shape* $\frac{X}{2\times(3-\epsilon)}$ *which would be enough for our purpose.*

3. Effective evaluations of average of arithmetical functions.

This part is divided in two : first we state and prove a general useful lemma which is a generalisation of a lemma of Riesel & Vaughan [11]and which provide us with good asymptotic expressions for averages of arithmetical functions. This lemma is followed by several applications we will require afterwards. Secondly, we will see how to obtain an upper bound for

$$\sum_{\substack{X<N\leq 2X\\ N\in\mathcal{A}}} \mathfrak{S}_2^{-1}(N) \quad \text{and} \quad \sum_{\substack{X<N\leq 2X\\ N\in\mathcal{A}}} \mathfrak{S}_2^{-2}(N)$$

where $\mathcal{A}$ is a set of positive density, without losing the information that $\mathcal{A}$ has positive density (i.e. without extending these summations to all integers). There we will follow Ruzsa [16].

We will often use the following elementary lemma :

Lemma 3.1. *Let* f *be a non-negative multiplicative real-valued function and let* d *be a positive integer. For all* $x \geq 0$, *we have*

$$\sum_{n\leq x} \mu^2(n)f(n) \leq \prod_{p|d}(1+f(p)) \sum_{\substack{n\leq x\\ (n,d)=1}} \mu^2(n)f(n) \leq \sum_{n\leq xd} \mu^2(n)f(n).$$

Proof. We have

$$\sum_{n\leq x} \mu^2(n)f(n) = \sum_{D|d} \sum_{\substack{n\leq x\\ (n,d)=D}} \mu^2(n)f(n) = \sum_{D|d} \mu^2(D)f(D) \sum_{\substack{n\leq x/D\\ (n,d)=1}} \mu^2(n)f(n),$$

and we conclude by using the non-negativity of f. ⋄⋄⋄

Next, we prove the aforementionned useful lemma.

Lemma 3.2. *Let* $(g_n)_{n\geq 1}$, $(h_n)_{n\geq 1}$ *and* $(k_n)_{n\geq 1}$ *be three complex sequences. Let* $H(s) = \sum_n h_n n^{-s}$, *and* $\overline{H}(s) = \sum_n |h_n| n^{-s}$. *We assume that* $g = h \star k$, *that* $\overline{H}(s)$ *is convergent for* $\Re(s) \geq -1/3$ *and further that there exist four constants* A, B, C *and* D *such that*

$$\sum_{n\leq t} k_n = A \operatorname{Log}^2 t + B \operatorname{Log} t + C + \mathcal{O}^*(Dt^{-1/3}) \quad \textit{for} \quad t > 0;$$

Then we have for all $t > 0$ *:*

$$\sum_{n\leq t} g_n = u \operatorname{Log}^2 t + v \operatorname{Log} t + w + \mathcal{O}^*(Dt^{-1/3}\overline{H}(-1/3))$$

with $u = AH(0)$, $v = 2AH'(0) + BH(0)$ *and* $w = AH''(0) + BH'(0) + CH(0)$. *We have also*

$$\sum_{n\le t} ng_n = Ut\operatorname{Log} t + Vt + W + \mathcal{O}^*(2.5Dt^{2/3}\overline{H}(-1/3))$$

with

$$\begin{cases} U = AH(0), \qquad V = -2AH(0) + 2AH'(0) + BH(0), \\ W = A(H''(0) - 2H'(0) + 2H(0)) + B(H'(0) - H(0)) + CH(0). \end{cases}$$

Proof. Write $\sum_{\ell\le t} g_\ell = \sum_m h_m \sum_{n\le t/m} k_n$, and all the regularity of our expressions comes from the fact that it is not necessary to impose $n \le t$ in $\sum_n k_n$. We then complete the proof without any trouble.

In order to estimate $\sum_{\ell\le t} \ell g_\ell$ for $t > 0$, we write

$$\sum_{\ell\le t} \ell g_\ell = t\sum_{\ell\le t} g_\ell - \int_1^t \sum_{\ell\le u} g_\ell du,$$

and we conclude by using the asymptotic expansion of $\sum_{\ell\le u} g_\ell$. ◇◇◇

In order to apply the preceeding lemma, we will require :

Lemma 3.3. *For all* $t > 0$, *we have*

$$\sum_{n\le t} \frac{1}{n} = \operatorname{Log} t + \gamma + \mathcal{O}^*\big(0.9105t^{-1/3}\big).$$

Denote by $\tau(n)$ *the number of divisors of* n. *For all* $t > 0$, *we have*

$$\sum_{n\le t} \frac{\tau(n)}{n} = \frac{1}{2}\operatorname{Log}^2 t + 2\gamma \operatorname{Log} t + \gamma^2 - \gamma_1 + \mathcal{O}^*\big(1.641t^{-1/3}\big),$$

with

$$\gamma_1 = \lim_{n\to\infty}\left(\sum_{m\le n}\frac{\operatorname{Log} m}{m} - \frac{\operatorname{Log}^2 n}{2}\right).$$

$(-0.072816 < \gamma_1 < -0.072815)$.

Proof. A proof of the second part of this lemma can be found in Riesel & Vaughan ([11],Lemma 1).

For the first part, we recall the classical

(3.1) $$|\sum_{n\le t}\frac{1}{n} - \operatorname{Log} t - \gamma| \le \frac{7}{12t} \quad \text{for} \quad t \ge 1.$$

For $0 < t < 1$, we choose $a > 0$ such that $\operatorname{Log} t + \gamma + a\, t^{-1/3} \ge 0$. This function decreases from 0 to $(a/3)^3$ then increases. This gives us the minimal value $a = 3\exp(-\gamma/3 - 1) \le 0.9105$. ◇◇◇

We will apply Lemma 3.2 with multiplicative sequences (g_n) satisfying $g_p = b/p + \mathcal{O}(1/p)$ with $b = 1$ or 2. In such a situation we choose $\sum k_n n^{-s} = \zeta(s+1)^b$ and (h_n) is multiplicative and determined by $\sum h_n n^{-s} = \sum g_n n^{-s}\zeta(s+1)^{-b}$.

Except in (3.24), the sequence (g_n) will be 0 over non-squarefree integers, giving

$$\sum_{m\geq 0} h_{p^m} p^{-ms} = (1 + g_p p^{-s})(1 - p^{-s-1})^b$$

thus enabling us to compute h_n and to show that the condition $\overline{H}(-1/3) < +\infty$ is met. The computations leading to (3.24) are similar.

Since in any case (h_n) is multiplicative, we will have

$$H(0) = \prod_p (1 + \sum_m h_{p^m}), \tag{3.2}$$

and

$$\frac{H'(0)}{H(0)} = \sum_p \frac{\sum_m m h_{p^m}}{1 + \sum_m h_{p^m}} (-\operatorname{Log} p), \tag{3.3}$$

and also

$$\frac{H''(0)}{H(0)} = \left(\frac{H'(0)}{H(0)}\right)^2 + \sum_p \left\{\frac{\sum_m m^2 h_{p^m}}{1 + \sum_m h_{p^m}} - \left(\frac{\sum_m m h_{p^m}}{1 + \sum_m h_{p^m}}\right)^2\right\} \operatorname{Log}^2 p. \tag{3.4}$$

We now apply Lemma 3.2.

Lemma 3.4. *For all $X > 0$ and all positive integers d, we have*

$$\sum_{\substack{n\leq X\\(n,d)=1}} \frac{\mu^2(n)}{\phi(n)} = \frac{\phi(d)}{d}\left\{\operatorname{Log} X + \gamma + \sum_{p\geq 2} \frac{\operatorname{Log} p}{p(p-1)} + \sum_{p|d} \frac{\operatorname{Log} p}{p}\right\} + \mathcal{O}^*(7.284 X^{-1/3} f_1(d))$$

with

$$f_1(d) = \prod_{p|d} (1 + p^{-2/3})\big(1 + \frac{p^{1/3} + p^{2/3}}{p(p-1)}\big)^{-1}.$$

Remark : The sum on the left is $G_d(X)$. The case $d = 1$ will be of special interest. The associated Dirichlet series is

$$\sum_n \frac{\mu^2(n)}{\phi(n) n^{s-1}} = \frac{\zeta(s)}{\zeta(2s)} \prod_{p\geq 2} \left(1 + \frac{1}{(p-1)(p^s+1)}\right) \tag{3.5}$$

and hence, we can see that the error term $\mathcal{O}(X^{-1/2})$ is admissible (in fact, our method could give $\mathcal{O}(X^{-1/2}\operatorname{Log}^2 X)$), and that we can not expect anything better than $\mathcal{O}(X^{-3/4})$.

Rosser & Schoenfeld ([13],(2.11)) give us

$$\gamma + \sum_{p\geq 2} \frac{\operatorname{Log} p}{p(p-1)} = 1.332\ 582\ 275\ 332\ 21... \tag{3.6}$$

Proof. Let us define the multiplicative function h_d by

$$h_d(p)=\frac{1}{p(p-1)},\quad h_d(p^2)=\frac{-1}{p(p-1)},\quad h_d(p^m)=0 \quad \text{if} \quad m\geq 3, \tag{3.7}$$

if p is a prime which is not a prime factor of d, and by $h_d(p^m)=\frac{\mu(p^m)}{p^m}$ for all $m\geq 1$ if p is a prime factor of d.

Then we have

$$\sum_{n\geq 1}\frac{h_d(n)}{n^s}\zeta(s+1)=\sum_{\substack{n\geq 1\\(n,d)=1}}\frac{\mu^2(n)}{\phi(n)n^s} \tag{3.8}$$

We now apply Lemma 3.2 and verify that

$$\prod_{p\geq 2}\left(1+\frac{p^{1/3}+p^{2/3}}{p(p-1)}\right)\leq 8 \tag{3.9}$$

which concludes the proof.

◇◇◇

Lemma 3.5.

(1) *For $z\geq 1$, we have $G(z)\leq \operatorname{Log} z+1.4709$.*
(2) *For $z\geq 6$, we have $1.06+\operatorname{Log} z\leq G(z)$.*
(3) *For $z\geq \exp(18)$ and $\alpha\geq 1.1$, we have $G(z^\alpha)\leq \alpha G(z)$.*

Proof. The first part follows from our asymptotic expression for $z\geq 146\ 050$. We could finish the proof by a hard computation but it would be very heavy. Instead, remark that we can modify Lemma 3.2 and take the exponent 0.45 instead of 1/3. Then we will have $G(z)-\operatorname{Log} z\leq 1.4708$ as soon as

$$\frac{1}{0.45}\operatorname{Log}\left(\overline{H}(-0.45)\frac{\exp(-1-0.45\gamma)}{0.45(1.4708-1.332583)}\right)\leq \operatorname{Log} z. \tag{3.10}$$

It is now a little difficult to compute $\overline{H}(-0.45)$, and we have to controle the error of computation. We have

$$\prod_{2\leq p\leq 200\ 000}(1+\frac{p^{0.45}+p^{0.9}}{p(p-1)})\leq 20.26 \tag{3.11}$$

and with $F(t)=(t^{0.45}+t^{0.9})/[t(t-1)\operatorname{Log} t]$ and $X=200\ 000$, we get

$$\operatorname{Log}\prod_{p>X}(1+\frac{p^{0.45}+p^{0.9}}{p(p-1)})\leq 1.001\ 093\big(XF(X)+\int_X^\infty F(t)dt\big)\leq 0.266\ 47 \tag{3.12}$$

since $\theta(t)\leq 1.001\ 093t$ if $t>0$ (cf [18]). Hence our first point is proved for $z\geq 42\ 300$. A direct verification shows that

$$\max_{z\geq 1}(G(z)-\operatorname{Log} z)=G(7)-\operatorname{Log} 7\leq 1.4709. \tag{3.13}$$

The second assertion is to be found in Montgomery & Vaughan ([7],Lemma 7) with a constant 1.07 instead of 1.06. We use only the slightly weakened form which is stated in the Lemma. It is one of the numerous places throughout this paper where results weaker than what is available are used. This looseness is introduced so that should a slight numerical mistake occur the results would still hold. The third assertion of our Lemma is an easy consequence of the two previous ones. ◇◇◇

Lemma 3.6. *For $y \geq 1$, we have*

$$\sum_{n \leq y} \frac{\mu^2(n)\kappa(1/2,n)}{\phi(n)} = \frac{3}{4}\left\{ \operatorname{Log} y + \gamma + \sum_p \frac{\operatorname{Log} p}{p(p-1)} - \frac{\operatorname{Log} 2}{6} \right\} + \mathcal{O}^*(4.280\ y^{-1/3}).$$

The following lower bound also holds :

$$\sum_{n \leq y} \frac{\mu^2(n)\kappa(1/2,n)}{\phi(n)} \geq \frac{3}{4} \operatorname{Log} y.$$

Finally we have for $\Delta \leq y$ and $y \geq \exp(18)$

$$\sum_{\substack{n \leq y \\ (n,\Delta)=1}} \frac{\mu^2(n)\kappa(1/2,n)}{\phi(n)} \leq \kappa(4/3,\Delta)\frac{\phi(\Delta)}{\Delta}\frac{3}{4}(2 \operatorname{Log} y + 1.220).$$

Proof. For the first part, we consider the multiplicative function h defined as h_1 of Lemma 3.4 on p^m if $p \neq 2$, and by

$$h(2) = 0, \quad h(4) = -\frac{1}{4}, \quad h(2^m) = 0 \quad \text{if} \quad m \geq 3, \tag{3.14}$$

and apply Lemma 3.2.

For the second part, let us denote the sum by $\overline{G}(y)$. Then $\overline{G}(y) = G(y) - G_2(y/2)/2$ and, with Lemma 3.1, $G_2(y/2) \leq G(y)/2$, hence, for $y \geq 6$, we get $\overline{G}(y) \geq \frac{3}{4} \operatorname{Log} y$. For $6 > y \geq 1$, a direct computation gives the result.

The third estimate follows by using first Lemma 3.1 and then the first estimate. $\diamond\diamond\diamond$

We will also use the following weighted version of the previous lemma :

Lemma 3.7. *Let A be a positive real number and $y \geq A^2$. We have for $\ell \leq y$ and $y \geq \exp(18)$*

$$\sum_{\substack{n \leq y \\ (n,\ell)=1}} \frac{\mu^2(n)\kappa(1/2,n)}{\phi(n)} \frac{1}{1+A/n} \leq \kappa(4/3,\ell)\frac{\phi(\ell)}{\ell}\frac{3}{4}(2 \operatorname{Log} y + 1.220 - \operatorname{Log} A).$$

Proof. Let us put

$$\begin{cases} \overline{G}(y) = \displaystyle\sum_{n \leq y} \frac{\mu^2(n)\kappa(1/2,n)}{\phi(n)}, \\ \overline{G}^*(y) = \displaystyle\sum_{n \leq y} \frac{\mu^2(n)\kappa(1/2,n)}{\phi(n)} \frac{1}{1+A/n}. \end{cases} \tag{3.15}$$

We first prove that $\overline{G}^*(y) \leq \overline{G}(y) - \frac{3}{4} \operatorname{Log} A$. We have

$$\overline{G}(y) - \overline{G}^*(y) = A \sum_{n \leq y} \frac{\mu^2(n)\kappa(1/2,n)}{\phi(n)} \frac{1}{n+A} = A\frac{\overline{G}(y)}{y+A} + A\int_1^y \overline{G}(t)\frac{dt}{(t+A)^2},$$

hence by Lemma 3.6

$$\overline{G}(y)-\overline{G}^*(y) \geq \frac{3A}{4}\left\{\frac{\operatorname{Log} y}{y+A}+\left[\frac{-\operatorname{Log} t}{t+A}\right]_1^y+\int_1^y \frac{dt}{t(t+A)}\right\} \geq \frac{3A}{4} \operatorname{Log}\left\{\frac{y}{y+A}\frac{1+A}{1}\right\}$$

which proves our estimate. Now, by a reasoning similar to that in Lemma 3.1, we get

$$\sum_{\substack{n\leq y\\(n,\ell)=1}} \frac{\mu^2(n)\kappa(1/2,n)}{\phi(n)}\frac{1}{1+A/n} \leq \kappa(4/3,\ell)\frac{\phi(\ell)}{\ell}\sum_{n\leq y\ell}\frac{\mu^2(n)\kappa(1/2,n)}{\phi(n)}, \tag{3.16}$$

and an appeal to the previous result concludes the proof. ⋄⋄⋄

Lemma 3.8. *For $y>0$ and a in $\{1/2, 2/3\}$, we have*

$$\sum_{n\leq y}\mu^2(n)\frac{\xi(n)\kappa(a,n)}{n} = k_9(a)\operatorname{Log}^2 y + k_{10}(a)\operatorname{Log} y + k_{11}(a) + \mathcal{O}^*\left(1.641 y^{-1/3}\epsilon(a)\right)$$

where $\xi(n) = \prod_{p|n}(2p-3)/(p-1)$. The constants $k_9(a)$, $k_{10}(a)$, $k_{11}(a)$ and $\epsilon(a)$ satisfy the inequalies

$0.0741 \leq k_9(1/2) \leq 0.0742$	$0.0790 \leq k_9(2/3) \leq 0.0791$
$0.650 \leq k_{10}(1/2) \leq 0.651$	$0.687 \leq k_{10}(2/3) \leq 0.689$
$0.725 \leq k_{11}(1/2) \leq 0.727$	$0.751 \leq k_{11}(2/3) \leq 0.753$
$\epsilon(1/2) = 20$	$\epsilon(2/3) = 21$

.

