Chapter 4

The local method of Landau et alia

We shall require some facts on the analytic continuation of the Riemann zeta function and most of them are recalled in the next chapter. As it turns out, the bound $|-\zeta'/\zeta(s)| \ll \operatorname{Log}|\Im s|$ valid when $|\Im s| \geq 2$ and $|1-\Re s|\operatorname{Log}|\Im s|$ is small enough will be of particular importance. We take this opportunity to develop some material and to expand on the historical side.

4.1 The Borel-Caratheodory Theorem

The Borel-Caratheodory Theorem which bounds the modulus of an analytic function in term of a bound for its real part is of fundamental importance in what follows. Caratheodory did not publish it anywhere: Landau says he owns a series a results in this vein from letters exchanged with him. See in particular (Landau, 1908, Satz I, section 5) and (Landau, 1926, Lemma 1). (Titchmarsh, 1932, section 5.5) attributes this Theorem to (Borel, 1897, page 365)* and to Caratheodory.

Theorem 4.1 (Borel-Caratheodory)**.** *Let F be an analytic function on $|s-s_0| \leq R$ such that $\Re F(s) \leq A$ in this disc. For any $r < R$, positive, we have*

$$\max_{|s-s_0|\leq r} |F(s)-F(s_0)| \leq \frac{Ar}{R-r}$$

and, for any $k \geq 1$,

$$\max_{|s-s_0|\leq r} |F^{(k)}(s)| \leq \frac{2k!R}{(R-r)^{k+1}}(A-\Re F(s_0)).$$

In fact (Landau, 1908, Satz I, section 5) has a slightly distinct version of the upper bound. See also (Titchmarsh, 1932, section 5.51), where the author follows roughly Landau's proof. This latter one relies on Schwarz's Lemma, though Landau does not cite the later (the proof is anyway quite obvious). I do not know whether Schwarz has anteriority or not. The proof we follow here is essentially the one due Borel, and can be found in (Tenenbaum, 1995, Theorem 11 and corollaries).

*Note that this proof has to be somewhat modified to meet our needs.

Proof. R We can assume with no loss of generality that $s_0 = 0$ and that $F(s_0) = 0$ (otherwise consider $F(s) - F(s_0)$). Note that under these assumptions, the corresponding A, namely $A - \Re F(s_0)$ is non-negative. We expand F is power series:

$$F(s) = \sum_{n\geq 1} a_n s^n$$

and write $a_n = |a_n| e^{i\theta_n}$. We have

$$\Re F(Re^{i\theta}) = \sum_{n\geq 1} |a_n| R^n \cos(n\theta + \theta_n).$$

We multiply this expression by $\cos(m\theta + \theta_m)$ and integrate it termwise to get

$$\pi |a_m| R^m = \int_0^{2\pi} \Re F(Re^{i\theta}) \cos(m\theta + \theta_m) d\theta.$$

When $m = 0$, this reads

$$0 = \int_0^{2\pi} \Re F(Re^{i\theta}) d\theta$$

which we combine with the above to obtain

$$\pi |a_m| R^m = \int_0^{2\pi} \Re F(Re^{i\theta})(1 + \cos(m\theta + \theta_m)) d\theta \leq 2\pi(A - \Re F(s_0)).$$

This readily yields, when $k \geq 1$,

$$\begin{aligned}|F^{(k)}(re^{i\theta})| &\leq \sum_{n\geq k} n(n-1)\cdots(n-k+1)|a_n| r^{n-k} \\ &\leq \frac{2(A - \Re F(s_0))}{R^k} \sum_{n\geq k} n(n-1)\cdots(n-k+1)(r/R)^{n-k} = 2\frac{Rk!(A - \Re F(s_0))}{(R-r)^{k+1}}.\end{aligned}$$

When $k = 0$, we use the additional fact that $a_0 = 0$ to get the claimed bound. □

4.2 The Landau local method

The proof and the statement of the following Lemma has taken some years to find a proper shape. One can find traces of it in (Landau, 1908), between equations (92) and (93), see the definition of F. It will evolve until (Landau, 1926, Lemma 1) to yield a bound on $\zeta'/\zeta(s)$ next to the line $\Re s = 1$. At the time, Gronwall and Landau were improving each other's bound. See also (Titchmarsh, 1951, section 3.9, Lemma α).

The circle of ideas we present below belongs to this realm.

