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A modular analogue of a problem of Vinogradov

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Abstract
Given a primitive, non-CM, holomorphic cusp form f with normalized Fourier coefficients $a(n)$ and given an interval $I \subset [-2, 2]$, we study the least prime p such that $a(p) \in I$. This can be viewed as a modular form analogue of Vinogradov's problem on the least quadratic non-residue. We obtain strong explicit bounds on p, depending on the analytic conductor of f for some specific choices of I.

Keywords Vinogradov's conjecture · Modular forms · Moebius function · Sato–Tate law

Mathematics Subject Classification 11M06 · 11N56 · 11N80

1 Introduction

The present article is concerned with understanding the distribution of the initial Fourier coefficients of primitive holomorphic cusp forms at primes. Suppose f is such a form of weight k for the group $\Gamma_0(N)$. We further assume that f is non-CM and has trivial nebentypus. The normalized Fourier coefficients of f at infinity are

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denoted by $(a(n))_{n\geq 1}$, so that $a(1)=1$ and

$$f(z)=\sum_{n=1}^{\infty} a(n) n^{\frac{k-1}{2}} e(nz),$$

where, as usual, $e(z)$ denotes $e^{2\pi i z}$ and with this normalization, the Ramanujan bound (proved by Deligne [6]) says $-2 \leq a(p) \leq 2$ for primes p. Furthermore, the function $n \mapsto a(n)$ is real-valued and multiplicative. We refer the reader to the text [12] for background information on holomorphic modular forms. The Sato–Tate conjecture for distribution of the angles θ_p, defined by $a(p)=2\cos\theta_p$, as p runs over primes, which is now a theorem of Clozel, Harris, Shepherd-Barron and Taylor [3, 9, 31], implies, in particular, that any interval of positive measure within $[-2,2]$ contains infinitely many values of $a(p)$. The goal of this article is to obtain bounds for the least prime p such that $a(p)$ lies in a fixed interval $I\subset[-2,2]$. This can be considered as an analogue of Vinogradov's problem of estimating, given a modulus $q\geq 1$, the size of the least quadratic non-residue modulo q (see [2, 32]). The quality of our bounds will be measured in terms of the analytic conductor $q(f)=Nk^2$ of the form f (see Sect. 2.1), and also separately in term of the weight k of the form, considering the level N to be fixed and in terms of the level N, considering the weight k to be fixed. We restrict our attention to forms with trivial nebentypus in order to clarify the presentation but the methods presented here can be extended to a more general setting.

Let $I\subset[-2,2]$. Theorem 1.6 of the paper [22] of Lemke-Oliver and Thorner implies that there exists a constant A depending only on I such that $a(p)\in I$ for some prime $p\leq q^A$. Their method relies on effective log-free zero density estimates for the L-function associated with f, and the Turán power-sum method. The value of the constant A is not stated explicitly in their paper but it is not hard to see that the constant is effective and can be worked out explicitly. However, the method is likely to produce quite large values of A. Our aim in the present work is to make the value of A as small as possible for some specific intervals.

We define, when κ is positive and $x\in[0,1]$:

$$\mathscr{F}(x;\kappa)=\int_0^{x/(1+x)}\frac{h^{\kappa-1}dh}{1-h}=\sum_{k\geq 0}\frac{1}{\kappa+k}\Big(\frac{x}{1+x}\Big)^{\kappa+k}. \tag{1.1}$$

Note that $\mathscr{F}(\cdot;\kappa)$ is increasing between $\mathscr{F}(0;\kappa)=0$ and $\mathscr{F}(1;\kappa)=\int_0^{1/2}\frac{h^{\kappa-1}dh}{1-h}$. We thus define a function $\mathscr{G}(\cdot;\kappa)$ with value in $[0,1]$ by

$$\mathscr{G}(y;\kappa)=\max\{x\in[0,1]:\mathscr{F}(x;\kappa)\leq 1/y\}. \tag{1.2}$$

The function $\mathscr{G}$ is non-increasing and we have $\mathscr{G}(y;\kappa)=1$ when $y\leq 1/\mathscr{F}(1;\kappa)$ and by convention $\mathscr{G}(\infty;\kappa)=0$.

We now state our main results which depend crucially on knowledge about the analytic properties of the symmetric power L functions associated to f (see Sect. 2.1 for definition). This is likely to change in future; only small changes would be required in our proofs to reflect any such improvement. Here is the assumption we rely on.

Hypothesis $\mathscr{H}_\ell$: The L-function $L(s, \mathrm{sym}^\ell(f))$ has analytic continuation to the entire complex plane and it satisfies the bound

$$L(1/2+it, \mathrm{sym}^\ell(f)) \ll_\varepsilon q(\mathrm{sym}^\ell(f), s)^{\lambda_\ell+\varepsilon}$$

for any $\varepsilon > 0$.

For holomorphic forms, the automorphy of $L(s, \mathrm{sym}^\ell f)$ has been known for $\ell \le 8$ by [4, 5, 7, 16–18], and has recently been proved for *all* ℓ when N is squarefree by Newton–Thorne [28]. As a result, these L-functions admit holomorphic continuation to the entire complex plane and by the convexity principle, $\mathscr{H}_\ell$ holds with $\lambda_\ell = 1/4$ (known as the convexity bound) for $\ell \le 8$ unconditionally and for all ℓ when N is squarefree.

Our results are the following.