Also

$$\sum_{n\leq y}\mu^2(n)\xi(n)\kappa(a,n) = k_3(a) y \operatorname{Log} y + k_4(a) y + k_5(a) + \mathcal{O}^*\left(4.1025 y^{2/3}\epsilon(a)\right)$$

with

$0.148 \leq k_3(1/2) \leq 0.149$	$0.158 \leq k_3(2/3) \leq 0.159$
$0.500 \leq k_4(1/2) \leq 0.502$	$0.529 \leq k_4(2/3) \leq 0.531$
$0.223 \leq k_5(1/2) \leq 0.225$	$0.220 \leq k_5(2/3) \leq 0.222$

($k_3(a) = 2k_9(a)$,$k_4(a) = -2k_9(a) + k_{10}(a)$ and $k_5(a) = 2k_9(a) - k_{10}(a) + k_{11}(a)$).

Proof. Let us define the multiplicative function h_a for p prime not equal to 2 by (3.17)

$$\begin{cases} h_a(p) = \dfrac{-1}{p(p-1)}, \\ h_a(p^2) = \dfrac{-3p+5}{p^2(p-1)}, \\ h_a(p^3) = \dfrac{2p-3}{p^3(p-1)}, \\ h_a(p^m) = 0 \text{ if } m \geq 4 \end{cases} \quad \text{and} \quad \begin{cases} h_{2/3}(2) = \dfrac{-2}{3},\ h_{2/3}(4) = \dfrac{-1}{12},\ \text{and}\ h_{2/3}(8) = \dfrac{1}{12}, \\ h_{1/2}(2) = \dfrac{-3}{4},\ h_{1/2}(4) = 0,\ \text{and}\ h_{1/2}(8) = \dfrac{1}{16}, \end{cases}$$

and $h_a(2^m)=0$ if $m\geq 4$.

Then

$$(3.18)\qquad \sum_n \frac{\mu^2(n)\xi(n)\kappa(a,n)}{n^{s+1}} = \sum_n \frac{h_a(n)}{n^s}\zeta^2(s+1) = H_a(s)\zeta^2(s+1).$$

and now we apply Lemma 3.2. ⋄⋄⋄

Lemma 3.9. *Let f_5 be the function defined by*

$$f_5(n) = \prod_{p|n}\left\{\frac{2p-3}{p-1}\frac{p^2}{1+p(p-1)}\right\}.$$

Then we have for $y\geq \exp(18)$

$$\sum_{500<n\leq y} \mu^2(n)f_5(n)\kappa(3/5,n)\left[\tfrac12 \operatorname{Log}\frac{y}{n}+1.9709\right] \leq 0.5140y\operatorname{Log} y + 2.0823y.$$

Proof. We first prove that for all $y\geq 1$, we have

$$\sum_{n\leq y}\mu^2(n)f_5(n)\kappa(3/5,n) = k_6 y\operatorname{Log} y + k_7 y + k_8 + \mathcal{O}^*\big(94.5y^{2/3}\big),$$

with $0.207\leq k_6\leq 0.208$, $0.623\leq k_7\leq 0.625$, and $0.123\leq k_8\leq 0.125$.

Let us define the multiplicative function h by, for p prime not equal to 2,

$$(3.19)\qquad \begin{cases} h(p) = \dfrac{p^2-4p+2}{p(p-1)(p^2-p+1)},\\[2mm] h(p^2) = \dfrac{-(3p^3-4p^2-2p+1)}{p^2(p-1)(p^2-p+1)},\\[2mm] h(p^3) = \dfrac{2p-3}{p(p-1)(p^2-p+1)},\\ h(p^m)=0 \ \text{ if } \ m\geq 4\end{cases} \quad\text{and}\quad \begin{cases} h(2)=\dfrac{-3}{5},\\[2mm] h(4)=\dfrac{-5}{16},\\[2mm] h(8)=\dfrac{1}{10},\\ h(2^m)=0\ \text{ if }\ m\geq 4.\end{cases}$$

Then

$$\sum_n\frac{\mu^2(n)\kappa(3/5,n)f_5(n)}{n^{s+1}} = \zeta^2(s+1)\sum_n\frac{h(n)}{n^s},$$

and Lemma 3.2 applies. From this we get

(3.20)

$$\begin{aligned}\sum_{n\leq y}\mu^2(n)f_5(n)\kappa(3/5,n)\operatorname{Log}\frac{y}{n} &= \int_1^y\Big\{k_6\operatorname{Log} t + k_7 + \frac{k_8}{t} + \mathcal{O}^*(94.5\ t^{-1/3})\Big\}dt\\ &= k_6 y\operatorname{Log} y - k_6 y + k_6 + k_7 y - k_7 + k_8\operatorname{Log} y\\ &\qquad + \mathcal{O}^*\big(\tfrac{3}{2}94.5(y^{2/3}-1)\big),\end{aligned}$$

which is now equal to

$$\begin{aligned}&2.4709k_6 y\operatorname{Log} y + [0.5(k_7-k_6)+1.9709k_7]y\\ &+0.5k_8\operatorname{Log} y + 1.9709k_8 + 1.4709(k_6-k_7) + \mathcal{O}^*\big(258y^{2/3}\big).\end{aligned}$$

The following points are easily checked :

(1)
$$\sum_{n\leq 500} \mu^2(n) f_5(n)\kappa(3/5,n)(0.5\,\mathrm{Log}\,\frac{y}{n}+1.9709) \geq 289\,\mathrm{Log}\,y - 818.9 + 0.5k_8\,\mathrm{Log}\,y.$$

(2) $0.5(k_7-k_6)+1.9709k_7 \leq 1.445$.

(3) $1.9709k_8 + 1.4709(k_6-k_7) \leq 0$.

By using these, we get an upper bound which is a non-decreasing function of z and a simple computation concludes the proof. ⋄⋄⋄

Lemma 3.10. *For all $Z>0$, we have*
$$\sum_{Z<n} \frac{\mu^2(n)}{n\phi(n)} \leq \frac{4}{Z} \quad \textit{and} \quad \sum_{\substack{Z<n\\(n,2)=1}} \frac{\mu^2(n)}{n\phi(n)} \leq \frac{8}{3Z}.$$

Proof. The identity
$$\frac{1}{\phi(n)} = \frac{1}{n}\sum_{\ell|n}\frac{\mu^2(\ell)}{\phi(\ell)},$$
implies that the sum we want to estimate is less than

$$\sum_{\ell}\frac{\mu^2(\ell)}{\ell^2\phi(\ell)}\sum_{Z/\ell<m}\frac{1}{m^2} \leq \frac{2}{Z}\prod_{p\geq 2}\Big(1+\frac{1}{p(p-1)}\Big), \tag{3.21}$$

and this is now less than $4/Z$. The proof of the second inequality is similar with the condition $(n,2)=1$ which we keep for the variable ℓ in (3.21) and drop for the variable m. ⋄⋄⋄

Average of a multiplicative function over a set of positive density

We study this general problem only in a special case we will need later on. We refer the reader to Ruzsa [16]for a more detailled study.

We are looking at (cf (2.7))
$$\sum_{\substack{X<N\leq 2X\\ N\in\mathcal{A}}} \mathfrak{S}_2^{-a}(N),$$
where a is 1 or 2 and $\mathcal{A}$ is the set of integers which are sums of two primes. We put $\mathcal{B}=\mathcal{A}\cap]X,2X]$ and apply Hölder's inequality with an exponent $\sigma\geq 1$:

$$\sum_{N\in\mathcal{B}}\mathfrak{S}_2^{-a}(N) \leq |\mathcal{B}|^{1-1/\sigma}\mathfrak{S}_2^{-a}\Big(\sum_{X<N\leq 2X}\prod_{p|N}\Big(\frac{p-2}{p-1}\Big)^{a\sigma}\Big)^{1/\sigma}. \tag{3.22}$$

For any real number $b\geq 1$, we put $g_b(N)=\prod_{p|N}\big(\frac{p-2}{p-1}\big)^b$. Following Lemma 3.2, we define the multiplicative function h_b by :

$$\begin{cases} h_b(p) = \dfrac{1}{p}(g_b(p)-1), \\ h_b(p^m) = 0 \quad \text{for all} \quad m\geq 2. \end{cases} \tag{3.23}$$

We denote by H_b its Dirichlet series and by $\overline{H}_b$ the Dirichlet series associated with $|h_b(n)|$. Lemma 3.2 gives us

$$\sum_{X<N\leq 2X}\prod_{p|N} g_{a\sigma}(N) = H_{a\sigma}(0)X + \mathcal{O}^*\big(2.277\overline{H}_{a\sigma}(-1/3)(1+2^{2/3})X^{2/3}\big), \tag{3.24}$$

and it remains to choose the best value of σ. We want to minimize

$$\frac{1}{|\mathcal{B}|}\times|\mathcal{B}|^{1-1/\sigma}\mathfrak{S}_2^{-a}\Big(\sum_{X<N\leq 2X} g_{a\sigma}(N)\Big)^{1/\sigma}.$$

And we want to do this minimisation with $|\mathcal{B}| = X/6$. We then work for $X \geq \exp(60)$, and compute several values. For each a, this function of σ varies very slowly. We obtain the following lemma :

Lemma 3.11. *For $X \geq \exp(60)$, we have*

$$\sum_{\substack{X<N\leq 2X\\ r_2(N)\neq 0}}\mathfrak{S}_2^{-1}(N)\leq 0.7174\delta^{36/37}X, \quad \textit{and} \quad \sum_{\substack{X<N\leq 2X\\ r_2(N)\neq 0}}\mathfrak{S}_2^{-2}(N)\leq 0.5159\delta^{18/19}X.$$

4. Study of the w_d.

As is clear in (2.18), we are in need of information about the w_d's. We will first get a more explicit formula for these w_d's by introducing the definition of the λ_d's in (2.4). This expression has the defect of introducing localized divisors of d. Our guess is that

$$0\leq \mu(d)\phi(d)G(z)w_d \ll 1, \tag{4.0}$$

which we are not able to prove.

For d small, we give an asymptotic expression as z goes to infinity which supports (4.0).

For d prime, or twice a prime or 6 times a prime ≥ 7, we prove (4.0) by using elementary means.

For the medium d's, we have upper bounds essentially of the shape $|\phi(d)G(z)w_d| \ll 3^{\nu(d)}$, but the introduction of this divisor function will prove to be numerically costly.

For the large d's (i.e. close to z^2), we use the fact that d has few decompositions of the form $d = [d_1, d_2]$ with $d_1, d_2 \leq z$ to avoid use of $3^{\nu(d)}$.

The last part of this section is devoted to the study of a function built from the w_d's and which will be of use in section 7.

4.1 An explicit expression.

By (2.2) and (2.4) and then substituting the definitions of $G_{d_i}(z/d_i)$ $(i = 1, 2)$, we obtain at once

$$G^2(z)w_d = \sum_{\substack{\ell\leq z\\ k\leq z}}\frac{\mu^2(\ell)}{\phi(\ell)}\frac{\mu^2(k)}{\phi(k)}L(d,\ell,k)$$

$$\text{where}\quad L(d,\ell,k) = \sum_{\substack{d_1|\ell\ ,\ d_2|k\\ d|[d_1,d_2]}}\mu(d_1)\mu(d_2)(d_1,d_2). \tag{4.1}$$

We now evaluate the function L.

Lemma 4.1. *For any positive squarefree integers d, k and ℓ, we have*

$$L(d,\ell,k)=\begin{cases}\mu(d)\mu((d,\ell,k))\phi\big([\ell,k]/d\big)\phi_2((d,\ell,k)) \\ \qquad\qquad\qquad \textit{if}\ \ d|[\ell,k]\ \textit{and}\ \ell/(\ell,d)=k/(k,d), \\ 0\quad \textit{otherwise.}\end{cases}$$

Proof. We obviously have $L(d,\ell,k)=0$ if $d\not|[\ell,k]$, hence from now on, we suppose $d|[\ell,k]$. We first prove that L is multiplicative in the following sense : if d, ℓ and k are squarefree integers, then

$$L(d,\ell,k)=\prod_p L((d,p),(\ell,p),(k,p)). \tag{4.2}$$

To do so, we introduce the function $\chi(a,b)$ which takes the value 1 if $a|b$ and 0 otherwise and its local version : for every prime p, we define $\chi_p(a,b)$ to be equal to $\chi((a,p),(b,p))$. Let M be an integer divisible by k and ℓ. For squarefree integers a and b dividing M, we have

$$\chi(a,b)=\prod_{p|M}\chi_p(a,b).$$

Assuming d, ℓ and k squarefree, we thus get

$$\begin{aligned}L(d,\ell,k)&=\sum_{\substack{d_1|M\\ d_2|M}}\mu(d_1)\mu(d_2)(d_1,d_2)\chi(d_1,k)\chi(d_2,\ell)\chi(d,[d_1,d_2])\\ &=\sum_{\substack{d_1|M\\ d_2|M}}\prod_{p|M}\mu((d_1,p))\prod_{p|M}\mu((d_2,p))\prod_{p|M}((d_1,p),(d_2,p))\\ &\qquad\prod_{p|M}\chi_p(d_1,k)\prod_{p|M}\chi_p(d_2,\ell)\prod_{p|M}\chi_p(d,[d_1,d_2]),\\ &=\prod_{p|M}\Big\{\sum_{\substack{d_1|p\\ d_2|p}}\mu(d_1)\mu(d_2)(d_1,d_2)\chi_p(d_1,k,p)\chi_p(d_2,\ell)\chi_p(d,[d_1,d_2])\Big\},\end{aligned}$$

out of which (4.2) follows readily. We define $L_p(d,\ell,k)=L((d,p),(\ell,p),(k,p))$, and find

(1) If $p|d$, then
  (i) If $p|(\ell,k)$ then $L_p(d,\ell,k)=p-2$;
  (ii) If $p|\ell$ or $p|k$, but $p\not|(\ell,k)$, then $L_p(d,\ell,k)=-1$;
(2) If $p\not|d$, then
  (i) If $p|(\ell,k)$ then $L_p(d,\ell,k)=p-1$;
  (ii) If $p|\ell$ or $p|k$, but $p\not|(\ell,k)$, then $L_p(d,\ell,k)=0$;

We now have to find a global expression out of the local ones. Let us write

$$\begin{cases}\ell=d_1d_3s,\quad \text{with}\quad (s,d)=1,\\ k=d_2d_3t,\quad \text{with}\quad (t,d)=1,\\ d=d_1d_2d_3.\end{cases}$$

Then, if $L(d,\ell,k)\neq 0$, the point (2.ii) above gives us $s=t$, that is to say $\frac{\ell}{(d,\ell)}=\frac{k}{(d,k)}$ and this last quantity is equal to $\frac{[\ell,k]}{d}$. The proof follows readily. ◇◇◇

Using the notations $\ell = d_1 d_3 s$, etc. from the proof of Lemma 4.1, we readily get

Lemma 4.2. *If w_d is given by (2.4) then*

$$w_d = \frac{\mu(d)}{G^2(z)\phi(d)} \sum_{\substack{s\le zd^{-1/2}\\(s,d)=1}} \frac{\mu^2(s)}{\phi(s)} \sum_{\substack{d_1d_2d_3=d\\d_1d_3s\le z\\d_2d_3s\le z}} \mu(d_3)\frac{\phi_2(d_3)}{\phi(d_3)}. \tag{4.3}$$

4.2 An asymptotic expression.

Lemma 4.3. *For all $z \ge 1$ and all integers d, we have*

$$G(z)w_d = \frac{\mu(d)}{\phi(d)}\Big\{1 + \frac{u_d}{G(z)} + \mathcal{O}^*\big(\frac{7.284}{G(z)z^{1/3}}(1+f_6(d))\big)\Big\}, \tag{4.4}$$

where

$$u_d = \frac{\phi(d)}{d}\sum_{tt'=d}\mu(t')\frac{\phi_2}{\phi}(t')\sum_{\substack{k|t\\k\ge\sqrt{t}}}\mathrm{Log}\,\frac{t}{k^2},$$

and

$$f_6(d) = f_1(d)\sum_{d_1d_2d_3=d}\frac{\phi_2}{\phi}(d_3)\max(d_1,d_2)^{1/3}d_3^{1/3},$$

f_1 having been defined in Lemma 3.4.