Lemma 4.1. *Let M be an upper bound for the holomorphic function F in $|s - s_0| \leq R$. Assume we know of a lower bound $m > 0$ for $|F(s_0)|$. Then*

$$\frac{F'(s)}{F(s)} = \sum_{|\rho - s_0|\leq R/2} \frac{1}{s-\rho} + \mathcal{O}^*\left(16\frac{\mathrm{Log}(M/m)}{R}\right)$$

for every s such that $|s - s_0| \leq R/4$ and where the summation variable ρ ranges the zeros ρ of F in the region $|\rho - s_0| \leq R/2$, repeated according to multiplicity.

This Lemma will be our main tool in what follows. The reader should look at the very interesting section 3 of (Heath-Brown, 1992a), and more precisely to (Heath-Brown, 1992a, Lemma 3.2). An expression for the real part of $F'(s)/F(s)$ in terms os the possible zeros is obtained there. In fact, the proof therein contains an expression for $F'(s)/F(s)$, but it seems necessary to take the real part to bound it solely in term of M/m (notations as above). See however subsection 4.4.3 of this chapter.

See also (Heath-Brown, 1992b) as well as (Ford, 2000, Lemma 2.1 and Lemma 2.2).

Proof. Let us consider

$$G(s) = \frac{F(s)}{\prod_{|\rho - s_0| \leq R/2}(s - \rho)}.$$

When $|s - s_0| = R$, we have $|s - \rho| \geq |s - s_0| - |\rho - s_0| \geq R/2 \geq |\rho - s_0|$ for the zeros under consideration, and thus, by the maximum principle, when $|s - s_0| \leq R$, we have

$$\left|\frac{G(s)}{G(s_0)}\right| = \left|\frac{F(s)}{F(s_0)} \prod_{|\rho - s_0| \leq R/2} \frac{s_0 - \rho}{s - \rho}\right| \leq M/m.$$

Since this function has no zeros inside $|s - s_0| \leq R/2$, we can write

$$G(s)/G(s_0) = e^{H(s)} \quad (|s - s_0| \leq R/2)$$

for an analytic function H that verifies $H(s_0) = 0$. Furthermore $\Re H(s) \leq \operatorname{Log}(M/m)$. By the Borel-Caratheodory Theorem, we deduce that

$$\left|\frac{G'(s)}{G(s)}\right| = |H'(s)| \leq \frac{8R}{(R - 2r)^2} \operatorname{Log}(M/m) \quad (|s - s_0| \leq r < R/2).$$

We have thus proved our assertion. □

4.3 Consequence for the Riemann zeta function

Here is the main consequence of Lemma 4.1:

Lemma 4.2. *Let $t_0 \geq 4$. We have*

$$\frac{\zeta'(s)}{\zeta(s)} = \sum_{|\rho - 1 - it_0| \leq 1} \frac{1}{s - \rho} + \mathcal{O}(\operatorname{Log} t_0) \quad (|s - 1 - it_0| \leq 1/2) \tag{4.1}$$

We use $F = \zeta$, $s_0 = 1 + it_0$ and $R = 2$ We only have to get some polynomial upper bound for $|\zeta(s)|$ when $-1 \leq \Re s \leq 3$ and $\Im s \geq 2$, as well as a lower bound for $|\zeta(s_0)|$. Concerning the upper bound, here is an expedient way to get one:

$$\zeta(s) = s \int_1^\infty [t] \frac{dt}{t^{s+1}} = \frac{s}{s-1} - \frac{1}{2} - s \int_1^\infty B_1(t) \frac{dt}{t^{s+1}}$$

where $[t]$ denotes the integer part of t, $\{t\}$ its fractional part, and $B_1(t)$ is the first Bernoulli function, defined by $B_1(t) = \{t\} - \frac{1}{2}$. We consider the higher order Bernoulli functions, B_2 and B_3:

$$\tfrac{1}{2}B_2(t) = \int_1^t B_1(u)du + \frac{1}{12} = \tfrac{1}{2}\{t\}^2 - \tfrac{1}{2}\{t\} + \frac{1}{12}.$$

This function is periodical of period 1 when $t \geq 1$ and has mean value 0 over a period (i.e. $\int_1^2 B_2(u)du = 0$). As a consequence

$$\tfrac{1}{3}B_3(t) = \int_1^t B_2(u)du = \tfrac{1}{3}\{t\}^3 - \tfrac{1}{2}\{t\}^2 + \tfrac{1}{6}\{t\}$$

is bounded. As it turns out, it is also of zero mean value over a period. Here is why we have introduced this set of functions:

