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École doctorale **ED Régionale SPI 72**
Unité de recherche **Laboratoire Paul Painlevé**

Thèse présentée par **Ramdinmawia VANLALNGAIA**

Soutenue le **6 juillet 2015**

En vue de l'obtention du grade de
Docteur
de
l'Université Lille 1

Discipline **Mathématiques**
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Fonctions de Hardy des séries L et sommes de Mertens explicites

Thèse dirigée par Gautami BHOWMIK

Composition du jury

Rapporteurs	Jörn STEUDING	professeur à l'Universität Würzburg
	Jie WU	chargé de recherche HDR au CNRS, Institut Élie Cartan, Nancy
Examinateurs	Emmanuel FRICAIN	professeur à l'Université Lille 1
	Olivier ROBERT	MCF à l'Université Jean Monnet, Saint-Étienne
	Vincent THILLIEZ	professeur à l'Université Lille 1
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Hardy's functions of L-series and explicit Mertens sums

Thesis supervised by Gautami BHOWMIK

Committee members

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Mots clefs : série de Dirichlet, fonction L de Dirichlet, fonction de Hardy, formule d'Atkinson, somme de Gauss, somme d'exponentielles, fonction sommatoire des nombres premiers, estimation explicite.

Keywords: Dirichlet series, Dirichlet L-function, Hardy's function, Atkinson's formula, Gauss sum, exponential sum, summatory function of primes, explicit estimate.

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À mes parents, au Prof. *MB Rege*

Sapience n'entre poinct en ame maliuole, & ſçience ſans conſcience n'eſt que ruyne de l'ame.

François RABELAIS

Alors, comme je n'étudiais rien, j'apprenais beaucoup.

Anatole FRANCE

FONCTIONS DE HARDY DES SÉRIES L ET SOMMES DE MERTENS EXPLICITES

Résumé

Cette thèse comporte deux parties principales.

Dans la première partie, nous étudions les fonctions de Hardy des fonctions L de Dirichlet. La fonction de Hardy $Z(t,\chi)$ liée à la fonction $L(s,\chi)$ est une fonction à valeurs réelles de la variable réelle t dont les zéros correspondent exactement aux zéros de $L(s,\chi)$ sur la droite critique $\Re s = \frac{1}{2}$; en effet, on a $|Z(t,\chi)| = |L(\frac{1}{2}+it,\chi)|$ pour tout $t \in \mathbb{R}$. Nous étudions sa primitive $F(T,\chi) \stackrel{\text{def}}{=} \int_0^T Z(t,\chi)\,\mathrm{d}t$. L'étude asymptotique de la fonction de Hardy liée à la fonction zêta de Riemann a connu récemment un regain d'activité grâce à Ivić [34] qui a montré entre autres la majoration

$$F(T) = O(T^{\frac{1}{4}+\epsilon})$$

et a conjecturé le comportement $F(T) = \Omega_\pm(T^{\frac{1}{4}})$. Cette dernière conjecture a été démontrée par Korolëv [48], qui a également exhibé un comportement surprenant de $F(T)$; en effet, il montre que $F(T)$ peut être approchée asymptotiquement par une fonction étagée périodique qui prend des valeurs positives et négatives. Ensuite, Jutila [39, 38] a donné une démonstration indépendante de ces résultats, avec un traitement plus uniforme de l'approximation et des termes d'erreur. Il montre d'abord une formule de type Atkinson pour $F(T)$ par le biais de la transformée de Laplace et en déduit les résultats de Korolëv. En suivant Jutila, nous étendons ces résultats aux fonctions L de Dirichlet, et nous montrons également que le comportement de $F(T,\chi)$ dépend notamment de la parité de χ et celle du conducteur.

Sauf en un point majeur où Jutila utilise très spécifiquement les coefficients de la série de Dirichlet (ici 1 !), ce travail suit assez fidèlement celui de Jutila, bien que de nombreuses modifications soient nécessaires. Au niveau de ce point majeur, notre travail s'écarte notoirement de celui de Jutila, et nous ne savons conclure que dans le cas des caractères pairs et pour les deux caractères impairs modulo 4 et 8. Ensuite, l'approximation par une fonction simple pose elle aussi plusieurs problèmes que nous traitons de façon satisfaisante aux points génériques, mais le comportement en certains points spéciaux pose encore de nombreuses questions.