Theorem 1.1 *For any $\delta \in (0, 2]$, let $\theta_1(\delta) = \mathscr{G}(2+\delta; \delta)$. The function θ_1 is increasing and we have $\theta_1(0+) = 0$ and $\theta_1(1) = 0.3956\cdots$. Suppose $\lambda_1 > 0$ is an exponent that satisfies the hypothesis $\mathscr{H}_\ell$ above for $\ell = 1$, and let $\varepsilon > 0$. Then for $q = N$ or k^2 sufficiently large, there exists a prime*

$$p \ll_\varepsilon q^{\frac{2\lambda_1}{1+\theta_1(\delta)}+\varepsilon}$$

with $a(p) \le \delta$.

Remark 1.2 The convexity bound (Phragmén–Lindelöf principle) allows taking $\lambda_1 = \frac{1}{4}$ but better exponents, called subconvex exponents are known in both the weight and the level aspects. For example, one may take $\lambda_1 = \frac{1}{6}$ when $N = 1$ by a result of Jutila and Motohashi [15].

Theorem 1.3 *For any $\delta \in (0, 1]$, let $\theta_2(\delta) = \mathscr{G}((1+\delta)^2; 2\delta+\delta^2)$. The function θ_2 is increasing when $\delta \le 0.5305\cdots$, and constant equal to 1 afterwards. We have $\theta_2(0+) = 0$, $\theta_2(1/2) = 0.9093\cdots$, $\theta_2(1) = 1$. Suppose $\lambda_2 > 0$ is an exponent that satisfies the hypothesis $\mathscr{H}_\ell$ above for $\ell = 2$, and let $\varepsilon > 0$. For any $\delta \in [0, 1]$, and for $q = N$ or k^2 sufficiently large, there exists a prime*

$$p \ll_\varepsilon q^{\frac{4\lambda_2}{1+\theta_2(\delta)}+\varepsilon}$$

with $|a(p)| \le 1+\delta$.

Remark 1.4 The convexity bound allows the choice $\lambda_2 = \frac{1}{4}$ and currently this is the best known exponent. Obtaining a subconvex estimate for the symmetric square L-function in the level or the weight aspect is a challenging problem.

It turns out that showing the existence of primes p of small size in terms of the conductor (i.e., weight and level) such that $a(p) \ge 0$ is rather difficult. By utilizing the fact that hypotheses $\mathscr{H}_\ell$ hold true for $1 \le \ell \le 5$, we are able to show the following result:

Theorem 1.5 *There is a prime $p \ll k^{24}N^{21}$ such that $a(p) \geq 0$.*

The results above are all obtained using a similar strategy and this is summarized in Theorem 1.11 below. For some specific intervals, however, we obtain better bounds by employing ad hoc techniques using L-functions as we now describe.

Theorem 1.6 *For any $\varepsilon > 0$, there is a prime $p = \mathcal{O}_{\varepsilon}(kN)^{1+\varepsilon}$ such that $a(p) < 0$.*

Corollary *The least prime such that $a(p) \neq 0$ is $\ll_{\varepsilon} (kN)^{1+\varepsilon}$, for any $\varepsilon > 0$.*

Remark 1.7 As the proof of the above theorem shows, the exponent 1 can be replaced by $4\lambda_2$ and any subconvex estimate $\lambda_2 < 1/4$ for the symmetric square L-function will lead to an improvement of the above result.

The next result relates the possibility of the initial coefficients at primes assuming extreme values with the size of $L(1, f)$. For $q = Nk^2$, let

$$\gamma^- := \liminf_{q\to\infty} \frac{\log L(1, f)}{\log\log q}, \qquad \gamma^+ := \limsup_{q\to\infty} \frac{\log L(1, f)}{\log\log q}.$$

From the zero-free region of $L(s, f)$ (See [11]), the standard techniques yield

$$-2 \leq \gamma^- \leq \gamma^+ \leq 2. \tag{1.3}$$

Theorem 1.8 *For any $\delta, \varepsilon > 0$, the least prime p such that $a(p) > \gamma^- - \delta$ is $\mathcal{O}(q^{\varepsilon})$. Similarly, the least prime p such that $a(p) < \gamma^+ + \delta$ is $\mathcal{O}(q^{\varepsilon})$.*

Remark 1.9 The bounds (1.3) seem to be the best known, and any improvement would yield a non-trivial result in Theorem 1.8. The quality of the upper-bound on p, namely $\mathcal{O}(q^{\varepsilon})$, compared to the above results, suggests that improving the bounds (1.3) is a difficult task. Under the Riemann Hypothesis for $L(s, f)$, one has the bounds

$$(\log\log q)^{-2} \ll L(1, f) \ll (\log\log q)^2,$$

at least in the case $N = 1$ (see [23, Thm. 3] for a precise and stronger statement), which yields conjecturally $\gamma^- = \gamma^+ = 0$. Furthermore, it is known that these bounds hold for almost all forms (see [24, Cor. 2] for a precise statement).

Several authors investigated the smallest *integer* n such that $a(n) < 0$, see for instance [13, 19, 21] or [25]. It follows from [25] that the least such n is $\mathcal{O}(q^{3/8})$, where $q = Nk^2$. A closer scrutiny of their proofs reveals that the integer n they produce is either a prime or the square of a prime. Indeed, all the above works make use of the contrast between the sizes of $a(p)$ and $a(p^2)$ forced by the Hecke relation $a(p)^2 - 1 = a(p^2)$ for primes p. Since we aim at localizing only $a(p)$'s, the coefficients at primes, we cannot rely on such procedures. In fact, the two methods we propose are *reverse*: from a localization on $a(p)$, we show that some polynomial in $a(p)$ has to be large for many primes p. This polynomial defines the value at p of a new function whose Dirichlet series we approximate with products of $L(s, \mathrm{sym}^{\ell} f)$ and it is by using the analytic properties of these latter that we reach a contradiction. To find an integer n such that $a(n) < 0$, only the analytic properties of $L(s, f)$ are required.