$$\begin{aligned}\zeta(s) &= \frac{s}{s-1} - \frac{1}{2} - s\int_1^\infty B_1(t)\frac{dt}{t^{s+1}} \\ &= \frac{s}{s-1} - \frac{1}{2} + \frac{s}{12} + \frac{s(s+1)}{2}\int_1^\infty B_2(t)\frac{dt}{t^{s+2}} \\ &= \frac{s}{s-1} - \frac{1}{2} + \frac{s}{12} - \frac{s(s+1)(s+2)}{6}\int_1^\infty B_3(t)\frac{dt}{t^{s+3}}\end{aligned}$$

which gives us an expression of the continuation of ζ to $\Re s > -2$. In particular, when $|s| \geq 2$ and $\Re s \geq -1$, we find that

$$|\zeta(s)| \leq \tfrac{13}{12}|s| + \frac{1}{2} + \tfrac{1}{20}\frac{|s|^3}{2} \leq \tfrac{2}{5}|s|^3.$$

Better bounds are available (in fact, $|\zeta(s)| \ll |s|^{3/2}$ under our conditions), but this will be enough for us.

We also need a lower bound for $1/|\zeta(1+it_0)|$. This is readily obtained as follows. We can modify the above proof to show that

$$|\zeta'(s)| \ll (\operatorname{Log} t)^2 \quad (\Re s \geq 1, t = \Im s \geq 2).$$

We use this bound to shift $\zeta(1+it_0)$ to $\zeta(\sigma+it_0)$ at a cost of $\mathcal{O}((\sigma-1)(\operatorname{Log} t_0)^2)$. We next recall the classical Mertens's inequality*:

$$1 \leq |\zeta(\sigma)|^3|\zeta(\sigma+it_0)|^4|\zeta(\sigma+2it_0)|.$$

*To prove it, notice that $-\zeta'/\zeta$ has a Dirichlet expansion with non-negative coefficients. Since $3 + 4\cos\theta + \cos(2\theta) = 2(1+\cos\theta)^2 \geq 0$, we find that

$$0 \leq 3\Re\frac{-\zeta'}{\zeta}(\sigma) + 4\Re\frac{-\zeta'}{\zeta}(\sigma+it_0) + \Re\frac{-\zeta'}{\zeta}(\sigma+2it_0).$$

Integrating this inequality in σ between σ_0 and ∞ yields the desired result. Alternatively, one could use directly the Dirichlet expansion of $\operatorname{Log}\zeta$, which also has non-negative coefficients.

Since $|\zeta(\sigma)| \ll 1/(\sigma-1)$ and $|\zeta(\sigma+2it_0)| \ll \operatorname{Log} t_0$, we get

$$(\sigma-1)^{3/4}(\operatorname{Log} t_0)^{-1/4} \ll |\zeta(\sigma+it_0)|.$$

We then take $\sigma = 1 + C(\operatorname{Log} t_0)^{-9}$ for a large enough constant C and get

$$|\zeta(\sigma+it_0)| \gg 1/(\operatorname{Log} t_0)^7.$$

This is an apriori bound that is enough for our purpose, but much better is known.

The reader has now all the elements to end the proof of Lemma 4.2. Note that the proof of

$$\frac{\zeta'(s)}{\zeta(s)} = \sum_{|\rho-1-(\operatorname{Log} t_0)^{-1}-it_0|\le 1} \frac{1}{s-\rho} + \mathcal{O}(\operatorname{Log} t_0) \quad (|s-1-it_0| \le 1/2)$$

would have been slightly simpler since the bound

$$|\zeta(1+(\operatorname{Log} t_0)^{-1}+it_0)|^{-1} \le |\zeta(1+(\operatorname{Log} t_0)^{-1})| \ll (\operatorname{Log} t_0)$$

would have been enough. It is however simply easier to write down the expression we have chosen!

4.4 Bounding $|\zeta'/\zeta|$ next to the line $\Re s = 1$

We do not reproduce here a proof of a zero-free region, though we have all the ingredients, but we content ourselves with citing (Kadiri, 2005):

Theorem 4.2 (Kadiri)**.** *The Riemann ζ-function has no zeros in the region*

$$\Re s \ge 1 - \frac{1}{R_0 \operatorname{Log}|\Im s|}, |\Im s| \ge 2, \quad \textit{with } R_0 = 5.69693.$$

The proof of this result is very intricate, but it is fairly easy to get a similar result with a much larger value of R_0, and even easier to prove it without specifying any admissible value for R_0.

Let s be in the region

$$\Re s \ge 1 - \frac{1}{2R_0 \operatorname{Log}(1+|\Im s|)}, \quad |\Im s| \ge 2. \tag{4.2}$$

Lemma 4.3. *We have*

$$\left|\zeta'/\zeta(s)\right| \ll \operatorname{Log} t$$

when $\Re s \ge 1 - \frac{1}{2R_0}(1+\operatorname{Log} t)^{-1}$ where $R_0 > 0$ is the constant of the zero free region given in Theorem 4.2.