Proof. We use section 3. First, (4.3) gives us

$$\mu(d)\phi(d)G^2(z)w_d = \sum_{d_1d_2d_3=d}\mu(d_3)\frac{\phi_2}{\phi}(d_3)G_d(\min(\frac{z}{d_1d_3},\frac{z}{d_2d_3})). \tag{4.5}$$

We now use Lemma 3.4 to write

$$G_d(\frac{z}{d_3\max(d_1,d_2)}) = \frac{\phi(d)}{d}\Big\{\mathrm{Log}\,(\frac{z}{d_3\max(d_1,d_2)}) + \gamma + \sum_p\frac{\mathrm{Log}\,p}{p(p-1)} + \sum_{p|d}\frac{\mathrm{Log}\,p}{p}\Big\}$$
$$+\mathcal{O}^*\big\{8\times 0.9105(\frac{z}{d_3\max(d_1,d_2)})^{-1/3}f_1(d)\big\}.$$

We collect all the error terms to get the claimed one. For the main term, we use:

$$\sum_{d_1d_2d_3=d}\mu(d_3)\frac{\phi_2}{\phi}(d_3) = \sum_{d_3|d}\mu(d_3)\frac{\phi_2}{\phi}(d_3)2^{\nu(d/d_3)} = \frac{d}{\phi(d)}$$

so that the expression

$$\sum_{d_1d_2d_3=d}\mu(d_3)\frac{\phi_2}{\phi}(d_3)(-\mathrm{Log}(d_3\max(d_1,d_2))) + \frac{d}{\phi(d)}\sum_{p|d}\frac{\mathrm{Log}\,p}{p}$$

is equal, by introducing $n = d_1d_3$, to

$$\sum_{\substack{d_3|n|d\\n^2\ge dd_3}}\mu(d_3)\frac{\phi_2}{\phi}(d_3)\,\mathrm{Log}\,\frac{1}{n} + \sum_{\substack{d_3|n|d\\n^2<dd_3}}\mu(d_3)\frac{\phi_2}{\phi}(d_3)\,\mathrm{Log}\,\frac{n}{dd_3} + \frac{d}{\phi(d)}\sum_{p|d}\frac{\mathrm{Log}\,p}{p}$$

which is

$$\sum_{\substack{d_3|n|d\\ n^2\ge dd_3}} \mu(d_3)\frac{\phi_2}{\phi}(d_3)\,\mathrm{Log}\,\frac{dd_3}{n^2}$$

$$-\frac{d}{\phi(d)}\,\mathrm{Log}\,d+\sum_{d_3|d}\mu(d_3)\frac{\phi_2}{\phi}(d_3)\sum_{m|\frac{d}{d_3}}\mathrm{Log}\,m+\frac{d}{\phi(d)}\sum_{p|d}\frac{\mathrm{Log}\,p}{p}$$

and we check that the sum of the last three summands is zero, which ends the study of the main term. ◇◇◇

Although I have not been able to prove that u_d is small and often < 0, that is what I expect, which would support (4.0). We can check that $u_{5005} > 0$, and that 5005 and 17017 are the first two counter-examples. It is also worth mentioning that numerical investigations seem to indicate that there are many cancellations in the expression defining u_d.

4.3 Three elementary estimates.

By applying elementary means, we can prove the following lemma :

Lemma 4.4. *If d is a prime, or 2 times a prime, or 6 times a prime different from 5, then, for all $z \ge 1$, we have*

$$0 \le \mu(d)\phi(d)G(z)w_d \le 1.$$

Proof.

∘ When $\nu(d) = 1$, that is, d is prime, we check easily that, in Lemma 4.2, $s \le z/d$ and that the inner sum equals $d/\phi(d)$. Hence, by (2.2),

$$w_d = \frac{\lambda_d}{G(z)\phi(d)} \quad \text{if} \quad \nu(d) = 1.$$

Now it is well known that $0 \le \mu(d)\lambda_d \le 1$ (see for instance Halberstam & Richert [3]), whence the result in this case.

∘ Next, suppose $d = 2p$, $p > 2$. In (4.3), $D \le z/p$ always. If $s \le z/(2p)$, the inner sum is $2p/\phi(2p)$; and if $z/(2p) < s \le z/p$, the inner sum is 2. Hence

$$\text{(4.6)} \qquad G^2(z)w_{2p} = \frac{\mu(2p)}{\phi(2p)}\Big\{\frac{2p}{\phi(2p)}G_{2p}\big(\frac{z}{2p}\big)+2\Big(G_{2p}\big(\frac{z}{p}\big)-G_{2p}\big(\frac{z}{2p}\big)\Big)\Big\}.$$

But

$$G(z) = G_{2p}(z)+\frac{1}{\phi(2)}G_{2p}\big(\frac{z}{2}\big)+\frac{1}{\phi(p)}G_{2p}\big(\frac{z}{p}\big)+\frac{1}{\phi(2p)}G_{2p}\big(\frac{z}{2p}\big),$$

hence

$$G(z) \ge \frac{2p}{\phi(2p)}G_{2p}\big(\frac{z}{2p}\big)+\big(1+\frac{1}{\phi(2)}+\frac{1}{\phi(p)}\big)\Big(G_{2p}\big(\frac{z}{p}\big)-G_{2p}\big(\frac{z}{2p}\big)\Big).$$

We conclude by noticing that $(1+\frac{1}{\phi(2)}+\frac{1}{\phi(p)}) > 2$.

∘ If $d = 6p$ with $p \geq 7$, then the ordered chain of divisors of d is 1, 2, 3, 6, p, $2p$, $3p$, $6p$ (Note that the hypothesis $p \geq 7$ is required here). We get

$$(4.7)\quad G^2(z)w_{6p} = \frac{\mu(6p)}{\phi(6p)}\Big\{\frac{6p}{\phi(6p)}G_{6p}\big(\frac{z}{6p}\big) + (3+\frac{2}{p-1})\Big(G_{6p}\big(\frac{z}{3p}\big) - G_{6p}\big(\frac{z}{6p}\big)\Big)\Big\}.$$

On the other hand, we check that

$$G(z) \geq \frac{6p}{\phi(6p)}G_{6p}\big(\frac{z}{6p}\big) + \big(\frac{6p}{\phi(6p)} - \frac{1}{\phi(6p)}\big)\Big(G_{6p}\big(\frac{z}{3p}\big) - G_{6p}\big(\frac{z}{6p}\big)\Big),$$

and the miracle is that $\frac{6p-1}{\phi(6p)} = 3 + \frac{5}{2(p-1)} \geq 3 + \frac{2}{p-1}$. ⋄⋄⋄

One could ask whether such an elementary approach could go any further, and for instance take care of the case $d = pq$ with $2 < p < q$ where one can check that $u_d < 0$. However for $d = 15$ (which is the first integer not covered by Lemma 4.4) and $z = 10$, we have computed that $\mu(d)\phi(d)G(z)w_d \geq 1.09$ and for $d = 35$ and $z = 42$, we have found that the above quantity (say $\rho(d)$) is greater than 1.05. This last example shows that even the stronger condition $d \leq z$ does not ensure $\rho(d) \leq 1$. As to more positive answers, no counter-example to the guess $\mu(d)w_d \geq 0$ has been found.

We finally give two identities easily derived from (2.4) that throw some light on this question

$$\begin{cases} \displaystyle\sum_{d\leq z^2} \mu(d)w_d = 1, \\ \displaystyle G(z)^2 \sum_{d\leq z^2} \phi(d)w_d = \Big(\sum_{d\leq z}\mu(d)\Big)^2. \end{cases}$$

4.4 Upper bounds.

We derive from (4.3) two kinds of upper bounds.

Lemma 4.5. *For $z \geq 1$ and any positive integer d, we have*

$$|G(z)w_d| \leq \frac{G(z\sqrt{d})}{G(z)}\prod_{p|d}\frac{3p-4}{p(p-1)} \leq \frac{G(z\sqrt{d})}{G(z)}\frac{30.31}{d^{7/10}}.$$

Proof. We start with (4.3) and omit the size conditions on d_1, d_2 and d_3. We then use Lemma 3.1 to write $G_d(z/\sqrt{d}) \leq \frac{\phi(d)}{d}G(z\sqrt{d})$. To prove the second assertion, we remark that the function $p \mapsto (3p-4)/(p-1)p^{1.3}$ is less than 1 if $p \geq 41$. ⋄⋄⋄

Lemma 4.6. *For $z \geq 1$ and any positive integer d, $|G(z)w_d|$ is less than*

$$\frac{G_d(z/\sqrt{d})}{G(z)}\sum_{\substack{\ell mn=d\\ \ell\leq z, mn\leq z}}\frac{\xi(\ell)}{\phi(\ell)}\frac{1}{m\phi(m)}\frac{1}{n} \leq \frac{G(z\sqrt{d})}{G(z)d}\sum_{\substack{\ell|d\\ d/z\leq\ell\leq z}}\xi(\ell),$$

with $\xi(\ell) = \prod_{p|\ell}(2p-3)/(p-1)$.

Proof. We start with (4.3), put $\Delta = d_1 d_3$, insert absolute values inside, and ignore the size conditions on d_1. A simple computation gives the upper bound

$$\frac{G_d(z/\sqrt{d})}{G(z)} \sum_{\substack{lk=d \\ l\leq z, k\leq z}} \frac{\xi(l)}{\phi(l)} \frac{1}{\phi(k)}.$$

To obtain the first inequality we use

$$\frac{k}{\phi(k)} = \sum_{mn=k} \frac{\mu^2(m)}{\phi(m)}$$

and to obtain the second one, we omit the size condition on k and use Lemma 3.1. ◇◇◇

4.5 A computational result.

In this section, the following assumptions and notations are used :

z is a real number $\geq \exp(30)$,

λ is a real number such that $\operatorname{Log}\lambda \leq \frac{39}{50}\operatorname{Log} z$,

q always denotes a positive odd squarefree integer,

$$t(q) = q \sum_{\substack{d\leq\lambda \\ d\equiv 0[q]}} \frac{|G(z)w_d|^2}{\phi(d)}. \tag{4.8}$$

First, let us note the following computational lemma which tells us that the "conjecture" $|G(z)\phi(d)w_d| \leq 1$ is not far from being true for small integers.

Define the deficiency $\Delta_k(x)$ by

$$\Delta_k(x) = \sum_{\substack{d\leq x \\ \nu(d)=k}} \frac{\mu^2(d)}{\phi(d)} \max(|G(z)w_d|^2 - \frac{1}{\phi^2(d)}, 0).$$

Lemma 4.7. *We have*

$$\Delta_1(50\ 000) = \Delta_3(50\ 000) = \Delta_6(50\ 000) = 0,$$
$$\Delta_2(50\ 000) \leq 1.02.10^{-16},$$
$$\Delta_4(50\ 000) \leq 1.54.10^{-12},$$
$$\Delta_5(50\ 000) \leq 1.14.10^{-13}.$$

Explanations of the computations may be found in the section 4.6.

We now introduce the following auxiliary notations :

$$f_2(d) = \prod_{p|d} (\frac{3p-4}{p-1})^2 \frac{1}{p(p-1)}, \tag{4.9}$$

(4.10) $$c_3 = \prod_{p\geq 2}(1+f_2(p)/p) \leq 3.027\ 251\ 641,$$

(4.11) $$f_3(q) = \prod_{p|q}\Big(\frac{f_2(p)}{1+f_2(p)/p}\Big),$$

(4.12) $$f_4(q) = \prod_{p|q}\Big(\frac{1}{p^2-3p+3}\Big),$$

Lemma 4.8. *We have*

(4.13) $$t(q) \leq (1.4)^2 c_3 f_3(q),$$

and

(4.14) $$t(q) \leq 2.301 f_4(q) + 2.10^{-12} q + (1.4)^2 f_3(q)\big(c_3 - \sum_{d\leq 50\ 000/q} \mu^2(d) f_2(d)/d\big).$$

Proof. By using Lemma 4.5 and Lemma 3.5, we get

$$|G(z)w_d| \leq 1.4\prod_{p|d}\frac{3p-4}{p(p-1)}.$$

Now for the $t(q)$, we have

$$\begin{aligned} t(q) \leq & q\sum_{\substack{d\leq A\\ d\equiv 0[q]}}\frac{|G(z)w_d|^2}{\phi(d)} \\ & + q\left(\sum_{d\equiv 0[q]}\frac{\mu^2(d)}{\phi(d)}\Big(1.4\prod_{p|d}\frac{3p-4}{p(p-1)}\Big)^2 - \sum_{\substack{d\leq A\\ d\equiv 0[q]}}\frac{\mu^2(d)}{\phi(d)}\Big(1.4\prod_{p|d}\frac{3p-4}{p(p-1)}\Big)^2\right), \end{aligned}$$

where A is a parameter to be chosen later. With our notations, we get
(4.15)
$$t(q) \leq q\sum_{\substack{d\leq A\\ d\equiv 0[q]}}\frac{|G(z)w_d|^2}{\phi(d)} + (1.4)^2 f_3(q)\left(c_3 - \prod_{p|q}\Big(1+\frac{f_2(p)}{p}\Big)\sum_{\substack{d\leq A/q\\ (d,q)=1}}\mu^2(d)\frac{f_2(d)}{d}\right).$$

To prove (4.13), we take $A<1$ in the above expression. To prove (4.14), we use our previous expression with $A = 50\ 000$ and

$$\sum_{\substack{d\leq 50\ 000\\ d\equiv 0[q]}}\frac{|G(z)w_d|^2}{\phi(d)} \leq \sum_{d\equiv 0[q]}\frac{\mu^2(d)}{\phi(d)^3} + \sum_{d\leq 50\ 000}\frac{\mu^2(d)}{\phi(d)}\max\Big(0, |G(z)w_d|^2 - \frac{1}{\phi(d)^2}\Big),$$

together with Lemma 4.7 . We then note that

$$c_3 - \prod_{p|q}\Big(1+\frac{f_2(p)}{p}\Big)\sum_{\substack{d\leq 50\ 000/q\\ (d,q)=1}}\mu^2(d)\frac{f_2(d)}{d} \leq c_3 - \sum_{d\leq 50\ 000/q}\mu^2(d)\frac{f_2(d)}{d},$$

and that $\sum_{d\geq 1}\mu^2(d)/\phi(d)^3 \leq 2.301$ to get (4.14). $\diamond\diamond\diamond$

Lemma 4.9. *If $\phi(q) \geq 61$ and $q \leq 3\ 000$, then $t(q) \leq 0.001$.*

Proof. If $q \leq 3\ 000$, then

$$c_3 - \sum_{d \leq 50\ 000/q} \mu^2(d) \frac{f_2(d)}{d} \leq 0.089\ 55,$$

and a direct computation using (4.14) yields the result for $q > 285$. If $q \leq 285$, we can replace 0.089 55 by 0.003 41 and a computation gives the result for the remaining q's. ◇◇◇

Lemma 4.10.