Dans la seconde partie, nous étudions certaines fonctions sommatoires des nombres premiers en vue d'estimations explicites dans la lignée de Rosser et Schoenfeld [72]. Nous donnons principalement des estimations explicites pour les termes d'erreur des sommes de Mertens $\sum_{p\le x}\frac{1}{p}, \sum_{p\le x}\frac{\log p}{p}, \sum_{n\le x}\frac{\Lambda(n)}{n}$ et des produits d'Euler $\prod_{p\le x}\left(1+\frac{z}{p}\right)$. La méthode utilisée est celle suggérée par un article récent de Ramaré [65] ; des estimations explicites très précises sont données au moyen d'une région explicite sans zéros de type de la Vallée Poussin pour la fonction zêta de Riemann.

Mots clefs : série de Dirichlet, fonction L de Dirichlet, fonction de Hardy, formule d'Atkinson, somme de Gauss, somme d'exponentielles, fonction sommatoire des nombres premiers, estimation explicite.

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HARDY'S FUNCTIONS OF L-SERIES AND EXPLICIT MERTENS SUMS

Abstract

The thesis consists of two main parts.

In the first part of this thesis, we study Hardy's functions of Dirichlet L-functions. The *Hardy's function* $Z(t,\chi)$ corresponding to a Dirichlet L-function $L(s,\chi)$ is a real-valued function of the real variable t whose zeros coincide exactly with the zeros of $L(s,\chi)$ on the critical line $\Re s = \frac{1}{2}$; in fact, we have $|Z(t,\chi)| = |L(\frac{1}{2}+it,\chi)|$ for all $t \in \mathbb{R}$. We study its primitive $F(T,\chi) \stackrel{\text{def}}{=} \int_0^T Z(t,\chi)\,\mathrm{d}t$. The asymptotic study of Hardy's function corresponding to the Riemann zeta function was recently revived by Ivić [34] who proved among others the asymptotic

$$F(T) = O(T^{\frac{1}{4}+\epsilon})$$

and conjectured the behaviour $F(T) = \Omega_{\pm}(T^{\frac{1}{4}})$. This conjecture was proved by Korolëv [48], who further exhibited a rather surprising property of $F(T)$; in fact, he proved that $F(T)$ can be asymptotically approximated by a periodic step function which takes both positive and negative values. Later, Jutila [39] gave another proof of these results, giving more uniform treatment of the approximation and the error terms. He first proves an Atkinson-like formula for $F(T)$ via the Laplace transform and therefrom derives the results of Korolëv. Following Jutila, we extend these results to Dirichlet L-functions, and show that the behaviour of $F(T,\chi)$ depends on the parity of the character as well as that of the conductor.

Except at one major point where Jutila very specifically uses the coefficients of the Dirichlet series (here equal to 1!), this adaptation quite faithfully follows the work of Jutila, although a number of modifications are necessary. Regarding this major point, our work deviates manifestly from that of Jutila, and we can conclude only for even characters and the two odd characters modulo 4 and 8. The approximation by a function also poses several problems which we treat in a satisfying manner at the generic points, but the behaviour at certain points still poses several questions.

In the second part, we study some summatory functions of primes in view of explicit estimates in the line of Rosser and Schoenfeld [72]. Principally, we give explicit estimates for the error terms in the Mertens sums $\sum_{p\le x}\frac{1}{p}, \sum_{p\le x}\frac{\log p}{p}, \sum_{n\le x}\frac{\Lambda(n)}{n}$ and the Euler products $\prod_{p\le x}\left(1+\frac{z}{p}\right)$. The method used is the one suggested by a recent paper of Ramaré [65]; precise explicit estimates are obtained using explicit de la Vallée Poussin zero-free regions for the Riemann zeta function.

Keywords: Dirichlet series, Dirichlet L-function, Hardy's function, Atkinson's formula, Gauss sum, exponential sum, summatory function of primes, explicit estimate.

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Qu'il me soit permis tout d'abord d'adresser mes plus vifs remerciements à Olivier Ramaré. Tout le travail contenu dans cette thèse a été commencé au début de 2014 sous sa direction ; grâce à sa grande disponibilité et son implication, cette thèse a pu être achevée et ensuite soutenue en tout juste un an. J'ai été marqué par sa vaste culture en Théorie des nombres et ses conseils de rédaction mathématique. Il s'est même chargé de la relecture d'une grande partie de cette thèse, m'accompagnant tout au long de la rédaction, en me suggérant des modifications mathématiques, typographiques et grammaticales. L'année 2014 a été une année intense, sans vacances pour lui, et sans même un weekend de repos pour moi. Je lui serais toujours reconnaissant pour m'avoir tiré d'un profond marasme mathématique et donc, n'hésitons pas à le dire, existentiel.