There exists essentially three ways to bound $\zeta'/\zeta(s)$ when $s = \sigma + it_0$ is well within the zero free region, i.e. in the region given by (4.2). One is due to Landau, another one to Linnik and a third one to Titchmarsh. We present the three of them. Let us set

$$s_1 = 1 + \frac{4}{R_0 \operatorname{Log}(1+t_0)}.$$

We apply Lemma 4.2 to s and s_1 and substract:

$$\frac{\zeta'}{\zeta}(s) - \frac{\zeta'}{\zeta}(s_1) = \sum_{\rho} \frac{s - s_1}{(s-\rho)(s_1-\rho)} + \mathcal{O}(\operatorname{Log} t_0) \tag{4.3}$$

When s in the region given by (4.2), we have, for any zero $\rho = \beta + i\gamma$ verifying $|s - \rho| \le 1$,

$$(1 - \tfrac{1}{3})(1-\beta) \ge \frac{1 - \frac{1}{3}}{R_0 \operatorname{Log}(1+t_0)} \ge \tfrac{1}{3}(\sigma_1 - 1) + (1 - \sigma)$$

and thus $\sigma - \beta \ge (\sigma_1 - \beta)/3$. As a consequence

$$\Big|\sum_{\rho} \frac{s - s_1}{(s-\rho)(s_1-\rho)}\Big| \ll \sum_{\rho} \frac{1/\operatorname{Log} t_0}{|s_1 - \rho|^2}.$$

4.4.1 Landau's way

(Landau, 1926, Hilfsatz 1) remarks that

$$\sum_{\rho} \frac{1/\operatorname{Log} t_0}{|s_1 - \rho|^2} \ll \sum_{\rho} \frac{\sigma_1 - \beta}{|s_1 - \rho|^2} = \sum_{\rho} \Re \frac{1}{s_1 - \rho} = \Re \frac{\zeta'}{\zeta}(s_1) + \mathcal{O}(\operatorname{Log} t_0).$$

Note that $\Re \frac{\zeta'}{\zeta}(s_1) < 0$ (a fact that Landau does not use). But to end the proof, we will anyway have to use

$$\Big|\frac{\zeta'}{\zeta}(s_1)\Big| \le \Big|\frac{\zeta'}{\zeta}(\sigma_1)\Big| \ll \operatorname{Log} t_0.$$

4.4.2 Linnik's way

(Linnik, 1944a) proposes a different conclusion that shed some more lights as to what happens. This is the one followed in (Ramaré, 2009). We first deduce from the above Linnik's density Lemma:

Lemma 4.4. *Let $n(t_0; r)$ be the number of zeros ρ of ζ such that $|\rho - 1 - it_0| \le r$. We have*

$$n(t_0; r) \ll 1 + r \operatorname{Log} t_0.$$

Proof. Assume $r > 0$. We use the formula (4.1) with $s = 1 + r + it_0$ and take real part. We get

$$\sum_{|\rho - 1 - it_0| \le 1} \frac{1 + r - \beta}{|s - \rho|^2} \ll r^{-1} + \operatorname{Log} t_0$$

by using

$$|\zeta'/\zeta(s)| \leq -\zeta'/\zeta(1+r) \ll r^{-1}.$$

We can discard any zeros, by positivity, from the left and thus restrict the summation to the zeros counted in $n(t_0; r)$. Note that

$$\frac{1+r-\beta}{|s-\rho|^2} = \frac{r}{(1-\beta+r)^2+(\gamma-t_0)^2} \geq \frac{r}{2(1-\beta)^2+2r^2+(\gamma-t_0)^2} \geq \frac{r}{4r^2} = \frac{1}{4r}.$$

The Lemma follows readily. □

Proof of Lemma 4.3 by Linnik. We again exploit (4.3) but we now remark that $|s-\rho| \gg |1+it_0-\rho|$ and that $|s_1-\rho| \gg |1+it_0-\rho|$. Thus

$$(\operatorname{Log} t_0)\left|\frac{\zeta'(s)}{\zeta(s)} - \frac{\zeta'(s_1)}{\zeta(s_1)}\right| \ll \sum_{|\rho-1-it_0|\leq 1} \frac{1}{|1+it_0-\rho|^2}.$$