(1) *For $q \geq 14\ 000$, we have $t(q) \leq 0.000\ 3$.*
(2) *If $\phi(q) \geq 61$, we have $t(q) \leq 0.001$.*
(3)

$$t(1) - \sum_{\substack{d \leq 2000 \\ (d,2)=1}} \frac{G^2(z)(|w_d|^2 + |w_{2d}|^2)}{\phi(d)} \leq 0.000\ 000\ 9.$$

Proof. We break up the proof into several steps according to the size of q.

∘ By using (4.14) and since $f_3(p)p^{1.25}$ is less than 1 if $p \geq 23$, we get

$$t(q) \leq (1.4)^2 \times 3.0273 \times \frac{38.1}{q^{1.25}} \leq \frac{227}{q^{1.25}}. \tag{4.16}$$

This yields immediately

$$t(q) \leq 0.000\ 3 \quad \text{if} \quad q \geq 51\ 000. \tag{4.17}$$

∘ If $q \leq 51\ 000$ then $\nu(q) \leq 5$ and we easily check that

$$f_3(q) \leq \frac{9^{\nu(q)}}{q^2} \frac{q}{\phi(q)} \leq \frac{9^{\nu(q)}}{q^2} 2.607 \text{ for } \nu(q) \leq 5. \tag{4.18}$$

This upper bound combined with (4.13) ensures us that $t(q) \leq 0.000\ 3$ as soon as $q \geq 14\ 000$ thus concluding the proof of the first part.

∘ If $q \leq 14\ 000$ then $\nu(q) \leq 4$ and we get with (4.14) and (4.18)

$$t(q) \leq 0.001 \quad \text{if} \quad q \geq 2\ 570. \tag{4.19}$$

∘ Lemma 4.9 concludes the proof of the first and second part of Lemma 4.10. The proof of the third part of Lemma 4.10 follows the same line as the proof of Lemma 4.8. We call $\mathcal{F}$ the set $\{d\ odd, d \leq 2000\} \cup \{2d,\ d\ odd, d \leq 2000\}$. Then

$$\sum_{\substack{d \leq \lambda \\ d \notin \mathcal{F}}} \frac{|G(z)w_d|^2}{\phi(d)} \leq \sum_{d \notin \mathcal{F}} \frac{\mu^2(d)}{\phi(d)^3} + \sum_{d \leq 50\ 000} \frac{\mu^2(d)}{\phi(d)} \max(0, |G(z)w_d|^2 - \frac{1}{\phi(d)})$$
$$+ (1.4)^2 \left(c_3 - \sum_{d \leq 50\ 000} \mu^2(d) \frac{f_2(d)}{d} \right),$$

which is checked to be $\leq 0.000\ 000\ 9$ as required. We did this computation by generating the d's : first the primes, then the product of two primes, up to the product of 6 primes, which is a bit annoying to program but very efficient since it avoids the factorization of d. ◇◇◇

4.6. About the computation of w_d.

Computing a "small" finite number of w_d with a decent accuracy is no problem, let us say if "small" is about 1000, and this by using the asymptotic expression. But we have to verify that $|G(z)w_d| \leq 1$ for a very large number of d's. In order to achieve this, we have worked with $\nu(d)$ fixed and generated the integers d instead of analysing their arithmetical structure. In practice, $\nu(d)$ is between 1 and 6. Now Lemma 4.4 tells us that we do not have to deal with some special cases. For the other ones, we have to compute u_d. The following remark enables us to implement a very fast way of computing w_d for $\nu(d) = 2$, 3, or 4. There will be few d's with $\nu(d) = 5$ or 6 in our range of interest so that they can be computed directly.

Put

$$A(t) = \sum_{\substack{m|t \\ m<\sqrt{t}}} \mathrm{Log}(\frac{m^2}{t}) = -2^{\nu(t)-1} \mathrm{Log}\, t + 2\,\mathrm{Log}\left(\prod_{\substack{m|t \\ m<\sqrt{t}}} m\right) \leq 0.$$

The introduction of this quantity simplifies the expression of u_d. We verify that

$$A(1) = 0, \qquad A(p) = -\mathrm{Log} p,$$

$$A(pq) = -2\mathrm{Log} q \text{ if } p < q,$$

$$A(pqr) = \begin{cases} -2\mathrm{Log}(pqr) \text{ if } p < q < r < pq, \\ -4\mathrm{Log} r \quad \text{if } p < q < pq < r, \end{cases}$$

$$A(pqrs) = \begin{cases} -8\,\mathrm{Log}\, s \text{ if } p < q < r < pqr < s, \\ -8\,\mathrm{Log}(pqrs) + 2\,\mathrm{Log}(p^3q^3r^3s) \text{ if } p < q < r < s < pqr \text{ and } qr < ps, \\ -4\,\mathrm{Log}(qrs) \text{ if } p < q < r < s < ps < qr. \end{cases}$$

Let us prove the last of these.

Proof. The problem is to find the divisor m of d such that

(1) $m \leq \frac{d}{m}$ and (2) there are at least 8 divisors of d which are less than m.

If $s > pqr$, $m = pqr$ is a convenient choice.

If $s < pqr$ and $qr < ps$ then qr is larger than $1, p, q, r, pq, pr$ and qr.

If $s > qr$, $m = s$ is an admissible choice.

If $s < qr$, $m = qr$ is an admissible choice.

If $s < pqr$ and $ps < qr$ then $m = ps$ is larger than $1, p, q, r, s, pq, pr$ and ps, and therefore is an admissible choice. $\diamond\diamond\diamond$

5. Proof of Proposition 1

Recall that

$$\mathcal{R} = \sum_{N\in]X,2X]} \mathfrak{S}_2^{-1}(N) r_2(N), \tag{2.14}$$

thus

$$\mathcal{R} = \sum_{N\in]X,2X]} \mathfrak{S}_2^{-1}(N)\Big(\sum_{\substack{p_1+p_2=N \\ p_1\leq X}} \mathrm{Log}\, p_1 - \sum_{\substack{p_1+p_2=N \\ p_1\leq X \\ p_2<\sqrt{X}}} \mathrm{Log}\, p_1\Big),$$

$$= \sum_{N\in]X,2X]} \mathfrak{S}_2^{-1}(N)\Big(\sum_{\substack{p_1+p_2=N \\ p_1\leq X}} \mathrm{Log}\, p_1\Big) + \mathcal{O}^*(\sqrt{X}\,\mathrm{Log}\, X).$$

We want to obtain a lower bound for this quantity. By expanding the arithmetical factor as a convolution product, we get, from (2.7),

$$\begin{aligned}\mathcal{R}+\mathcal{O}^*(\sqrt{X}\operatorname{Log}X) &\geq \sum_{N\in]X,2X]}\mathfrak{S}_2^{-1}(N)\sum_{\substack{p_1+p_2=N\\ p_1\leq X}}\operatorname{Log}p_1\frac{\operatorname{Log}p_2}{\operatorname{Log}2X},\\ &\geq \mathfrak{S}_2^{-1}\sum_{\substack{d\leq 2X\\ (d,2)=1}}\frac{\mu(d)}{\phi(d)}(B_d(X)+\mathcal{O}^*(C_d(X))),\end{aligned}$$

where

$$\text{(5.1)}\quad \begin{aligned}B_d(X) &= \sum_{\substack{N\in]X,2X]\\ N\equiv 0[2d]}}\sum_{\substack{p_1+p_2=N\\ p_1\leq X\\ (p_1p_2,d)=1}}\operatorname{Log}p_1\frac{\operatorname{Log}p_2}{\operatorname{Log}2X},\\ &= \sum_{a \bmod^* d}\left(\sum_{\substack{p_1\equiv a[d]\\ p_1\leq X}}\operatorname{Log}p_1\sum_{\substack{p_2\equiv -a[d]\\ X-p_1<p_2\leq 2X-p_1}}\frac{\operatorname{Log}p_2}{\operatorname{Log}2X}\right),\end{aligned}$$

and

$$C_d(X)=\frac{\operatorname{Log}^2 d}{\operatorname{Log}2X}\quad\text{if } d \text{ is prime, and 0 otherwise.}$$

We will study $B_d(X)$ according to the size of d.

For the small d's, that is to say, those which are in $\mathcal{D}$ (see *Notations*), we have, by Lemma 0,

$$B_d(X)=\frac{1+\mathcal{O}^*(3\epsilon_d)}{\phi(d)}\frac{X\theta(X)}{\operatorname{Log}2X},$$

and

$$\begin{aligned}\mathfrak{S}_2^{-1}\sum_{d\in\mathcal{D}}\frac{\mu(d)}{\phi^2(d)}(1+\mathcal{O}^*(3\epsilon_d))\frac{\operatorname{Log}X}{\operatorname{Log}2X} &\geq \frac{0.66025-0.00369}{1.320323}\times 0.9897,\\ &\geq 0.4921.\end{aligned}$$

For the lower medium range of d's, we use the Brun-Titschmarch inequality of Montgomery & Vaughan :

$$\mathfrak{S}_2^{-1}\sum_{\substack{d\leq 4000\\ d\notin\mathcal{D}\\ (d,2)=1}}\frac{\mu^2(d)}{\phi(d)}B_d(X)\leq 2\mathfrak{S}_2^{-1}\frac{X}{\operatorname{Log}X}\sum_{\substack{d\leq 4000\\ d\notin\mathcal{D}\\ (d,2)=1}}\frac{\mu^2(d)}{\phi^2(d)}\frac{\theta(X)}{1-\frac{\operatorname{Log}d}{\operatorname{Log}X}},$$

which is less than $0.01288\frac{X^2}{\operatorname{Log}X}$. Note that since we are only looking for a lower bound of $\mathcal{R}$, we could have discarded the d's for which $\mu(d)=1$; this process would enable us to replace 0.01288 by 0.00651, but it is not required for our final result.

For the upper medium range of d's, we put $Q = 4000\,\mathrm{Log}\,X$ and we have

$$\mathfrak{S}_2^{-1}\sum_{\substack{4000<d\le Q\\(d,2)=1}}\frac{\mu^2(d)}{\phi(d)}B_d(X)\le 2\mathfrak{S}_2^{-1}\frac{X}{\mathrm{Log}\,2X}\frac{\theta(X)}{1-\frac{\mathrm{Log}\,Q}{\mathrm{Log}\,X}}\sum_{\substack{4000<d\le Q\\(d,2)=1}}\frac{\mu^2(d)}{\phi^2(d)},$$

$$\le 2\frac{X}{\mathrm{Log}\,2X}\frac{\theta(X)}{1-\frac{\mathrm{Log}\,Q}{\mathrm{Log}\,X}}\mathfrak{S}_2^{-1}\Big(\prod_{p\ge 3}(1+\frac{1}{(p-1)^2})-\sum_{\substack{d\le 4000\\(d,2)=1}}\frac{\mu^2(d)}{\phi^2(d)}\Big),$$

$$\le 0.0002691\frac{X^2}{\mathrm{Log}\,X}.$$

For the large d's, we have

$$\mathfrak{S}_2^{-1}\sum_{\substack{Q<d\le 2X\\(d,2)=1}}\frac{\mu^2(d)}{\phi(d)}B_d(X)\le \mathfrak{S}_2^{-1}\sum_{\substack{Q<d\le 2X\\(d,2)=1}}\frac{\mu^2(d)}{\phi(d)}\theta(X)(\frac{X}{d}+1),$$

$$\le \mathfrak{S}_2^{-1}(X\theta(X)\frac{8}{3Q}+\theta(X)G(2X)),$$

$$\le 0.000\,505\frac{X^2}{\mathrm{Log}\,X},$$

by Lemma 3.10 and Lemma 0.

Putting all these estimates together, we obtain

$$\mathcal{R}\ge 0.478\frac{X^2}{\mathrm{Log}\,X}.$$

6. The main term. Proof of Proposition 2.

Let us recall

$$\mathcal{R}_1^* = \sum_{\substack{N\in]X,2X]\\ r_2(N)\neq 0}}\mathfrak{S}_2^{-1}(N)\Big\{\sum_{d\le\lambda}w_d\sum_{a \bmod^* d}\frac{\mu(d)}{\phi(d)}\tilde{\theta}(X)e\big(-N\frac{a}{d}\big)\Big\}. \tag{2.20}$$

Following the "Conjectural behaviour of $R_2(N)$" at the end of 2.2, we will see that the first d's lead to the main contribution. The other ones will be discarded in a simple way. A little problem arises because $G(z)w_d$ is not $\mu(d)/\phi(d)$ but has an asymptotic expression of the shape $\frac{\mu(d)}{\phi(d)}(1+v_d/G(z))$ where v_d is numerically comparable to -1. But $G(z)$ is about 30 and, hence, we can not look at v_d just as an error term. We will also see how to take advantage of the sign of $c_d(N)$ to deal with v_d.

We define

$$\mathcal{G} = \big\{d\le 2000/\ d \text{ odd and } \mu^2(d)=1\big\}, \tag{6.1}$$

and $\mathcal{F}=\mathcal{G}\cup 2\mathcal{G}$. Then we split $\mathcal{R}_1^*$ in three parts :

$$\mathcal{R}_1^* = \mathcal{R}_{11}^*+\mathcal{R}_{12}^*+\mathcal{R}_{13}^*, \tag{6.2}$$

with

$$\mathcal{R}_{11}^* = \tilde{\theta}(X) \sum_{\substack{N\in]X,2X] \\ r_2(N)\neq 0}} \mathfrak{S}_2^{-1}(N) \Big\{ \sum_{d\in\mathcal{F}} \frac{\mu^2(d)}{G(z)\phi^2(d)} c_d(N) \Big\}, \tag{6.3}$$

$$\mathcal{R}_{12}^* = \tilde{\theta}(X) \sum_{\substack{N\in]X,2X] \\ r_2(N)\neq 0}} \mathfrak{S}_2^{-1}(N) \Big\{ \sum_{d\in\mathcal{F}} \frac{\mu(d)}{\phi(d)} (w_d - \frac{\mu(d)}{G(z)\phi(d)}) c_d(N) \Big\}, \tag{6.4}$$

and (cf (2.17))

$$\mathcal{R}_{13}^* = \tilde{\theta}(X) \sum_{\substack{d\leq\lambda \\ d\notin\mathcal{F}}} \frac{\mu(d)}{\phi(d)} w_d \sum_{a \bmod^* d} \bar{U}(a/d). \tag{6.5}$$

6.1. Preliminary lemmas.

The following lemma will be useful for dealing in an explicit way with incomplete sums of multiplicative functions. The error terms which arise here will be in practice very small.

A complex-valued function h is said to be strongly multiplicative if

(1) $h(mn) = h(m)h(n)$ whenever m and n are coprime integers ;
(2) $h(p^k) = h(p)$ for every prime p and every positive integer k.

Lemma 6.1. *Let X and Y be positive real numbers with $X \leq Y$. Let f and g be complex-valued functions with compact support. Let h be a strongly multiplicative function with the following property : if h_1 is defined by $h_1 = h \star \mu$, then we have*

(i) $h_1(p) > -p$ for all primes p,

(ii) there exists a real number c such that $\sum_{Z<\ell} |h_1(\ell)|/\ell \leq c/Z$ for all $Z > 0$.