Regarding bounds conditional on the Riemann Hypothesis, Ankeny [1] has proved that for any non-trivial character $\chi \mod q$, if the Riemann hypothesis is true for $L(s, \chi)$, then the least n such that $\chi(n) \neq 1$ is $\mathcal{O}((\log q)^2)$. It is not difficult to show that the analogous phenomenon holds in our setting:

Theorem 1.10 *Assume that for all $\ell \geq 1$, the function $L(s, \mathrm{sym}^\ell f)$ is entire and satisfies the Riemann hypothesis. Then for any interval $I \subseteq [-2, 2]$ of positive measure, the least prime p such that $a(p) \in I$ satisfies $p \ll_I (\log q)^2$.*

Let us now state our general theorem depending on the hypothesis $\mathscr{H}_\ell$. Note that this result implies Theorems 1.1, 1.3 and 1.5.

Theorem 1.11 *(Generic theorem) Let $(b_\ell)_{1\leq \ell\leq L}$ be non-negative integers, Let $\kappa > 0$ and F be real, and let $I \subset [-2, 2]$ be such that*

$$\begin{cases} \forall x \in [-2,2] \smallsetminus I, & \displaystyle\sum_{1\leq \ell\leq L} b_\ell U_\ell(x/2) \geq \kappa > 0, \\ \forall x \in [-2,2], & \displaystyle\sum_{1\leq \ell\leq L} b_\ell U_\ell(x/2) \geq F, \end{cases} \tag{1.4}$$

where U_ℓ are the Chebyshev polynomials of the second kind. Then, on assuming $(\mathscr{H}_\ell)_{\ell\leq L}$, the least prime p such that $a(p) \in I$ satisfies

$$\frac{\log p}{\log N} \leq \frac{2\sum_\ell \ell b_\ell \lambda_\ell}{1 + \mathscr{G}(\kappa - F; \kappa)} + \varepsilon, \tag{1.5}$$

for any $\varepsilon > 0$ and N large enough with respect to the weight k and ε; and

$$\frac{\log p}{\log k} \leq \frac{2\sum_\ell (\ell + \epsilon(\ell)) b_\ell \lambda_\ell}{1 + \mathscr{G}(\kappa - F; \kappa)} + \varepsilon. \tag{1.6}$$

for any $\varepsilon > 0$ and k large enough with respect to the level N and ε. Here $\epsilon(\ell) = \frac{1-(-1)^\ell}{2} \in \{0, 1\}$ is the parity of ℓ.

The intervals $[\alpha, \beta]$ for which there is a linear combination with non-negative coefficients of $U_1, \ldots, U_8$ which takes positive values outside $[\alpha, \beta]$ delimit a curve in (α, β), whose exact determination is an interesting question (without the non-negativity condition, the analogue for $U_1, \ldots, U_4$ was solved in Appendix A of [22]). Between this curve and the diagonal $\alpha = \beta$, Theorem 1.11 yields an upper-bound on $\frac{\log p}{\log q}$, which gets smaller as one moves away from the diagonal. This is represented in Figure 1, which was obtained by case-by-case analysis of all linear combinations with $\sum_{\ell\leq 8} \ell b_\ell \leq 42$. On the left, darker colors indicate a larger upper-bound.

Theorem 1.11 should be compared with Theorem 1.8 of [22]. In both cases, we are given an interval $I \subset [-2, 2]$, and we are looking for the least prime p such that $a(p) \in I$. In Theorem 1.8 of [22], the authors obtain an exponent depending on the

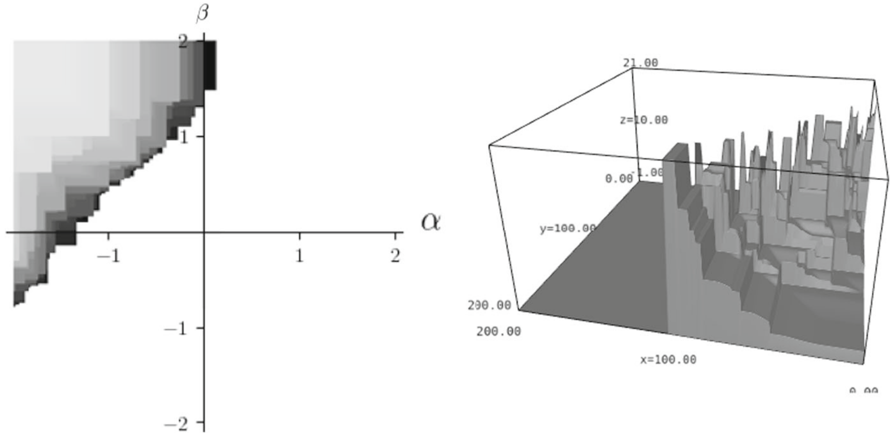

Fig. 1 Upper-bound on $\frac{\log p}{\log N}$ in Theorem 1.11 for $I = [\alpha, \beta]$

quality with which the indicator function $\mathbb{1}_I$ can be minorized by a linear combination of $U_0, U_1, U_2, \ldots$. In Theorem 1.11, we obtain an exponent depending on the quality with which the complementary indicator function $\mathbb{1}_{[-2,2]\setminus I}$ is minorized by a linear combination *with non-negative coefficients* of $U_1, U_2, \ldots$. An inconvenient of our method is that there is no clear description of the allowable intervals I. Theorems 1.1–1.5 indicate that, when it can be applied, the method described here yields non-trivial numerical results.