Let us use diadic decomposition on this last sum. Set $r_k = 2^k/(R_0 \operatorname{Log}(t_0+1))$ when $k \geq 0$, so that:

$$\sum_{|\rho-1-it_0|\leq 1} \frac{1}{|1+it_0-\rho|^2} = \sum_{k\geq 0} \sum_{r_k<|\rho-1-it_0|\leq r_{k+1}} \frac{1}{|1+it_0-\rho|^2}$$
$$\ll \sum_{k\geq 0} \sum_{r_k<|\rho-1-it_0|\leq r_{k+1}} \frac{1+r_{k+1}\operatorname{Log} t_0}{r_k^2} \ll \sum_{k\geq 0} \frac{(\operatorname{Log} t_0)^2}{2^k} \ll (\operatorname{Log} t_0)^2.$$

We conclude the proof as before. □

4.4.3 Titchmarsh's way

(Titchmarsh, 1951, Lemma γ, section 3.9) uses yet another way: first get an upper bound for $-\Re\zeta'(s)/\zeta(s)$ and then apply the Borel-Caratheodory Theorem to this function.

4.5 Bounding $|1/\zeta|$ next to the line $\Re s = 1$

The analytical upper bound we produce for the mean of the Barban & Vehov weights relies on a lower bound for $|\zeta(s)|$ when s is in the vicinity of the line $\Re s = 1$. This is again a consequence our upper bound for $|-\zeta'/\zeta(s)|$. The reader will first notice that it is obvious when $\Re s \geq 1 + 1/(\operatorname{Log} 2 + |\Im s|)$. Otherwise, let $s = \sigma_0 + it_0$ be in the region given by (4.2). We have

$$\operatorname{Log} \frac{\zeta(\sigma_0+it_0)}{\zeta(\sigma_1+it_0)} = \int_{\sigma_1}^{\sigma_0} \frac{\zeta'}{\zeta}(\sigma+it_0)d\sigma$$

with $\sigma_1 = 1 + (\operatorname{Log} t_0)^{-1} + it_0$. We bound the integrand by $\mathcal{O}(\operatorname{Log} t_0)$, and thus, on taking real parts:

$$\operatorname{Log}|\zeta(\sigma_0+it_0)| = \operatorname{Log}|\zeta(\sigma_1+it_0)| + \mathcal{O}(1). \quad (t_0 \geq 2) \tag{4.4}$$

The right-hand side is $\leq \operatorname{Log}\operatorname{Log} t_0 + \mathcal{O}(1)$ in absolute value, hence our bound for $1/|\zeta(\sigma_0 + it_0)|$. Note that we could have taken real parts earlier and thus relied only on an upper bound for $-\Re\zeta'/\zeta(s)$*.

4.6 Some other consequences

As a matter of fact, (4.4) gives more information than just a lower bound. Let us start by recalling the following Lemma, which is a direct consequence of (Montgomery & Vaughan, 1974, Corollary 2).

Lemma 4.5.

$$\int_0^T \Big|\sum_n a_n n^{it}\Big|^2 dt = \sum_n |a_n|^2\big(T + \mathcal{O}^*(3\pi n)\big).$$

Lemma 4.5 together with (4.4) lead directly to:

Lemma 4.6. *Let $T \geq 2$ and $\sigma \geq 1 - (12\operatorname{Log}(1+T))^{-1}$. We have*

$$\int_1^T |\zeta(\sigma+it)|^{\pm 1} dt \asymp T, \qquad \int_1^T |\zeta(\sigma+it)|^{\pm 2} dt \asymp T.$$

Proof. We use (4.4) with $\sigma_0 = \sigma$ and σ_1. This leads immediately to

$$\int_0^T |\zeta(\sigma+it)|^{\pm 1} dt \asymp \int_0^T |\zeta(\sigma_1+it)|^{\pm 1} dt$$

and the same with ± 2.† By Cauchy inequality, we have

$$\left|\int_0^T |\zeta(\sigma_1+it)|^{\pm 1} dt\right|^2 \leq T\int_0^T |\zeta(\sigma_1+it)|^{\pm 2} dt = T\sum_{n\geq 1}\frac{T+\mathcal{O}(n)}{n^{2\sigma_1}} \asymp T^2$$

by appealing to Lemma 4.5 below. Furthermore

$$\left|\int_1^T dt\right|^2 \leq \int_1^T |\zeta(\sigma+it)| dt \int_1^T |\zeta(\sigma+it)|^{-1} dt$$

hence the lower bounds. □

4.7 Better bounds

Using Vinogradov-Korobov – Full-fledged density estimates –

*The minus sign is here because $\sigma_0 \leq \sigma_1$.

†Here $\pm$ means we choose a sign $+$ or $-$ and stick to it on both sides of the relation.

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