Then the double sum

$$\sum_{\substack{n\in]X,Y] \\ d\geq 1}} h(n)f(d)g((d,n)) =$$

$$(Y-X)\prod_{p\geq 2}\big(1+\frac{h_1(p)}{p}\big)\sum_d \Big\{\frac{f(d)}{d}\prod_{p|d}\frac{1}{1+\frac{h_1(p)}{p}}\sum_{r|d} g(r)h(r)\phi(d/r)\Big\}$$
$$+\mathcal{O}^*\Big(c(1-\frac{X}{Y})\sum_d \frac{|f(d)|}{d}\sum_{r|d}|g(r)h(r)|\phi(d/r)\Big)$$
$$+\mathcal{O}^*\Big(\sum_d |f(d)|\prod_{p|d}\frac{1}{1+\frac{|h_1(p)|}{p}}\big(\sum_{\ell\leq Yd}|h_1(\ell)|\big)\sum_{r|d}|g(r)h(r)|\phi(d/r)\Big).$$

Proof. Let us note first that $h_1(\ell) = 0$ if $\mu(\ell) = 0$ because of the strong multiplicativity of h and also that the hypothesis (ii) ensures the convergence of $\prod_{p\geq 2}\big(1 + h_1(p)/p\big)$.

Let us call S our sum. We have

$$S = \sum_d f(d) \sum_{r|d} g(r) \sum_{\substack{n\in]X,Y] \\ (n,d)=r}} h(n).$$

Let $H(d,r)$ be the innermost sum. Then we put $n = r\tilde{r}m$ with $(m,r) = 1$ and $[p|\tilde{r} \implies p|r]$. We have $h(n) = h(r\tilde{r})h(m) = h(r)h(m)$ since h is strongly multiplicative. We then use $h(m) = \sum_{\ell|m} h_1(\ell)$ and write the condition $\ell|m$ as $\ell|n$ and $(\ell, r) = 1$. We thus get

$$H(d,r) = \sum_{\substack{m\in]X/r,Y/r] \\ (m,d/r)=1}} h(r) \sum_{\substack{\ell|m \\ (\ell,r)=1}} h_1(\ell).$$

We see that even $(\ell, d) = 1$. Hence

$$\begin{aligned} H(d,r) =& h(r) \sum_{\substack{(\ell,d)=1 \\ \ell\le Y/r}} h_1(\ell) \sum_{\substack{m\in]X/r,Y/r] \\ (m,d/r)=1 \\ \ell|m}} 1, \\ =& h(r) \sum_{\substack{(\ell,d)=1 \\ \ell\le Y/r}} h_1(\ell)\Big(\frac{Y-X}{d\ell}\phi(d/r) + \mathcal{O}^*(\phi(d/r))\Big). \end{aligned}$$

We then remark that

$$\sum_{\substack{(\ell,d)=1 \\ \ell\le Y/r}} \frac{h_1(\ell)}{\ell} = \sum_{(\ell,d)=1} \frac{h_1(\ell)}{\ell} + \mathcal{O}^*(cr/Y),$$

and from this the stated result follows in a straightforward manner. ⋄ ⋄ ⋄

We will also require the following computational lemma :

Lemma 6.2. *We have*

$$\prod_{p\ge3} \Big(1 - \frac{1}{p(p-1)}\Big) = 0.74791 + \mathcal{O}^*(10^{-5}) \ ;$$

$$\prod_{p\ge3} \Big(1 + \frac{1}{(p-1)^2}\Big) = 1.4132 + \mathcal{O}^*(10^{-4}) \ ;$$

$$\prod_{p\ge3} \Big(1 + \frac{2}{p^2-2p+2}\Big) = 1.7668 + \mathcal{O}^*(10^{-4}) \ ;$$

$$\prod_{p\ge3} \Big(1 + \frac{2p-3}{(p-1)(p^2-p-1)}\Big) = 1.5801 + \mathcal{O}^*(10^{-4}) \ ;$$

and

$$\sum_{\substack{d\le2000 \\ (d,2)=1}} \mu^2(d) \prod_{p|d} \frac{2p-3}{(p-1)(p^2-p-1)} = 1.5789 + \mathcal{O}^*(10^{-4});$$

$$\sum_{\substack{d\le2000 \\ (d,2)=1}} \mu^2(d) \prod_{p|d} \frac{p^2-p-1}{p(p-1)^2} = 4.3559 + \mathcal{O}^*(10^{-4});$$

$$\sum_{\substack{d\le2000 \\ (d,2)=1}} \mu^2(d) \prod_{p|d} \frac{2p-3}{p(p-1)} = 8.1512 + \mathcal{O}^*(10^{-4}).$$

6.2. An estimate for $\mathcal{R}_{11}^*$

Define

$$\begin{cases} \rho = \sum_{\substack{N\in]X,2X]\\ N\ even}} \mathfrak{S}_2^{-1}(N) \sum_{\substack{2000<d\\ d\ odd}} \frac{\mu^2(d)}{\phi(d)^2}\phi((d,N)), \\ \rho^* = \sum_{\substack{N\in]X,2X]\\ N\ even}} \mathfrak{S}_2^{-1}(N) \sum_{\substack{d\leq 2000\\ d\ odd}} \frac{\mu^2(d)}{\phi(d)^2}\phi((d,N)). \end{cases}$$

Then, by using $c_{2d}(N) = c_d(N)$ for odd d and even N, and $|c_d(N)| \leq \phi((d,N))$, we get (cf (6.3))

$$\mathcal{R}_{11}^* = \frac{\tilde{\theta}(X)}{G(z)}\{\delta X + 2\mathcal{O}^*(\rho)\}.$$

Now $\rho+\rho^*$ is easily evaluated and an estimation of ρ^* is given by Lemmas 6.1, 3.10 and 6.2. In this way, we get an upper bound for ρ.

$$\begin{aligned}\rho+\rho^* &= \mathfrak{S}_2^{-1}\prod_{p\geq 3}\Big(1+\frac{1}{(p-1)^2}\Big)\sum_{\substack{N\in]X,2X]\\ N\ even}}\prod_{\substack{p|N\\ p\neq 2}}\frac{p(p-2)}{p(p-2)+2},\\ &= \mathfrak{S}_2^{-1}\prod_{p\geq 3}\Big(1+\frac{1}{(p-1)^2}\Big)\sum_{(d,2)=1}\prod_{p|d}\frac{-2}{p(p-2)+2}\Big(\frac{X}{2d}+\mathcal{O}^*(1)\Big).\end{aligned}$$

Thus

$$\begin{aligned}\rho+\rho^* = &\frac{X}{2}\mathfrak{S}_2^{-1}\prod_{p\geq 3}\Big(1+\frac{1}{(p-1)^2}\Big)\prod_{p\geq 3}\Big(1-\frac{2}{p(p(p-2)+2)}\Big)\\ &+\mathcal{O}^*\Big[\mathfrak{S}_2^{-1}\prod_{p\geq 3}\Big(1+\frac{1}{(p-1)^2}\Big)\prod_{p\geq 3}\Big(1+\frac{2}{p(p(p-2)+2)}\Big)\Big].\end{aligned}$$

On the other hand, Lemma 6.1 gives us

$$\mathfrak{S}_2\rho^* = \frac{X}{2}\prod_{p\geq 3}\Big(1-\frac{1}{p(p-1)}\Big)\sum_{\substack{d\leq 2000\\ (d,2)=1}}\mu^2(d)\prod_{p|d}\frac{2p-3}{(p-1)(p^2-p-1)}$$

$$+\mathcal{O}^*\left(\frac{4}{2}\sum_{\substack{d\leq 2000\\ (d,2)=1}}\mu^2(d)\prod_{p|d}\frac{p^2-p-1}{p(p-1)^2}\right)+\mathcal{O}^*\left(G(2000\cdot 2X)\sum_{\substack{d\leq 2000\\ (d,2)=1}}\mu^2(d)\prod_{p|d}\frac{2p-3}{p(p-1)}\right).$$

From this and Lemma 6.2, we deduce that

$$|\mathcal{R}_{11}^*| \leq \frac{\tilde{\theta}(X)}{G(z)}X(\delta+0.000794). \tag{6.6}$$

6.3. An estimate for $\mathcal{R}_{12}^*$.

Define

$$v_d = \mu(d)\phi(d)G^2(z)(w_d - \frac{\mu(d)}{G(z)\phi(d)}), \tag{6.7}$$

and

$$H(N) = \sum_{d\in\mathcal{F}} \frac{\mu^2(d)}{\phi^2(d)} v_d c_d(N) = \sum_{\substack{d\leq 2000\\(d,2)=1}} \frac{\mu^2(d)}{\phi^2(d)}(v_d + v_{2d})c_d(N).$$

With this, we have

$$\mathcal{R}_{12}^* = \tilde{\theta}(X) \sum_{\substack{N\in]X,2X]\\ r_2(N)\neq 0}} \mathfrak{S}_2^{-1}(N)\frac{H(N)}{G^2(z)}. \tag{6.8}$$

Let us comment on these definitions : v_d depends also on z and, according to Lemma 4.3, it has a limit as z goes to infinity. We have conjectured that v_d is "very often" negative and we have checked that this is true for small d's, provided z is greater than exp(30). Let us also recall that $v_1 = 0$.

In order to obtain an upper bound for $\mathcal{R}_{12}^*$, we want to find the maximum of $H(N)$. If we use $|c_d(N)| \leq \phi((d, N))$, we will lose the fact that $c_d(N)$ and v_d vary in sign. Let us write $H(N)$ in another way :

$$H(N) = \sum_{\substack{D\leq 2000\\(D,2)=1\\ D|N}} D\mu(D) \sum_{\substack{d\leq 2000\\(d,2)=1\\ D|d}} \mu(d)\frac{v_d + v_{2d}}{\phi^2(d)}. \tag{6.9}$$

Now we define

$$a_D = D\mu(D) \sum_{\substack{d\leq 2000\\(d,2)=1\\ D|d}} \mu(d)\frac{v_d + v_{2d}}{\phi^2(d)}, \tag{6.10}$$

so we can simply write $H(N) = \sum_{D|N} a_D$. What can we say about the sign of a_D ? Because the series which defines a_D is convergent and the terms v_d and v_{2d} are conjectured to vary very slowly, we may think that a_D is of the sign of its first term, that is to say "probably negative", except for a_1 because $v_1 = 0$. If such a thing happens, it will be easy to get $\max_{N\ even} H(N)$. But verifying this requires only a finite (reasonnably small) number of computations.

We have made the required computations and found that

$$\begin{cases} 0 \leq a_1 \leq 0.3209, \\ a_D \leq 0 \ \text{ for } \ 1 \leq D \leq 2000 \ \text{ and } \ D \neq 323 = 17 \times 19, \\ a_{17} + a_{19} + a_{323} \leq 0. \end{cases}$$

From this, we get

$$\max_{N\ even} H(N) = a_1,$$

and, hence with (6.8) and Lemma 3.11,

$$\mathcal{R}_{12}^* \le \frac{\tilde\theta(X)X}{G(z)}\frac{0.2319}{G(z)}\,\delta^{36/37}. \tag{6.11}$$

Remark : In fact, here we have choosen the set $\mathcal{F}$ and it happens that, if we do not impose that $[d \in \mathcal{F}$ and d odd$]$ implies that $2d$ is also in $\mathcal{F}$, then we no longer control the sign of the a_D's if D is too large.

6.4. An estimate for $\mathcal{R}_{13}^*$.

Let us recall that

$$\mathcal{R}_{13}^* = \tilde\theta(X)\sum_{\substack{d\le\lambda\\ d\notin\mathcal{F}}}\frac{\mu(d)}{\phi(d)}w_d\sum_{a \bmod^* d}\bar U\big(\frac{a}{d}\big). \tag{6.5}$$

We apply Cauchy's inequality and the large sieve inequality to get

$$|\mathcal{R}_{13}^*|^2 \le \frac{\tilde\theta(X)^2}{G^2(z)}\sum_{\substack{d\le\lambda\\ d\notin\mathcal{F}}}\frac{|G(z)w_d|^2}{\phi(d)}\|U\|_2^2(X+\lambda^2).$$

Now

$$\sum_{\substack{d\le\lambda\\ d\notin\mathcal{F}}}\frac{|G(z)w_d|^2}{\phi(d)} \le 0.000\ 000\ 9$$

by Lemma 4.10. By using Lemma 3.11, we get

$$|\mathcal{R}_{13}^*| \le \frac{X\tilde\theta(X)}{G(z)}(\delta^{18/19}0.5159\times 0.000\ 000\ 9)^{1/2},$$

hence

$$|\mathcal{R}_{13}^*| \le 0.0008\frac{X^2}{G(z)}\delta^{9/19}. \tag{6.12}$$

6.5. Conclusion.

We have

$$\mathcal{R}_1^* \le |\mathcal{R}_{11}^*| + \mathcal{R}_{12}^* + |\mathcal{R}_{13}^*|,$$

and, by using (6.6), (6.11), and (6.12), we obtain

$$\mathcal{R}_1^* \le \frac{X^2}{G(z)}\{\delta + 0.0008 + \frac{0.232}{G(z)}\delta^{36/37} + 0.0008\ \delta^{9/19}\},$$

which concludes the proof of Proposition 2.

7. An upper bound for the dispersion. Proof of Proposition 3.

Here we deal with $\mathcal{R}_2^*$ which is given by

$$\mathcal{R}_2^* = \sum_{d\le\lambda}\sum_{a \bmod^* d}\Big[w_d\big(T(a/d) - \frac{\mu(d)}{\phi(d)}\tilde\theta(X)\big)\Big]\bar U(a/d). \tag{2.22}$$

7.1. Preliminary.

First, it is nicer, as will be clear in a moment, to work with

$$\tilde{T}(\alpha) = \sum_{\sqrt{X}<p\leq X} \operatorname{Log} p \; e(p\alpha). \tag{7.1}$$

This gives rise to a neglegible error term :

$$\mathcal{R}_{21}^* = \sum_{d\leq\lambda} \sum_{a \bmod^* d} \left[w_d T_1(a/d) \bar{U}(a/d) \right], \tag{7.2}$$

with $T_1(\alpha) = T(\alpha) - \tilde{T}(\alpha)$. We treat (7.2) by using Cauchy's inequality

$$|\mathcal{R}_{21}^*|^2 \leq \sum_{d\leq\lambda} \sum_{a \bmod^* d} |U(a/d)|^2 \sum_{d\leq\lambda} \sum_{a \bmod^* d} w_d^2 |T_1(a/d)|^2.$$

We apply the large sieve inequality to treat the first factor. As for the second, we use $|G(z)w_d| \leq 60.62 \; d^{-7/10}$ proved by combining Lemma 4.6 and Lemma 3.5. Then we integrate by parts the resulting expression and apply the large sieve inequality. Thus

$$\sum_{d\leq\lambda} \sum_{a \bmod^* d} w_d^2 |T_1(a/d)|^2 \leq 60.62^2 \frac{7}{5} \int_1^\lambda \sum_{d\leq t} \sum_{a \bmod^* d} |T_1(a/d)|^2 \frac{dt}{t^{12/5}} + \frac{60.62^2}{\lambda^{7/5}} \sum_{d\leq\lambda} \sum_{a \bmod^* d} |T_1(a/d)|^2$$

which is now not more than

$$\frac{60.62^2}{G^2(z)} \|T_1\| (\sqrt{X} + \frac{10}{3} \lambda^{3/5}).$$

Collecting these estimates together with Lemma 3.11 and $\theta(t) < 1.002t$ for $t > 0$ (cf [21]), we obtain

$$|\mathcal{R}_{21}^*|^2 \leq 60.62^2 \cdot 1.002 \cdot 0.5159 \frac{1 + \frac{10}{3} X^{-8/25}}{X} (1 + X^{-2/5}) \frac{X^4}{G^2(z)} \leq 10^{-25} \frac{X^4}{G^2(z)}.$$

7.2. Dispersion and multiplicative characters.

Now we study the main part of $\mathcal{R}_2^*$ which we call $\mathcal{R}_{22}^*$:

$$\mathcal{R}_{22}^* = \mathcal{R}_2^* - \mathcal{R}_{21}^*. \tag{7.4}$$

Let us remark first that $\tilde{\theta}(X) = \tilde{T}(0)$ and prove the following lemma which uses well-known ideas :

Lemma 7.1. *Let d be a positive integer. If $S(\alpha)=\sum_n a_n e(n\alpha)$ is such that $a_n=0$ as soon as n and d have a common prime factor, then*

$$\sum_{a \bmod^* d}\left|S(a/d)-\frac{\mu(d)}{\phi(d)}S(0)\right|^2=\frac{1}{\phi(d)}\sum_{\substack{\chi mod\ d\\ \chi\neq\chi_0}}|\tau(\chi)|^2\left|\sum_n a_n\chi(n)\right|^2,$$

where χ_0 is the principal character modulo d, and

$$\tau(\chi)=\sum_{a \bmod^* d}\chi(a)e(a/d).$$

We also have, if $q\neq 1$,

$$\sum_{\chi mod^* q}\left|\sum_n a_n\chi(n)\right|^2=\sum_{d|q}\mu(\frac{q}{d})\phi(d)\sum_{b \bmod^* d}\left|\sum_{n\equiv b[d]}a_n-\frac{S(0)}{\phi(d)}\right|^2.$$

Proof. Let us recall that, if d is squarefree, $|\tau(\chi)|^2$ is equal to the conductor of χ. We have, for all b modulo d,

$$\sum_{n\equiv b[d]}a_n=\frac{1}{\phi(d)}\sum_{\chi mod\ d}\bar{\chi}(b)\sum_n a_n\chi(n),$$

hence, for all a coprime with d :

$$S(a/d)=\frac{1}{\phi(d)}\sum_{\chi mod\ d}\chi(a)\tau(\bar{\chi})\sum_n a_n\chi(n).$$

Now we have $\tau(\chi_0)=\mu(d)$ and the orthogonality of characters concludes the proof of the first part.