1.1 Notation

Our notation is quite standard. We follow the usual practice of denoting by p an arbitrary prime and by ε an arbitrarily small positive real number which need not be the same in every occurrence. For any set $X \subset \mathbb{R}$ and maps $F : X \mapsto \mathbb{C}$ and $G : X \mapsto [0, \infty)$, we write

$$F(x) \ll G(x) \text{ or } F(x) = \mathcal{O}(G(x))$$

if there exists a $C > 0$ such that $|F(x)| \leq CG(x)$ for all $x \in X$. Sometimes, the implied constant C depends on some parameters and this dependence is shown in the subscript. For example, often the implied constant depends on the parameter ε, an arbitrarily small positive real number, and we display this dependence by writing $\ll_\varepsilon$ or $\mathcal{O}_\varepsilon$. Sometimes, the dependence is not shown when it is clear from the context in order to avoid making the notation too cumbersome. By $\check{\eta}$, we denote the Mellin transform of a function η:

$$\check{\eta}(s) = \int_0^\infty \eta(t) t^{s-1} dt. \tag{1.7}$$

2 Background on modular forms and L-functions

2.1 Symmetric power L-functions

For a primitive form f, as in the introduction, its normalized coefficients $a_f(p) = a(p)$ can be written as

$$a(p) = \alpha_f(p) + \beta_f(p)$$

where, for $p \nmid N$, $\alpha_f(p) = 1/\beta_f(p)$ and both are complex numbers of absolute value 1. For each $\ell \in \mathbb{N}$, the ℓth symmetric power L-function of f is defined, for $\Re s > 1$, by

$$L(s, \mathrm{sym}^\ell f) = \prod_p \prod_{0 \le j \le \ell} \left(1 - \alpha_f(p)^{\ell-j} \beta_f(p)^j / p^s\right)^{-1} =: \sum_{n \ge 1} \frac{a_{\mathrm{sym}^\ell f}(n)}{n^s}. \quad (2.1)$$

We have $\mathrm{sym}^1 f = f$ and it is convenient to set $\mathrm{sym}^0 f = \mathbb{1}$ so that $L(s, \mathrm{sym}^0 f) = \zeta(s)$. It is expected from a general conjecture of Langlands [20] that for every ℓ, there is a cuspidal automorphic representation of $GL_{\ell+1}(\mathbb{A}_{\mathbb{Q}})$ that corresponds to the L-function $L(s, \mathrm{sym}^\ell f)$. For $1 \le \ell \le 8$, this was shown in [7] (for $\ell = 2$), [17] (for $\ell = 3$), [16, 18] (for $\ell = 4$) and [4, 5] (for $5 \le \ell \le 8$). When N is squarefree, this has been announced for all $\ell \ge 0$ in [28].

Following [14, Eq.(5.5)]), we define the analytic conductor of $L(s, \mathrm{sym}^\ell f)$ as

$$q(s, \mathrm{sym}^\ell(f)) = N^\ell (|t| + 2)^{\ell+1} k^{\ell+\epsilon(\ell)}, \quad (2.2)$$

with $\epsilon(\ell) = \frac{1-(-1)^\ell}{2}$ being 1 or 0 according to whether ℓ is odd or even, as in the statement of Theorem 1.11.

Once we know that a symmetric power L-function comes from an automorphic representation, the analytic continuation and functional for that L-function follows from [8] and thus the Phragmén–Lindelöf convexity principle (or the approximate functional equation [14, eq. (5.20)]) implies that for $1 \le \ell \le 8$, the hypothesis $\mathscr{H}_\ell$ holds with the value $\lambda = 1/4$. This is known as the convexity bound. Giving a bound on an L-function that is stronger than the convexity bound is a challenging problem which has been solved in a few cases (see [27] and the references therein) and this is known as the subconvexity problem. Sometimes we are interested in the size of the L-functions in terms of only the size of the variable t, or the weight k or the level N. A result of Jutila and Motohashi [15] says that taking $\lambda_1 = 1/6$ is permissible in the weight and the t-aspect. We further define

$$q(\mathrm{sym}^\ell(f)) := N^\ell k^{\ell+\epsilon(\ell)}. \quad (2.3)$$

In particular, $q(f) = Nk^2$ and $q(\mathrm{sym}^2(f)) = N^2 k^2$. Note that in the weight aspect, $q(f)$ and $q(\mathrm{sym}^2(f))$ are of the same order.

For the coefficients of the symmetric ℓth power L-function of f, we have the following relation for every prime p:

$$a_{\mathrm{sym}^\ell f}(p) = a\big(p^\ell\big) = U_\ell(\cos\theta(p)) = U_\ell(a(p)/2) = \frac{\sin((\ell+1)\theta(p))}{\sin\theta(p)}, \quad (2.4)$$

where U_ℓ is the Chebyshev polynomial of second kind, whose properties we recall next.

2.2 Chebyshev polynomials of the second kind

We recall that the Chebyshev polynomials of second kind $(U_\ell)_{\ell\geq 0}$ are defined by

$$U_0 = 1, \qquad U_1 = 2x, \qquad U_{\ell+1} - 2xU_\ell + U_{\ell-1} = 0. \quad (2.5)$$

These polynomials form an orthonormal basis in the space of polynomials on the interval $[-1, 1]$ relative to the Hermitian product

$$\langle f, g\rangle = \int_{-1}^{1} f(x)\overline{g(x)}\tfrac{2}{\pi}\sqrt{1-x^2}dx. \quad (2.6)$$

The first few are given by

$$\begin{aligned}
U_2 &= 4x^2 - 1,\\
U_3 &= 8x^3 - 4x,\\
U_4 &= 16x^4 - 12x^2 + 1,\\
U_5 &= 32x^5 - 32x^3 + 6x,\\
U_6 &= 64x^6 - 80x^4 + 24x^2 - 1,\\
U_7 &= 128x^7 - 192x^5 + 80x^3 - 8x,\\
U_8 &= 256x^8 - 448x^6 + 240x^4 - 40x^2 + 1.
\end{aligned}$$

The last equality in Eq. (2.4) comes from the relation

$$U_n(\cos\theta) = \frac{\sin((n+1)\theta)}{\sin\theta}.$$

3 Auxiliary lemmas

3.1 Convolutions

Lemma 3.1 *Assume $(\mathscr{H}_\ell)_{1\leq\ell\leq L}$. Let $L \geq 1$ be an integer and let $(b_\ell)_{0\leq\ell\leq L}$ be a collection of non-negative integers. Then, we have the equality*

$$\prod_p \left(1 + \frac{\sum_\ell b_\ell a(p^\ell)}{p^s}\right) = \prod_{0 \le \ell \le L} L(s, \mathrm{sym}^\ell f)^{b_\ell} H(s),$$

where H is a function that is holomorphic and bounded by a constant in the region $\Re s \geq \frac{1}{2} + \varepsilon$ for any $\varepsilon > 0$.

Proof This follows easily by comparing the pth Euler factors. □

We recall that, in the half-plane of absolute convergence, we have

$$L(s, f) = \prod_p \left(1 - \frac{a(p)}{p^s} + \frac{1}{p^{2s}}\right)^{-1} = \prod_p \left(1 - \frac{\alpha(p)}{p^s}\right)^{-1} \left(1 - \frac{\beta(p)}{p^s}\right)^{-1} \tag{3.1}$$

as well as

$$L(s, \mathrm{sym}^2 f) = \prod_p \left(1 - \frac{a(p)^2 - 1}{p^s} + \frac{a(p)^2 - 1}{p^{2s}} - \frac{1}{p^{3s}}\right)^{-1}. \tag{3.2}$$

3.2 Averages of multiplicative functions

We quote Theorem 21.2 of [29] which follows an idea of Wirsing [33].

Lemma 3.2 *Let f be a non-negative multiplicative function and κ be a non-negative real parameter such that*

$$\begin{cases} \displaystyle\sum_{\substack{p \geq 2, \nu \geq 1 \\ p^\nu \leq Q}} f(p^\nu) \log(p^\nu) = \kappa Q + \mathcal{O}(Q/\log(2Q)) \qquad (Q \geq 1), \\ \displaystyle\sum_{p \geq 2} \sum_{\substack{\nu, \ell \geq 1, \\ p^{\nu+\ell} \leq Q}} f(p^\ell) f(p^\nu) \log(p^\nu) \ll \sqrt{Q}, \end{cases}$$

then we have

$$\sum_{d \leq D} f(d) = \kappa\, C \cdot D\, (\log D)^{\kappa - 1}\, (1 + o(1)),$$

where

$$C = \frac{1}{\Gamma(\kappa + 1)} \prod_p \left\{ \left(1 - \frac{1}{p}\right)^\kappa \sum_{\nu \geq 0} f(p^\nu) \right\}. \tag{3.3}$$

Lemma 3.3 *Under the same hypotheses of Lemma 3.2 we have, for any continuously differentiable function η with $\int_0^1 \eta(u)du \neq 0$:*

$$\sum_{d\leq D} f(d)\eta(d/D) = \kappa C(1+o(1)) \int_2^D (\log u)^{\kappa-1}\eta(u/D)du$$

as $D \longrightarrow \infty$.

The condition on η is obviously satisfied if, as will be the case for us, η is non-negative with support inside the interval [0, 1].

Proof Using Lemma 3.2, we find that

$$\begin{aligned}\sum_{d\leq D} f(d)\eta(d/D) &= \sum_{d\leq D} f(d)\eta(1) - \sum_{d\leq D} f(d)\int_{d/D}^1 \eta'(t)dt\\ &= \sum_{d\leq D} f(d)\eta(1) - \int_0^1 \sum_{d\leq tD} f(d)\eta'(t)dt\\ &= \kappa\eta(1)C\cdot D\,(\log D)^{\kappa-1}\\ &\quad -\int_2^D \kappa C u(\log u)^{\kappa-1}\eta'(u/D)du/D + o(D(\log D)^{\kappa-1})\end{aligned}$$

as $D\to\infty$. Hence, by partial summation,

$$\begin{aligned}&\sum_{d\leq D} f(d)\eta(d/D)(\kappa C)^{-1}\\ &\quad=\int_2^D (\kappa-1+\log u)(\log u)^{\kappa-2}\eta(u/D)du + o(D(\log D)^{\kappa-1})\\ &\quad=\int_2^D (\log u)^{\kappa-1}\eta(u/D)du + o(D(\log D)^{\kappa-1}).\end{aligned}$$