For the second statement, we first note that Parseval's identity gives us

$$\sum_{d|q}\sum_{\chi mod^* d}\left|\sum_n a_n\chi(n)\right|^2=\phi(q)\sum_{b \bmod^* q}|\sum_{n\equiv b[q]}a_n|^2.$$

The latter sum is equal to

$$\sum_{b \bmod^* q}|\sum_{n\equiv b[q]}a_n-\frac{S(0)}{\phi(q)}|^2+\frac{|S(0)|^2}{\phi(q)},$$

and Moëbius' inversion formula establishes our result. ◇◇◇

We apply Cauchy's inequality to $\mathcal{R}_{22}^*$ to obtain

$$|\mathcal{R}_{22}^*|^2 \leq \|U\|_2^2(X+\lambda^2)B(\lambda,X), \tag{7.5}$$

where

$$B(\lambda,X)=\sum_{d\leq\lambda}|w_d|^2\sum_{a\bmod^* d}|\tilde{T}(a/d)-\frac{\mu(d)}{\phi(d)}\tilde{T}(0)|^2. \tag{7.6}$$

Because of the factor $|w_d|^2$, the terms with d small in (7.6) give a good approximation to $B(\lambda,X)$. But we can do better by using multiplicative characters and see that the distribution modulo small d's has an even greater influence. It is convenient to introduce the notation

$$\langle\tilde{T},\chi\rangle=\sum_n a_n\overline{\chi(n)}.$$

By using Lemma 7.1, we get

$$B(\lambda,X)=\sum_{\substack{1\neq q\leq\lambda\\(q,2)=1}}\mu^2(q)\frac{t(q)}{G^2(z)}\sum_{\chi mod^* q}|\langle\tilde{T},\chi\rangle|^2, \tag{7.7}$$

where

$$t(q)=q\sum_{\substack{d\leq\lambda\\d\equiv0[q]}}\frac{|G(z)w_d|^2}{\phi(d)},$$

which has already been studied. In (7.7), we have included a $\mu^2(q)$ to remind the reader that $t(q)$ vanishes when q is not squarefree, and we have added the condition $(q,2)=1$ because otherwise, there are no primitive characters modulo q (q is squarefree).

The large sieve inequality provides us with the following estimate (very good for Q^2 near to X), valid for any positive real number Q,

$$\sum_{q\leq Q}G(Q/q)\sum_{\chi mod^* q}|\langle\tilde{T},\chi\rangle|^2\leq\|\tilde{T}\|_2^2(X+Q^2). \tag{7.8}$$

We take $Q=\sqrt{X/10}$ in the previous inequality and substract from each side the term corresponding to $q=1$. By Lemma 3.4, $G(Q)\geq\frac{1}{2}\operatorname{Log}X$, whence

$$\sum_{1\neq q\leq Q}G(Q/q)\sum_{\chi mod^* q}|\langle\tilde{T},\chi\rangle|^2\leq 0.6\tilde{\theta}(X)X\operatorname{Log}X. \tag{7.9}$$

Then, our upper bound for $B(\lambda,X)$ will follow from the two following facts : If q is sufficiently large, $t(q)$ is small, and hence the corresponding contribution is small; if q is small, Lemma 0 ensures that $\langle\tilde{T},\chi\rangle$ is small, and hence the corresponding contribution is small.

If q is in $\mathcal{D}$, we have, for every b coprime with q,

$$|\sum_{\substack{\sqrt{X}<p\leq X\\p\equiv b[q]}}\operatorname{Log}p-\frac{\tilde{T}(0)}{\phi(q)}|\leq\epsilon_q\frac{X}{\phi(q)}.$$

By using Lemma 4.6, (7.6) and (7.8), we get

$$
\begin{aligned}
B(\lambda, X) \leq & \frac{1}{G^2(z)} \sum_{\substack{q \in \mathcal{D} \\ q \neq 1}} \mu^2(q) t(q) \sum_{\chi mod^* q} |\langle \tilde{T}, \chi \rangle|^2 \\
& + \frac{0.001 - 0.0003}{G^2(z)} 0.6\tilde{\theta}(X) X \frac{\operatorname{Log} X}{G(Q/14000)} \\
& + \frac{0.0003}{G^2(z)} 0.6\tilde{\theta}(X) X \frac{\operatorname{Log} X}{G(Q/\lambda)},
\end{aligned} \tag{7.10}
$$

and, for q in $\mathcal{D}$, we use the second part of Lemma 7.7 together with

$$t(q) \leq 2.301 f_4(q) + 2.10^{-12} q + 0.00125 f_3(q)$$

which is written with the notations of Lemma 4.8 and which follows from (4.14) by noticing that $50\ 000/q \geq 50\ 000/105$. We give now some partial computations

$$
\begin{cases}
\sum_{\substack{q \in \mathcal{D} \\ q \neq 1}} t(q) \sum_{\chi mod^* q} |\langle \tilde{T}, \chi \rangle|^2 \leq 0.000\ 001\ 1, \\
G(Q/\lambda) \geq 0.2 \operatorname{Log} X, \\
G(Q/14000) \geq 0.443 \operatorname{Log} X.
\end{cases}
$$

The fact that by using Lemma 7.1, ϵ_d^2 appears instead of ϵ_d is a remarkable feature of this proof.

We finally get

$$B(\lambda, X) \leq 0.00185 \frac{X^2}{G^2(z)}. \tag{7.11}$$

Rm : In order to see the strength of (7.10), it is worth saying that the involved idea may be used for proving an effective Barban-Davenport-Halberstam Theorem with a saving of a small constant over the trivial estimate (instead of saving any power of $\operatorname{Log} X$). This has been pointed to me by professor Iwaniec during some valuable discussions.

7.3. End of the proof of Proposition 3.

By using (7.3), (7.5), (7.11) and Lemma 3.11, we get

$$|\mathcal{R}_2^*| \leq 0.0309 \frac{X^2}{G(z)} \delta^{9/19},$$

which concludes the proof of Proposition 3.

8. A weighted large sieve inequality. Proof of Proposition 4.

The main result of this section lies in the following theorem :

Theorem 8.1. *Let $T(\alpha) = \sum_{n\leq X} a_n e(n\alpha)$ be a trigonometric polynomial with complex coefficients and such that*

$$\text{either } \{n/a_n \neq 0\} \subset 2\mathbb{N}, \quad \text{or } \{n/a_n \neq 0\} \subset 2\mathbb{N}+1.$$

Then, for $z \geq \exp(18)$ satisfying $0.5\,\mathrm{Log}\,X - 0.5 \geq \mathrm{Log}\,z \geq 0.422\,\mathrm{Log}\,X$, and any real number λ in $[1, \sqrt{X}]$, we have

$$\sum_{\lambda<d\leq z^2} |G(z)w_d| \sum_{a \bmod^* d} |T(a/d)|^2 \leq$$

$$\|T\|_2^2 \Big\{ 52.9\frac{X}{\lambda^{7/10}} + 7.464(Xz\,\mathrm{Log}\,z)^{\frac{13}{23}}$$

$$+ 0.3708\sqrt{Xz}\,\mathrm{Log}^2 z + 0.1898\frac{X}{z}\,\mathrm{Log}^3 z + 1.251z^2 \Big\}$$

Principle of the proof :

If $|G(z)w_d| \leq C_\epsilon d^{-1+\epsilon}$ for an $\epsilon > 0$ where C_ϵ is a constant which depends on ϵ, then an integration by parts and the usual large sieve inequality yield the bound

$$C_\epsilon \frac{\|T\|_2^2}{G(z)} \{ \frac{X}{\lambda^{1-\epsilon}} + 2z^{2(1+\epsilon)} \}$$

The preceeding theorem is a more precise version of this, by using better upper bounds for $|w_d|$, one of the difficulties being to get z^2 without any power of $\mathrm{Log}\,z$ and with a small constant.

Before proving this theorem, we show how to deduce Proposition 4 from it. We are working under $(Hyp.)$. We apply Cauchy's inequality to

$$\sum_{\lambda<d\leq z^2} |w_d| \sum_{a \bmod^* d} |TU(a/d)|$$

in order to separate T and U and apply Theorem 8.1 to each of the resulting sums; a numerical application concludes the argument. We limit ourselves to the main term :

$$\left(\frac{\|T\|_2^2 \|U\|_2^2}{G^2(z)} (1.251)^2 z^4 \right)^{\frac{1}{2}}$$

which, by using Lemma 3.11, is less than

$$\left(0.5159\frac{X^2}{G^2(z)} \frac{\theta(X)}{X} (1.251)^2 \frac{X^2}{15000} \right)^{\frac{1}{2}} \delta^{9/19}$$

i.e. $0.0074X^2\delta^{9/19}/G(z)$.

Throughout the next two sections, the following notations is used :

$$W(d) = \sum_{a \bmod^* d} |T(a/d)|^2, \tag{8.5}$$

where T is any trigonometric polynomial $\sum_{n\leq X} a_n e(n\alpha)$ such that $\{n/a_n \neq 0\}$ is included either in $2\mathbb{N}$ or in $2\mathbb{N}+1$, not identically zero and normalised by $\|T\|_2^2 = 1$.

8.1. Lemmas concerning the polynomial T.

We prove here the inequalities which will contain our knowledge about the polynomial T. Our main tools are the large sieve inequality and its weighted version of Montgomery & Vaughan, the latter with the refinement du to Preismann [8]. We recall his result :

Lemma 8.1 (Preismann). *If $S(\alpha)=\sum_{n\le X} a_n e(n\alpha)$ is any trigonometric polynomial, and Q is any positive real number, then we have*

$$\sum_{q\le Q}\frac{1}{X+\rho qQ}\sum_{a \bmod^* q}|S(a/q)|^2\le \|S\|_2^2,$$

with $\rho=\sqrt{1+\frac{2}{3}\sqrt{6/5}}(\le 4/3)$.

Our second tool when dealing with T is a control of parity of the integers n such that $a_n\neq 0$. To achieve this, let us recall the definition of the function $\kappa(a,\Delta)$ which depends on a parameter a : $\kappa(a,\Delta)=1$ if $(\Delta,2)=1$ and $=a$ otherwise.

Lemma 8.2. *Let Δ be an integer not divisible by 4 and t a positive real number. We have*

$$\sum_{\substack{(d,\Delta)=1\\ d\le t}}\frac{W(\Delta d)}{X+\rho t\Delta d}\le \kappa(1/2,\Delta),$$

and

$$\sum_{\substack{(d,\Delta)=1\\ d\le t}} W(\Delta d)\le \kappa(1/2,\Delta)(X+\Delta t^2),$$

with ρ as in Lemma 8.1.

Remark : If we have no parity control, the factor $\kappa(1/2,\Delta)$ disappears.

Proof. We shall suppose Δ to be even, otherwise both our inequalities are simple if we remark that the set of points

$$\Big\{\frac{a}{\Delta d}\ \ /\ \ (d,\Delta)=1,\ d\le t,\ a \bmod^* \Delta d\Big\}$$

is Δt^2-well-spaced.

∘ First step :

We first show that, if $2|\Delta$ then $W(\Delta d)=W(\Delta d/2)$ if $\Delta/2$ is odd and d any integer. By the Chinese remainder Theorem, we have

$$\sum_{a \bmod^* \Delta d}|T^2\big(\frac{a}{\Delta d}\big)|=\sum_{b \bmod^* 2}\Bigg(\sum_{c \bmod^* \Delta d/2}|T^2\big(\frac{b}{2}+\frac{c}{\Delta d/2}\big)|\Bigg).$$

But with our parity control,

$$|T\big(\frac{b}{2}+\frac{c}{\Delta d/2}\big)|=|T\big(\frac{c}{\Delta d/2}\big)|$$

hence our result.

∘ Second step :

Let us write $\Delta = 2\Delta'$. Then

$$\sum_{\substack{(d,\Delta)=1\\ d\le t}} \frac{W(\Delta d)}{X+\rho t\Delta d} = \frac{1}{2}\Big(\sum_{\substack{(d,\Delta)=1\\ d\le t}} \frac{W(\Delta d)}{X+\rho t\Delta d} + \sum_{\substack{(d,\Delta)=1\\ d\le t}} \frac{W(\Delta' d)}{X+\rho t\Delta d}\Big). \tag{8.1}$$

Now, let d and d' be two integers prime to Δ and $\le t$, and and a and a' be two integers respectively prime to Δd and $\Delta d'$; then

$$|\frac{a}{\Delta' d} - \frac{a'}{\Delta' d'}| \ge |\frac{a}{\Delta d} - \frac{a'}{\Delta d'}| \ge \frac{1}{\Delta d t},$$
$$|\frac{a}{\Delta d} - \frac{a'}{\Delta' d'}| \ge \frac{1}{\Delta' 2 d t} = \frac{1}{\Delta d t}.$$

The conclusion follows from the large sieve inequality (weighted or not). ⋄ ⋄ ⋄

Lemma 8.3. *Let A et B be two real positive constants such that $A \le B$. Then, for any integer Δ not divisible by 4, the following inequalities hold :*

$$\sum_{\substack{(d,\Delta)=1\\ A<d\le B}} \frac{W(\Delta d)}{\Delta d} \le \kappa(1/2,\Delta)\{\frac{X}{\Delta A} + \rho B\};$$

$$\sum_{\substack{(d,\Delta)=1\\ A<d\le B}} \frac{W(\Delta d)}{\Delta d} \le \kappa(1/2,\Delta)\Big\{\frac{X}{\Delta A}\frac{1}{1+\rho B A\Delta/X} + \frac{2X}{\rho B\Delta}(\frac{1}{2} + \operatorname{Log}\frac{B}{A}) + \rho B\Big\};$$

$$\sum_{\substack{(d,\Delta)=1\\ A<d\le B}} \frac{W(\Delta d)}{\Delta d}\operatorname{Log}\frac{z^2}{\Delta d} \le \kappa(1/2,\Delta)\Big\{\frac{X}{\Delta A}\operatorname{Log}\frac{z^2}{\Delta A} + \rho B\big[\operatorname{Log}\frac{z^2}{\Delta B} + 1\big]\Big\}$$

for $\quad \Delta B \le z^2$.

Remark : If we use the large sieve inequality instead of its weighted version, we have on the right of the first inequality the factor $X/\Delta A + 2B$.

Proof. Let us put

$$S = \frac{1}{\rho B}\sum_{\substack{(d,\Delta)=1\\ A<d\le B}} \frac{W(\Delta d)}{\Delta d}, \tag{8.2}$$

which can be rewritten as

$$S = \sum_{\substack{(d,\Delta)=1\\ A<d\le B}} \frac{W(\Delta d)}{X+\rho B\Delta d} + X\sum_{\substack{(d,\Delta)=1\\ A<d\le B}} \frac{W(\Delta d)}{(X+\rho B\Delta d)\rho B\Delta d},$$

and thus

$$S \le \kappa(1/2,\Delta) + X\sum_{\substack{(d,\Delta)=1\\ A<d\le B}} \frac{W(\Delta d)}{(X+\rho B\Delta d)\rho B\Delta d}.$$

The last summation can be handled in two different ways.