However, we also have

$$\begin{aligned}&\int_2^D (\log u)^{\kappa-1}\eta(u/D)du\\ &\quad=\mathcal{O}\Big(\int_2^{D/\log D}(\log u)^{\kappa-1}du\Big) + \int_{D/\log D}^D (\log u)^{\kappa-1}\eta(u/D)du\\ &\quad=\mathcal{O}\left(D(\log D)^{\kappa-2}\right) + D(\log D)^{\kappa-1}\int_{1/\log D}^1 \Big(1+\frac{\log v}{\log D}\Big)^{\kappa-1}\eta(v)dv\\ &\quad=\mathcal{O}\left(D(\log D)^{\kappa-2}\right) + D(\log D)^{\kappa-1}\int_{1/\log D}^1 \eta(v)dv\\ &\quad\sim\Big(\int_0^1 \eta(v)dv\Big)D(\log D)^{\kappa-1}\end{aligned}$$

as $D \to \infty$, since $\int_0^1 \eta(u)du \neq 0$. In the third line we have used the uniform estimate $(1+(\log v)/\log D)^{\kappa-1} = 1+O(\log(1/v)/\log D)$ for $1/\log D < v < 1$. Hence our claimed estimate

$$\sum_{d\leq D} f(d)\eta(d/D) = \kappa C(1+o(1)) \int_2^D (\log u)^{\kappa-1}\eta(u/D)du$$

follows. □

4 A general average bound

Lemma 4.1 *Let $L \in \mathbb{N}_{>0}$, and assume $(\mathscr{H}_\ell)_{1\leq\ell\leq L}$. Let $(b_\ell)_{0\leq\ell\leq L}$ be a collection of non-negative integers. Given a primitive form $f(z) = \sum_{n\geq1} a(n)e(nz)$ as in the introduction, let us define a multiplicative function h_f by the equality*

$$\sum_n \frac{h_f(n)}{n^s} = \prod_p \left(1 + \frac{\sum_\ell b_\ell a(p^\ell)}{p^s}\right)$$

Then h_f is supported on squarefree integers and there exists a polynomial P_L of degree at most $b_0 - 1$ such that, for any $\varepsilon > 0$, we have

$$\sum_{n\geq1} h_f(n)\eta(n/X) = XP_L(\log X) + \mathcal{O}\left(X^{\frac{1}{2}+\varepsilon} \prod_{1\leq\ell\leq L} q(\mathrm{sym}^\ell(f))^{b_\ell\lambda_\ell+\varepsilon}\right) \quad (4.1)$$

for any compactly supported twice continuously differentiable non-negative function η.

Proof Let us denote by S the left-hand side of (4.1). By taking Mellin transforms (*e.g.* p.90 of [14]), we get

$$S = \frac{1}{2i\pi}\int_{2-i\infty}^{2+i\infty} X^s\check{\eta}(s)^s \sum_{n\geq1}\frac{h_f(n)}{n^s}ds.$$

The fact that η is twice continuously differentiable ensures us that its Mellin transform verifies $\check{\eta}(s) \ll 1/(1+|s|^2)$ uniformly in any closed vertical strip in the half plane $\Re s > 0$. Lemma 3.1 gives us an expression for the Dirichlet series $\sum_{n\geq1} h_f(n)/n^s$ from which we see that we can shift the line of integration to $\Re s = \frac{1}{2}+\varepsilon$ obtaining that the error term is at most

$$\mathcal{O}\left(X^{\frac{1}{2}+\varepsilon} \prod_{1\leq\ell\leq L} q(\mathrm{sym}^\ell(f))^{b_\ell\lambda_\ell+\varepsilon}\right),$$

by our hypothesis $(\mathscr{H}_\ell)_{1\le\ell\le L}$ and the convexity principle. The residue at 1 gives the claimed main term, and the lemma follows readily. □

5 A general Lemma around Vinogradov's trick

Lemma 5.1 *Let g be a real-valued multiplicative function supported on the squarefree integers. We assume further that $g(p) \ge F$ for every prime p, and that for every prime $p \le P$, we have $g(p) \ge \kappa > 0$. Let η be a non-negative, continuously differentiable function with support within $[0, 1]$ such that $\int_0^1 \eta(v)dv = 1$. We have, for $M = P^\theta$ for some $\theta \in [0, 1]$,*

$$\sum_{n\ge1} \mu^2(n)g(n)\eta\Big(\frac{n}{PM}\Big) \ge (1+o(1))\kappa\, CMP(\log MP)^{\kappa-1}\big(1-(\kappa-F)\mathscr{F}(\theta;\kappa)\big),$$

where C is given by (3.3) and $\mathscr{F}$ is defined in (1.1)

The factor $\mu^2(n)$ is only here to remind the reader that the variable n is restricted to squarefree values. It can be omitted!

Proof We set

$$S = \sum_{n\ge1} g(n)\eta\Big(\frac{n}{PM}\Big). \tag{5.1}$$

By our hypotheses, we find that

$$\begin{aligned} S &= \sum_{\substack{n\le PM,\\ P^+(n)\le P}} g(n)\eta\Big(\frac{n}{PM}\Big) + \sum_{P<p\le PM} g(p) \sum_{n\le PM/p} g(n)\eta\Big(\frac{pn}{PM}\Big) \\ &\ge \sum_{\substack{n\le PM,\\ P^+(n)\le P}} g(n)\eta\Big(\frac{n}{PM}\Big) + F \sum_{P<p\le PM}\ \sum_{n\le PM/p} \mu^2(n)\kappa^{\omega(n)}\eta\Big(\frac{pn}{PM}\Big) \\ &\ge \sum_{n\le PM} \mu^2(n)\kappa^{\omega(n)}\eta\Big(\frac{n}{PM}\Big) + (F-\kappa) \sum_{P<p\le PM}\ \sum_{n\le PM/p} \mu^2(n)\kappa^{\omega(n)}\eta\Big(\frac{pn}{PM}\Big). \end{aligned}$$

Here $P^+(n)$ denotes the greatest prime divisor of n and $\omega(n)$ the number of prime divisors of n. We appeal to Lemma 3.3 with $f(n) = \mu^2(n)\kappa^{\omega(n)}$ and get

$$\begin{aligned} S/(C\kappa) \ &\ge (1+o(1)) \int_2^{PM} (\log u)^{\kappa-1}\eta\Big(\frac{u}{PM}\Big)du \\ &\qquad +(F-\kappa+o(1)) \sum_{N<p\le PM} \int_2^{PM/p} (\log u)^{\kappa-1}\eta\Big(\frac{up}{PM}\Big)du. \end{aligned}$$

Note that the change of variable $vPM = u$ shows that

$$\int_2^{PM} (\log u)^{\kappa-1}\eta\Big(\frac{u}{PM}\Big)du = PM(\log PM)^{\kappa-1}\int_0^1 \eta(v)dv(1+o(1)).$$

We use this estimate with M replaced by M/t and the prime number theorem to infer that

$$\sum_{N<p\le PM}\int_2^{PM/p} (\log u)^{\kappa-1}\eta\Big(\frac{up}{PM}\Big)du$$
$$= PM(1+o(1))\int_0^1 \eta(v)dv\int_N^{PM}\Big(\log\frac{PM}{t}\Big)^{\kappa-1}\frac{dt}{t\log t}$$

while this last integral equals, with the change of variable $v = (PM)^h$ and $M = N^\theta$,

$$\int_1^M \frac{(\log v)^{\kappa-1}dv}{v(\log(PM)-\log v)} = (\log PM)^{\kappa-1}\int_0^{\theta/(1+\theta)}\frac{h^{\kappa-1}dh}{1-h}.$$

Recall that $\int_0^1 \eta(v)dv = 1$. We thus find that

$$\frac{(1+o(1))S}{C\kappa PM(\log PM)^{\kappa-1}} \ge 1+(F-\kappa)\int_0^{\theta/(1+\theta)}\frac{h^{\kappa-1}dh}{1-h}$$
$$= 1-(\kappa-F)\mathscr{F}(\theta,\kappa).$$

□

6 Proof of Theorems 1.11, 1.1, 1.3, 1.5

Suppose $a(p) \notin I$ for every $p \le P$. Under the assumptions of Theorem 1.11, let $\theta \in [0, 1]$ be such that

$$\frac{1}{\kappa - F} > \mathscr{F}(\theta;\kappa); \tag{6.1}$$

for instance, we may take $\theta = \max(\mathscr{G}(\kappa - F;\kappa) - \varepsilon, 0)$. Consider the sum

$$S = \sum_{n\ge1} h_f(n)\eta(n/PM),$$

where $M \in [1, P]$. From the upper and the lower bound of S as given by Lemma 4.1 and 5.1, respectively, and noting that $b_0 = 0$, we obtain,

$$(PM)^{\frac12+\varepsilon}\prod_{1\le\ell\le L} q(\mathrm{sym}^\ell(f))^{b_\ell\lambda_\ell+\varepsilon} \gg PM.$$

Therefore, with $M = P^{\theta}$ for some $\theta \in [0, 1]$ satisfying (6.1), we have

$$P \ll_k N^{\frac{2\sum_\ell \ell b_\ell \lambda_\ell}{1+\theta}+\varepsilon}.$$

This leads to the estimate (1.5) and the other estimate (1.6) is proved in a similar manner.

Let us inspect what this gives to us under the convexity bound for $\lambda_\ell = 1/4$. Since the quantity $2\sum_{\ell\geq 1} \ell b_\ell \lambda_\ell$ takes all the values that are half-positive integers, we may inspect the first of them one by one. As we did above, we focus on the level N.

6.1 First case $(1/2)\sum_{\ell\geq 1} \ell b_\ell = 1/2$

This is only possible with the choice $b_1 = 1$, all other b_ℓ's being 0. We have $\sum_{1\leq\ell\leq L} b_\ell U_\ell(x/2) = x$ which is positive when $x = a(p) > 0$. On assuming $a(p) \geq \delta$ when $p \leq P$, we see that we may take $\kappa = \delta$ and $F = -2$ and get, for $N \geq N_0(\varepsilon)$,

$$\frac{\log P}{\log N} \leq \frac{2\lambda_1}{1 + \mathscr{G}(2+\delta; \delta)} + \varepsilon. \tag{6.2}$$

Hence Theorem 1.1.

6.2 Second case $(1/2)\sum_{\ell\geq 1} \ell b_\ell = 1$

This is only possible with the choice $b_2 = 1$, all other b_ℓ's being 0. We have $\sum_{1\leq\ell\leq L} b_\ell U_\ell(x/2) = x^2 - 1$ which is positive when $x = a(p) \notin [-1, 1]$. On assuming $|a(p)| \geq 1 + \delta$ when $p \leq P$, we see that we may take $\kappa = 2\delta + \delta^2$ and $F = -1$ and get, for $N \geq N_0(\varepsilon)$,

$$\frac{\log P}{\log N} \leq \frac{4\lambda_2}{1 + \mathscr{G}(1 + 2\delta + \delta^2; 2\delta + \delta^2)} + \varepsilon. \tag{6.3}$$

Hence Theorem 1.3.