(1) Define the decreasing function φ by

$$\varphi(t) = \frac{1}{(X+\rho B\Delta t)\rho B\Delta t}.$$

By summation by parts, we get :

$$S \leq \kappa(1/2,\Delta) + X\varphi(B) \sum_{\substack{(d,\Delta)=1\\ A<d\leq B}} W(\Delta d) + X\int_A^B \sum_{\substack{(d,\Delta)=1\\ A<d\leq t}} W(\Delta d)(-\varphi'(t))dt,$$

and another appeal to Lemma 8.2 yields the bound

$$S \leq \kappa(1/2,\Delta)\Big\{1 + X^2\varphi(A)(X+\Delta A^2) + 2X\Delta\int_A^B t\varphi(t)dt\Big\}. \tag{8.3}$$

We finally get

$$\rho BS \leq \kappa(1/2,\Delta)\Big\{\rho B + \frac{X(X+\Delta A^2)}{\Delta A(X+\rho BA\Delta)} + \frac{2X\Delta}{\rho B\Delta}\operatorname{Log}\Big(\frac{X+\rho B^2\Delta}{X+\rho AB\Delta}\Big)\Big\}.$$

(2) We could have handled our sum in a different way :

$$\begin{aligned} S \leq \kappa(1/2,\Delta) &+ \frac{X}{\rho B\Delta}\int_A^B \sum_{\substack{(d,\Delta)=1\\ A<d\leq t}} \frac{W(\Delta d)}{(X+\rho B\Delta d)}\frac{dt}{t^2} \\ &+ \frac{X}{\rho B\Delta}\sum_{\substack{(d,\Delta)=1\\ A<d\leq B}} \frac{W(\Delta d)}{(X+\rho B\Delta d)}\frac{1}{B} \end{aligned} \tag{8.4}$$

which gives us

$$\sum_{\substack{(d,\Delta)=1\\ A<d\leq B}} \frac{W(\Delta d)}{\Delta d} \leq \kappa(\frac{1}{2},\Delta)\{\frac{X}{\Delta A} + \rho B\}.$$

(3) The last inequality follows from the latter one by writing :

$$\sum_{\substack{(d,\Delta)=1\\ A<d\leq B}} \frac{W(\Delta d)}{\Delta d}\operatorname{Log}\frac{z^2}{\Delta d} = \int_1^{z^2/\Delta} \sum_{\substack{(d,\Delta)=1\\ A<d\leq B}} \frac{W(\Delta d)}{\Delta d}\frac{dt}{t}$$

and the result follows. $\diamond\diamond\diamond$

8.2. Other lemmas.

We introduce some further notations :

$$V(A,B) = \sum_{A<d\le B} |G(z)w_d| W(d). \tag{8.5}$$

We consider four parameters $\lambda \le Q_0 \le Q \le z^2$ *and assume* $Q/z \ge 500$, $z \ge \exp(18)$ *and* $X \ge ez^2$.

c_1 *is an upper bound for* $G(z\sqrt{Q_0})/G(z)$, c_2 *is an upper bound for* $G(z\sqrt{Q})/G(z)$.

Finally ρ *is the constant* $\sqrt{1+\frac{2}{3}\sqrt{6/5}}$.

Lemma 8.4.

$$V(\lambda, Q_0) \le 30.31 c_1 \big\{X\lambda^{-7/10} + \frac{20}{13} Q_0^{13/10}\big\}.$$

Proof. We use Lemma 4.5 to write

$$|G(z)w_d| \le 30.31 d^{-7/10} c_1,$$

and get Lemma 8.4 by a simple integration by parts. ◇◇◇

Lemma 8.5.

$$V(Q_0, Q) \le c_2\big\{0.1917 Xz\ Q_0^{-1} \operatorname{Log} z + 0.1452 Q \operatorname{Log}^2 z\big\}.$$

Proof. We use Lemma 4.6 to write

$$|G(z)w_d| \le \frac{c_2}{d} \sum_{\substack{\ell | d \\ \ell \le z}} \xi(\ell),$$

where $\xi(\ell) = \prod_{p|\ell}(2p-3)/(p-1)$.

One has

$$V(Q_0, Q) \le c_2 \sum_{Q_0<d\le Q} \frac{\mu^2(d)}{d} \sum_{\ell | d} \xi(\ell) W(d),$$

hence, by using Lemma 8.3, first part,

$$V(Q_0, Q) \le \big\{X\ Q_0^{-1} \sum_{\ell \le z} \mu^2(\ell)\xi(\ell)\kappa(1/2,\ell) + \rho Q \sum_{\ell\le z} \mu^2(\ell)\frac{\xi(\ell)}{\ell}\kappa(1/2,\ell)\big\}. \tag{8.6}$$

Now the result follows from Lemma 3.8 with $a = 1/2$. ◇◇◇

Now comes the more difficult

Lemma 8.6.

$$V(Q,z^2) \leq 1.566XQ\ z^{-1}\big[1.471\operatorname{Log}\frac{Xz}{Q} + 1.391 + (\operatorname{Log} z + 0.610)\operatorname{Log}\frac{z^2}{Q}\big] + \\ + 0.3785\frac{X\operatorname{Log}^2 z}{z}\big(\tfrac{1}{2} + \operatorname{Log}\frac{z^2}{Q}\big) + 1.251z^2.$$

Proof. It is convenient to put $k_1 = 0.5$ and $k_2 = 1.4709$. We have by Lemma 3.5(1): $G(y) \leq k_1 \operatorname{Log}(y^2) + k_2$ for all $y \geq 1$. We use Lemma 4.6 to write

(8.7)
$$|G(z)w_d| \leq \frac{G(z/\sqrt{d})}{G(z)} \sum_{\substack{\ell mn = d\\ \ell\leq z, mn\leq z}} \frac{\xi(\ell)}{\ell}\frac{1}{m\phi(m)}\frac{1}{n},$$

which gives us
(8.8)
$$V(Q,z^2)G(z) \leq \sum_{\substack{Q/z<\ell\leq z\\ m\leq z\\ (m,\ell)=1}} \frac{\mu^2(\ell)\xi(\ell)\ell}{\phi(\ell)}\frac{\mu^2(m)}{\phi(m)} \sum_{\substack{Q/(\ell m)<n\leq z/m\\ (n,\ell m)=1}} \frac{k_1\operatorname{Log}\frac{z^2}{\ell mn} + k_2}{\ell mn} W(\ell mn).$$

For the innermost sum, we apply Lemma 8.3, part (3) to the part multiplied by k_1, and part (2) to the sum multiplied by k_2 and get the upper bound :
(8.9)
$$\kappa(1/2,\ell m)\left\{\frac{X}{Q}\big[k_1\operatorname{Log}\frac{z^2}{Q} + \frac{k_2}{1+\rho\frac{zQ}{mX}}\big] + \frac{2k_2X}{\rho z\ell}\big(\tfrac{1}{2} + \operatorname{Log}\frac{z\ell}{Q}\big) + \frac{\rho z}{m}\big(k_1\operatorname{Log}\frac{z}{\ell} + k_1 + k_2\big)\right\}$$

We also notice that $1/2 + \operatorname{Log}(zl/Q) \leq 1/2 + \operatorname{Log}(z^2/Q)$. We are thus left with the question of finding upper bounds for

$$\Sigma_1 = k_1\frac{X}{Q}\operatorname{Log}\frac{z^2}{Q}\sum_{Q/z<\ell\leq z}\frac{\mu^2(\ell)\xi(\ell)\ell}{\phi(\ell)}\kappa(1/2,\ell)\sum_{\substack{m\leq z\\ (m,\ell)=1}}\frac{\mu^2(m)}{\phi(m)}\kappa(1/2,m),$$

$$\Sigma_2 = \frac{k_2X}{Q}\sum_{Q/z<\ell\leq z}\frac{\mu^2(\ell)\xi(\ell)\ell}{\phi(\ell)}\kappa(1/2,\ell)\sum_{\substack{m\leq z\\ (m,\ell)=1}}\frac{\mu^2(m)}{\phi(m)}\frac{\kappa(1/2,m)}{1+\rho zQ/(mX)},$$

$$\Sigma_3 = k_2\frac{2X}{\rho z}\big(\tfrac{1}{2} + \operatorname{Log}\frac{z^2}{Q}\big)\sum_{Q/z<\ell\leq z}\frac{\mu^2(\ell)\xi(\ell)}{\phi(\ell)}\kappa(1/2,\ell)\sum_{\substack{m\leq z\\ (m,\ell)=1}}\frac{\mu^2(m)}{\phi(m)}\kappa(1/2,m),$$

$$\Sigma_4 = \rho z\sum_{Q/z<\ell\leq z}\frac{\mu^2(\ell)\xi(\ell)\ell}{\phi(\ell)}\kappa(1/2,\ell)\big(k_1\operatorname{Log}\frac{z}{\ell} + k_1 + k_2\big)\sum_{\substack{m\leq z\\ (m,\ell)=1}}\frac{\mu^2(m)\kappa(1/2,m)}{m\phi(m)}.$$

Now we have only to compute all these averages, which will be rather long (we have to be precise) but without any major difficulty.

<u>*The summation over m :*</u>

Using Lemma 3.7 and 3.8, the identity

$$(8.10)\qquad \sum_{(m,\ell)=1}\frac{\mu^2(m)\kappa(1/2,m)}{m\phi(m)}=\tfrac{5}{6}\prod_{p\geq 2}\bigl(1+\frac{1}{p(p-1)}\bigr)\prod_{p|\ell}\frac{p(p-1)}{1+p(p-1)}\kappa(6/5,\ell),$$

and $500\leq Q/z$, we get the upper bounds

(8.11)

$$\begin{cases}\Sigma_1 \leq k_1\dfrac{X}{Q}\operatorname{Log}\dfrac{z^2}{Q}\tfrac{3}{4}\bigl(2\operatorname{Log} z+1.220\bigr)\displaystyle\sum_{500<\ell\leq z}\mu^2(\ell)\xi(\ell)\kappa(2/3,\ell),\\ \Sigma_2 \leq \dfrac{k_2 X}{Q}\tfrac{3}{4}\bigl(\operatorname{Log}\dfrac{Xz}{\rho Q}+1.220\bigr)\displaystyle\sum_{500<\ell\leq z}\mu^2(\ell)\xi(\ell)\kappa(2/3,\ell),\\ \Sigma_3 \leq k_2\dfrac{2X}{\rho z}\bigl(\dfrac{1}{2}+\operatorname{Log}\dfrac{z^2}{Q}\bigr)\tfrac{3}{4}\bigl(2\operatorname{Log} z+1.220\bigr)\displaystyle\sum_{500<\ell\leq z}\dfrac{\mu^2(\ell)\xi(\ell)}{\ell}\kappa(2/3,\ell),\\ \Sigma_4 \leq \rho z\tfrac{5}{6}\displaystyle\prod_{p\geq 2}\bigl(1+\dfrac{1}{p(p-1)}\bigr)\sum_{500<\ell\leq z}\dfrac{\mu^2(\ell)f_5(\ell)}{\ell}\kappa(3/5,\ell)\bigl(k_1\operatorname{Log}\dfrac{z}{\ell}+k_1+k_2\bigr)\end{cases}$$

with

$$f_5(\ell)=\xi(\ell)\prod_{p|\ell}\frac{p^2}{1+p(p-1)}.$$

The summation over ℓ :

By using Lemma 3.8, we get

$$\sum_{\ell\leq z}\mu^2(\ell)\xi(\ell)\kappa(2/3,\ell)\leq 0.159z\operatorname{Log} z+0.531z+0.222+86.2z^{2/3},$$

but we have $\sum_{\ell\leq 500}\mu^2(\ell)\xi(\ell)\kappa(2/3,\ell)\geq 9\ 400$, whence

$$(8.12)\qquad \sum_{500<\ell\leq z}\mu^2(\ell)\xi(\ell)\kappa(2/3,\ell)\leq 0.159z\operatorname{Log} z+0.7446z.$$

Lemma 3.8 also gives us

$$(8.13)\qquad \sum_{1<\ell\leq z}\mu^2(\ell)\frac{\xi(\ell)\kappa(2/3,\ell)}{\ell}\leq 0.0791\operatorname{Log}^2 z+0.689\operatorname{Log} z.$$

Lemma 3.9 gives us

$$(8.14)\qquad \sum_{500<\ell\leq z}\mu^2(\ell)f_5(\ell)\kappa(3/5,\ell)\bigl[\tfrac{1}{2}\operatorname{Log}\frac{z}{\ell}+1.9709\bigr]\leq 0.514z\operatorname{Log} z+2.0823z.$$

Gathering these results, we get that $V(Q,z^2)G(z)$ is less than

(8.15)

$$\Bigl\{\frac{3Xz}{4Q}[k_2(\operatorname{Log}\frac{Xz}{Q}+0.9455)+k_1(2\operatorname{Log} z+1.220)\operatorname{Log}\frac{z^2}{Q}][0.159\operatorname{Log} z+0.7446]$$
$$+\frac{3}{2\rho}\frac{X}{z}k_2(2\operatorname{Log} z+1.220)(\tfrac{1}{2}+\operatorname{Log}\frac{z^2}{Q})(0.0791\operatorname{Log} z+0.689)\operatorname{Log} z$$
$$+\rho\tfrac{5}{6}\prod_{p}\bigl(1+\frac{1}{p(p-1)}\bigr)z^2(0.514\operatorname{Log} z+2.0823)\Bigr\}.$$

Last Reduction :
Keeping in mind that $G(z) \geq \operatorname{Log} z + 1.332\,582 - 7.284 z^{-1/3}$, we have

$$\tfrac{3}{4}\frac{[0.159 \operatorname{Log} z + 0.7446]}{G(z)} \leq 0.1566,$$

$$\frac{3}{2\rho} k_2 \frac{(\operatorname{Log}(z^2) + 1.220)(0.0791 \operatorname{Log} z + 0.689)}{G(z) \operatorname{Log} z} \leq 0.3795,$$

and

$$\rho \tfrac{5}{6} \prod_p \big(1 + \frac{1}{p(p-1)}\big) \frac{(0.514 \operatorname{Log} z + 2.0823)}{G(z)} \leq 1.251$$

and obtain Lemma 8.6. ⋄ ⋄ ⋄

8.3. Proof of Theorem 8.1 and small improvements.
We are seeking for an upper bound for $V(\lambda, z^2)$. We write

$$V(\lambda, z^2) = V(\lambda, Q_0) + V(Q_0, Q) + V(Q, z^2)$$

and apply Lemmas 8.4, 8.5 and 8.6. We now have to choose the parameters Q_0 and Q. The term $0.3795X(1/2 + \operatorname{Log}(z^2/Q))/z$ is an error term and is not taken into account in our choice.

We take

$$Q = \sqrt{Xz} \geq z^{3/2}$$

and verify that the sum of the two quantities which depend on Q is less than

$$0.3708\sqrt{Xz} \operatorname{Log}^2 z.$$

We have $\operatorname{Log} z\sqrt{Q} \leq 1.843 \operatorname{Log} z$, hence we take $c_2 = 1.843$ by Lemma 3.5. We then choose

$$Q_0 = \left(\frac{1}{60.62} 0.1917 \frac{1.843}{1.749} Xz \operatorname{Log} z\right)^{10/23}.$$

We have $Q_0 \leq \big(0.00583 Xz \operatorname{Log} z\big)^{10/23}$, thus

$$\begin{aligned}\frac{\operatorname{Log} z\sqrt{Q_0}}{\operatorname{Log} z} &\leq 1 + \frac{5}{23} \frac{\operatorname{Log}(0.00583 Xz \operatorname{Log} z)}{\operatorname{Log} z} \\ &\leq 1.7326 + \frac{5}{23} \frac{\operatorname{Log}\operatorname{Log} z - 5.144}{\operatorname{Log} z} \leq 1.749\end{aligned}$$

hence, by Lemma 3.5, we take $c_1 = 1.749$. We then verify that the sum of the two quantities which depend on Q_0 is less than

$$7.464\big(Xz \operatorname{Log} z\big)^{13/23}.$$

This concludes the proof of Theorem 8.1.