6.3 Finding non-negative values

Let $I = [0, 2]$. A numerical computation found the coefficients $(b_\ell)_{0\leq\ell\leq 5} = (0, 0, 3, 5, 4, 1)$, which satisfy (1.4) with $\kappa \geq 1/3$ and $F = -10$. Then Theorem 1.5 follows from the bounds (1.5) and (1.6).

7 Proof of Theorem 1.6

Let $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ be smooth, compactly supported and such that $\mathbb{1}_{[0,1]} \geq \eta \geq \mathbb{1}_{[1/3,2/3]}$, let $\varepsilon > 0$, and consider

$$T(X) = \sum_n \mu^2(n)a(n)\eta(n/X) \qquad (X \geq 1).$$

By Lemma 4.1, we get

$$T(X) \ll X^{1/2}(k^2N)^{1/4+\varepsilon}. \tag{7.1}$$

Suppose that $a(p) \geq 0$ for all primes $p \leq X$. If the inequality

$$\sum_{\substack{n:\\ a(n)\geq 1}} \mu^2(n)a(n)\eta(n/X) \geq X^{1-\varepsilon} \tag{7.2}$$

holds then we easily have

$$T(X) \geq X^{1-\varepsilon}. \tag{7.3}$$

Otherwise, suppose that (7.2) does not hold. We write

$$\begin{aligned} T(X) &= \sum_{\substack{n:\\ 0\leq a(n)<1}} \mu^2(n)a(n)\eta(n/X) + \sum_{\substack{n:\\ a(n)\geq 1}} \mu^2(n)a(n)\eta(n/X) \\ &\geq \sum_{\substack{n:\\ 0\leq a(n)<1}} \mu^2(n)a(n)^2\eta(n/X) + \sum_{\substack{n:\\ a(n)\geq 1}} \mu^2(n)a(n)\eta(n/X) \\ &= \sum_n \mu^2(n)a(n)^2\eta(n/X) + \sum_{\substack{n:\\ a(n)\geq 1}} \mu^2(n)a(n)(1-a(n))\eta(n/X). \end{aligned}$$

Now the last sum is $\mathcal{O}(X^{1-\varepsilon/2})$ by Deligne's bound $|a(p)| \leq 2$ and the negation of (7.2). The first sum can be handled by Rankin–Selberg method (Lemma 4.1) and is $\gg L(1, \mathrm{sym}^2 f)X + \mathcal{O}(X^{1/2}(k^2N^2)^{1/4+\varepsilon})$. Thus we have, using the lower bound $L(1, \mathrm{sym}^2 f) \gg 1/\log(kN)$ due to Hoffstein and Lockhart [10],

$$T(X) \gg X/\log(kN) + \mathcal{O}(X^{1/2}(k^2N^2)^{1/4+\varepsilon}) + \mathcal{O}(X^{1-\varepsilon/2}). \tag{7.4}$$

One of the equations (7.3) and (7.4) must hold and either, in conjunction with equation (7.1), implies the theorem.

8 Proof of Theorem 1.8

By equation (3) of [30], the Deligne bound $|a(p)| \leq 2$ and Mertens' theorem (see [14, Eq. (2.15)]), we have

$$\log L(1, f) = \mathcal{O}_\varepsilon(1) + \sum_{p \leq q^\varepsilon} \frac{a(p)}{p},$$

and therefore

$$\sum_{p \leq q^\varepsilon} \frac{1}{p}\Big(a(p) - \frac{\log L(1, f)}{\log\log q}\Big) = \mathcal{O}_\varepsilon(1).$$

However, if we had $a(p) < \gamma^- - \delta$ for $p \leq q^\varepsilon$, then we would also have

$$\sum_{p \leq q^\varepsilon} \frac{1}{p}\Big(a(p) - \frac{\log L(1, f)}{\log\log q}\Big) \leq \mathcal{O}_{\varepsilon,\delta}(1) - \frac{\delta}{2}\log\log q,$$

which is a contradiction for q large enough, and therefore there must be a prime $p \leq q^\varepsilon$ such that $a(p) \geq \gamma_- - \delta$. An identical argument shows the existence of $p \leq q^\varepsilon$ such that $a(p) \leq \gamma_+ + \delta$.

9 Conditional bounds: proof of Theorem 1.10

By the Stone-Weierstrass theorem, the fact that (U_ℓ) forms a basis of $\mathbb{R}[X]$, and the relation (2.4), we may find $L \geq 1$ and real coefficients $b_0, \ldots, b_L$ depending on I, with $b_0 > 0$, such that

$$\sum_{p \leq x} \mathbb{1}(a(p) \in I)(1 - \tfrac{p}{x}) \log p \geq \sum_{\ell=0}^{L} b_\ell \sum_{p \leq x} a_{\mathrm{sym}^\ell f}(p)(1 - \tfrac{p}{x}) \log p. \qquad (9.1)$$

By Chebyshev's estimate, the contribution of the term $\ell = 0$ is

$$b_0 \sum_{p \leq x} (1 - \tfrac{p}{x}) \log p \gg_I x$$

with an absolute constant. To show that the right-hand side of (9.1) is positive for some $x = \mathcal{O}_I((\log q)^2)$, it therefore suffices to show that for all integer $\ell \geq 1$ and all real $x \geq 1$, we have

$$\sum_{p \leq x} a_{\mathrm{sym}^\ell f}(p)(1 - \tfrac{p}{x}) \log p = \mathcal{O}_\ell(x^{1/2} \log q).$$

This is an immediate consequence of the explicit formula [14, eq. (5.33)] (with an additional smoothing, as in [26, eq. (13.28)]) along with classical zero density estimates [14, Theorem 5.8].

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