Small improvements.

Using $z \geq \exp(30)$ instead of $z \geq \exp(18)$ yields of course an improvement of our constants. Also, we could have discussed in (8.7) whether ℓmn is odd or even and used in the latter case

$$G_{\ell mn}(z/\sqrt{\ell mn}) \leq G_2(z/\sqrt{\ell mn})$$

instead of the cruder

$$G_{\ell mn}(z/\sqrt{\ell mn}) \leq G(z/\sqrt{\ell mn}).$$

9. An additive theorem in addition of sequences. Proof of Theorem 1.

Our main tool is the following effective version of a theorem due to Ostmann. This version has been obtained by J.-M. Deshouillers and we are happy to thank him for this helpful result.

As a matter of notations, if $\mathcal{A}$ is a sequence of natural numbers, and x is any real number, then $A(x)$ is the number of elements of $\mathcal{A}$ which lie in $[1, x]$ (usual notation used in [4]). We also define $A(n, m)$ to be $A(m) - A(n-1)$.

One key of the proof of Theorem 9.1 is the following lemma which permits one to "transfer" elements from one summand to the other.

Lemma 9.1 (Dyson's transform). *Let $\mathcal{A} = \{a_1 < a_2 < \dots\}$ and $\mathcal{B} = \{0 = b_1 < b_2 < \dots\}$ be two sequences of natural integers. For any e in $\mathcal{A}$, we define*

$$\mathcal{A}' = \mathcal{A} \cup \{\mathcal{B} + e\} \quad \textit{and} \quad \mathcal{B}' = \mathcal{B} \cap \{\mathcal{A} - e\}.$$

We have

$$(\alpha)\ \mathcal{A}' + \mathcal{B}' \subset \mathcal{A} + \mathcal{B}$$
$$(\beta)\ \{e\} + \mathcal{B}' \subset \mathcal{A}'$$
$$(\gamma)\ 0 \in \mathcal{B}'$$
$$(\delta)\ A'(m) + B'(m-e) = A(m) + B(m-e)$$

Proof. See [4], Chap. 1. ⋄⋄⋄

Theorem 9.1. *Let $\mathcal{A}$ be a sequence of natural numbers containing 0. We assume that there exist a real number σ, integers H, K and n_0 such that*

(a) For $n \geq n_0$, one has $A(n) \geq \sigma n + \frac{K(K+1)}{2}(H-1)$,

(b) $\{0, 1, \dots, K\} \in \mathcal{A}$, and $\{n_0, n_0+1, \dots, n_0+K\} \in \mathcal{A}$,

(c) $(K+1)H\sigma \geq K + H$.

Then, every integer $\geq Hn_0$ is a sum of at most H elements of $\mathcal{A}$.

Rm : The hypothesis and conclusion are similar to the usual Theorem of Mann with the two differences that only asymptotic results on $\mathcal{A}$ are available and that the lower density σ is $> H^{-1}$ (It is no restriction if σ is not the inverse of an integer). The assumption (a) is not enough to ensure the result, for we have to avoid the case of an arithmetic progression. It is striking that only the weak assumption (b) is enough to get rid of this case, just as in Mann's Theorem, 0 and 1 in $\mathcal{A}$ are enough.

Proof. By induction, one defines $\mathcal{A}_h^l$ for $1 \leq l \leq K(H-1)$ and $1 \leq l \leq H$. We start with

$$\mathcal{A}_h^1 = \mathcal{A} \ \text{ for } \ 1 \leq h \leq H.$$

We assume that $\left(\mathcal{A}_h^{l-1}\right)_{1 \leq h \leq H}$ has been defined for some $l \leq K(H-1)$, write $l = (k-1)(H-1) + r$ with $1 \leq r \leq H-1$ and $1 \leq k \leq K$, and define

$$\begin{aligned} \mathcal{A}_1^l &= \mathcal{A}_1^{l-1} \cup \{\mathcal{A}_{r+1}^{l-1} + k\}, \\ \mathcal{A}_{r+1}^l &= \mathcal{A}_{r+1}^{l-1} \cup \{\mathcal{A}_1^{l-1} - k\}, \\ \mathcal{A}_s^l &= \mathcal{A}_s^{l-1} \ \text{ if } \ s \neq 1 \ \text{ and } \ s \neq r+1. \end{aligned}$$

By the properties of Dyson's transform, we have

$$(i)\ \mathcal{A}_1^l + \cdots + \mathcal{A}_H^l \subset \mathcal{A}_1^{l-1} + \cdots + \mathcal{A}_H^{l-1}$$

$$(ii)\ \begin{cases} \{0,1,\ldots,k-1\} + \mathcal{A}_s^l \subset \mathcal{A}_1^l & \text{for } r+1 < s \\ \{0,1,\ldots,k\} + \mathcal{A}_s^l \subset \mathcal{A}_1^l & \text{for } s \le r+1 \end{cases}$$

$$(iii)\ A_1^l(m) + A_{r+1}^l(m-k) = A_1^{l-1}(m) + A_{r+1}^{l-1}(m-k),\ \forall m \ge 0$$

$$(iv)\ \{0, n_0\} \subset \mathcal{A}_1^l \cap \cdots \cap \mathcal{A}_H^l \quad , \quad \{n_0, n_0+1, \ldots, n_0+K\} \subset \mathcal{A}_1^l.$$

From (iii), we deduce

$$A_1^l(m) + \cdots + A_H^l(m) \ge A_1^{l-1} + \cdots + A_H^{l-1}(m) - k.$$

We finally define $\mathcal{B}_h = \mathcal{A}_h^{K(H-1)}$, for $1 \le h \le H$. We have

(9.1) $\mathcal{B}_1 + \cdots + \mathcal{B}_H \subset H\mathcal{A}$

(9.2) $\{0,1,\ldots,K\} + (\mathcal{B}_2 \cup \cdots \cup \mathcal{B}_H) \subset \mathcal{B}_1$

(9.3) For $n \ge n_0$, one has $B_1(n) + \cdots + B_H(n) \ge \dfrac{K+H}{K+1} n$

(9.4) $\{0, n_0\} \subset \mathcal{B}_1 \cap \cdots \cap \mathcal{B}_H$.

If the second inequality in (9.3) always holds, we define n_1 to be 1 ; otherwise, we define n_1 to be the smallest n for which the second inequality in (9.3) holds. Because of (9.2), $n_1 \in \mathcal{B}_1$. For $n \ge n_1$, we have

$$B_1(n_1, n) + \cdots + B_H(n_1, n) \ge \frac{K+H}{K+1}(n - n_1 + 1). \tag{9.5}$$

For $h = 2, \ldots, H$, we let n_h denote the smallest integer in $\mathcal{B}_h$ which is at least equal to n_1. With no loss of generality, we may assume that $n_1 \le n_2 \le \cdots \le n_H$, and because of (4), that $n_H \le n_0$. We then define

$$\mathcal{A}_h = \mathcal{B}_h - n_h, \text{ for } h = 1, 2, \ldots, H.$$

We have

$$\sum_{h=1}^H \mathcal{A}_h = \sum_{h=1}^H \mathcal{B}_h - \sum_{h=1}^H n_h \qquad \text{with} \quad \sum_{h=1}^H n_h \le H n_0 \tag{9.6}$$

and the counting function $A_h(n)$ satisfies

$$A_h(n) = B_h(1 + n_h, n + n_h) = B_h(n_1, n + n_h) - 1,$$

since n_h is the only $a \in \mathcal{B}_h$ with $n_1 \le a < n_h + 1$. We thus have

$$A_h(n) \ge B_h(n_1, n + n_1) - 1. \tag{9.7}$$

We want to prove that $S(n) = \sum_{h=1}^H A_h(n) \ge n$, for each $n \ge 1$, and we distinguish three cases.

A. $n_1 \le n_1 + n < n_2$.

$$\begin{aligned} S(n) \ge A_1(n) &= B_1(n_1, n+n_1) - 1 = \sum_{h=1}^{H} B_1(n_1, n+n_1) - 1 \\ &\ge \frac{K+H}{K+1}(n+1) - 1 \ge n. \end{aligned}$$

B. $n_2 \le n_1 + n < n_2 + K + 1$
Because of (2), n_2, $n_2 + 1$, ... , $n_1 + n$ are in $\mathcal{B}_1$, and so

$$\begin{aligned} S(n) \ge A_1(n) &= B_1(n_1, n_1 + n) - 1 \\ &= B_1(n_1, n_2 - 1) + n_1 + n - n_2 + 1 - 1 \\ &= \sum_{h=1}^{H} B_h(n_1, n_2 - 1) + n_1 + n - n_2 \\ &\ge \frac{K+H}{K+1}(n_2 - n_1) - (n_2 - n_1) + n \ge n. \end{aligned}$$

C. $n_2 + K + 1 \le n_1 + n$
We have $\{n_1, n_1+1, \ldots, n_1+K\} \subset \mathcal{B}_1$, so that $S(n) \ge K$, which is all right if $n \le K$. Otherwise, we have $n \ge K+1$, and (7) and (5) lead to

$$\begin{aligned} S(n) &\ge \frac{K+H}{K+1}(n+1) - H = \frac{K+1}{K+1}(n+1) + (H-1)\frac{n+1}{K+1} - H \\ &\ge n + 1 + H - 1 + \frac{H-1}{K+1} \ge n. \end{aligned}$$

For all n, we have $S(n) \ge n$, so that Dyson's Theorem implies that $\mathbb{N} = \sum_{h=1}^{H} \mathcal{A}_h$, which implies that every integer $\ge \sum_{h=1}^{H} n_h$ is in $\sum_{h=1}^{H} \mathcal{B}_h$, and so every integer $\ge Hn_0$ is in $H\mathcal{A}$. ◇◇◇

Let us deduce our Theorem 1 from Theorem 2 and Theorem 9.1. First of all, $\mathcal{A}$ is the sequence of all numbers $(g-6)/2$ where g is a sum of two odd primes. We choose $H = 3$, $K = 39$, $\sigma = \frac{7}{20}$ and $2n_0 = 1.002.10^{30}$.

We have $2n_0 \ge 8\exp(67)$ and , for $Y \ge n_0$,

$$A(Y) = \sum_{k=0}^{2} \left[A(Y2^{-k}) - A(Y2^{-k-1})\right] + A(Y/8),$$

hence

$$A(Y) \ge \sum_{k=0}^{2} \Big[\sum_{\substack{\frac{Y}{2^k}+3 < n \le 2(\frac{Y}{2^k}+3) \\ r(n) \ne 0}} 1 - 2\Big] + 10^{12} - 3$$

because $A(Y/8) \ge A(10^{12} - 3) = 10^{12} - 3$ by a result of Granville, te Riele and van de Lune [2].

Hence, by Theorem 2, we get

$$A(Y) \ge \frac{7}{20}Y + \frac{K(K+1)}{2}(H-1).$$

Now a direct computation shows that the assumption (b) holds and we conclude that every integer larger than $6n_0$ is a sum of at most 6 odd primes.

A greedy algorithm will complete the proof easily : Let N be an even integer less than $6n_0$. Then by using [9], we find a function f_7 such that the interval $[(N-3)-f_7(N-3), N-3]$ contains at least one prime p_1. A slight difficulty arises because f_7 is not necessarily non-decreasing ; hence we build the non-decreasing function f_8 which is the largest non-decreasing function less than f_7. Then $N-p_1$ is less than $f_8(N-3) \le f_8(6n_0)$. By repeating this process at most four times, we get an integer M which is less than 2.10^{10}. Hence, either M is even and the sum of at most 2 primes by [2], or M is odd, which implies that we have only used three primes, and M is a sum of at most 3 primes.

Limit of the method :

We assume here that we are able to check the Riemann Hypothesis for any modulus less than a given bound and up to an height arbitrarily large but also less than a given bound (for instance, for all moduli less than 10 000 up to an height of 10^8). Then following the method used in this paper, we can show that the lower aymptotic density of the sums of two primes is not less than $1/(4+\epsilon)$ for any fixed positive ϵ. Using the fact that the sequence of primes is an essential component (since it is an asymptotic basis), we can show that the asymptotic lower density of the sums of three primes is not less than $1/(4-4/25+\epsilon')$ where ϵ' is a function of ϵ going to 0 with ϵ (cf Theorem 5 (chapter 1, section 3) of [4]). We can then conclude that every large enough odd integer is a sum of at most 5 primes. The fact is that the argument using essential components requires a very small ϵ to work which can not be reached by today's computers.

References

[1]. Chen Jingrun & Wang Tianze, *On the odd Goldbach problem*, Acta Math. Sin. **32 5** (1989), 702–718.

[2]. Granville, de Lune, te Riele, *Number Theory and Applications*, (Nato) Ed. Mollin (1989).

[3]. H.Halberstam & H.-E.Richert, *Sieves methods*, Academic Press, London (1974).

[4]. H.Halberstam & K.F.Roth, *Sequences*, Clarendon Press, Oxford (1966).

[5]. C.Hooley, *Applications of sieves methods to the theory of numbers*, Cambridge University Press, no 70 (1976).

[6]. Yu.V.Linnik, *The Dispersion Method in Binary Additive Problems*, Translations of Mathematical Monographs. A.M.S. **4** (1963).

[7]. H.Montgomery & R.C.Vaughan, *The large sieve*, Mathematika **20 no 2** (1973), 119–133.

[8]. E.Preissmann, *Sur une inégalité de Montgomery & Vaughan*, L'enseignement mathématique **30** (1984), 95–113.

[9]. O.Ramaré, *Short effective intervals containing primes*, submitted to J. Number Theory.

[10].O.Ramaré & R.Rumely, *Primes in arithmetic progressions*, Math.Comp. (to appear).

[11].H.Riesel & R.C.Vaughan, *On sums of primes*, Arkiv för mathematik **21** (1983), 45–74.

[12].J.B.Rosser, *Explicit bounds for some functions of prime numbers*, Amer. J. Math. **63** (1941), 211–232.

[13].J.B.Rosser & L.Schoenfeld, *Approximate formulas for some functions of prime numbers*, Illinois J. Math. **6** (1962), 64–94.

[14].J.B.Rosser & L.Schoenfeld, *Sharper bounds for the Chebyshev functions ψ and θ. I*, Math. Comp. **29 129** (1975), 243–269.

[15].R.Rumely, *Numerical Computations Concerning the ERH*, To appear in Math. Comp..

[16].I.Z.Ruzsa, *An additive property of squares and primes*, Acta Arith. **49 4** (1987), 281–289.

[17].H.N.Shapiro & J.Warga, *On the representation of large integers as sums of primes*, Comm. Pure Appl. Math. **3** (1950), 153–176.

[18].L.Schoenfeld, *Sharper bounds for the Chebyshev functions ψ and θ. II*, Math. Comp. **30 no 134** (1976), 337–360.

[19].R.C.Vaughan, *A note on Šnirel'man's approach to Goldbach's problem*, Bull. London Math. Soc. **8** (1976), 245–250.

[20].R.C.Vaughan, *On the estimation of Schnirelman's constant*, J. Reine Angew. Math. **290** (1977), 93–108.

Table 1

d	ϵ_d	**d**	ϵ_d
1	$1.7285.10^{-5}$	39	0.00913
3	0.00212	41	0.01037
5	0.00232	43	0.01043
7	0.00243	47	0.01053
11	0.00257	51	0.01006
13	0.00262	53	0.01066
15	0.00774	55	0.01004
17	0.00931	57	0.01021
19	0.00944	59	0.01079
21	0.00827	61	0.01083
23	0.00967	65	0.01027
29	0.00995	69	0.01044
31	0.01004	77	0.01059
33	0.00890	87	0.01071
35	0.00941	93	0.01079
37	0.01025	105	0.01037