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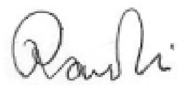

Le 17 août 2015
Villeneuve d'Ascq

acti labores jucundi

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Notations and conventions

> Pour moy i'oublie fouuent l'vn & l'autre de ces vains offices, comme ie retranche en ma maifon toute ceremonie.
>
> MONTAIGNE, *Essais*

Although most of the notation is standard, we recall them here for ease of reference.

Symbol	Name	Definition
$\zeta(s)$	The Riemann zeta function	See § 1.6.
$L(s,\chi)$	Dirichlet L-function	See § 1.6.
$Z(t,\chi)$	Hardy's Z-function	See § 1.9.
$N(T)$	—	Number of nontrivial zeros of $\zeta(s)$ upto height T.
$\Lambda(n)$	von Mangoldt function	$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^k \text{ is a prime power,} \\ 0 & \text{otherwise.} \end{cases}$
$\mu(n)$	Möbius function	$\mu(1) = 1, \mu(n) = (-1)^k$ if n is a product of k distinct primes, $\mu(n) = 0$ if n is not squarefree.
$\pi(x)$	Prime counting function	Number of primes $\leqslant x$.
$\vartheta(x)$	Chebyshef ϑ-function	$\vartheta(x) = \sum_{p\leqslant x} \log p$.
$\psi(x)$	Chebyshef ψ-function	$\psi(x) = \sum_{n\leqslant x} \Lambda(n)$.
$\varphi(n)$	Euler totient	Number of positive integres $\leq n$ relatively prime to n.
$\Gamma(s)$	The Gamma function	
$\lfloor x \rfloor$	Greatest integer function	$\lfloor x \rfloor = \max\{n \in \mathbb{Z} : n \leqslant x\}$.
$\{x\}$	Fractional part	$\{x\} = x - \lfloor x \rfloor$.

Symbol	Name	Definition
$\|x\|$	—	Distance from x to $\mathbb{Z}$; $\|x\| = \min\{\|n-x\| : n \in \mathbb{Z}\}$.
$\mathfrak{a}$	Parity	See § 1.2.
$\mathfrak{w}$	Root number	See § 1.3.
$B_n(x)$	n^{th} Bernoulli function	See § 1.4.
$\Re z$	Real part	The real part of a complex number z.
$\Im z$	Imaginary part	The imaginary part of a complex number z.
$\arg z$	Argument	The argument of a complex number z.
$e(z)$	—	$e^{2\pi iz}$
$\hat{f}(y)$	Fourier transform	$\hat{f}(y) = \int_{\mathbb{R}} f(x)e(-xy)\,\mathrm{d}x$.
$\check{f}(p)$	Laplace transform	$\check{f}(p) = \int_0^\infty f(t)e^{-pt}\,\mathrm{d}t$.
$f = O_{\mathbf{p}}(g)$	Landau big-oh notation	There is a constant c, eventually depending on the parameter list **p**, such that $\|f\| \leqslant c\|g\|$ in a certain domain, often clear from context.
$f \ll_{\mathbf{p}} g$	Vinogradov notation	Same as $f = O_{\mathbf{p}}(g)$.
$f = O^*(g)$	—	$\|f\| \leqslant \|g\|$ in the domain implied by context.
$f(x) = o(g(x))$ as $x \to x_0$	Landau little-oh notation	$\frac{f(x)}{g(x)} \to 0$ as $x \to x_0$. Of course, $g(x)$ is assumed to be nonzero in a neighbourhood of x_0.
$f(x) = \Omega(g(x))$ as $x \to x_0$	—	$\limsup_{x\to x_0} \frac{\|f(x)\|}{g(x)} > 0, g(x)$ being a positive function.
$f(x) = \Omega_+(g(x))$ as $x \to x_0$	—	$\limsup_{x\to x_0} \frac{f(x)}{g(x)} > 0, g(x)$ being a positive function.
$f(x) = \Omega_-(g(x))$ as $x \to x_0$	—	$\liminf_{x\to x_0} \frac{f(x)}{g(x)} < 0, g(x)$ being a positive function.
$f(x) = \Omega_\pm(g(x))$ as $x \to x_0$	—	Both $f(x) = \Omega_+(g(x))$ and $f(x) = \Omega_-(g(x))$ are true as $x \to x_0$.
$f \asymp_{\mathbf{p}} g$ for $x \in E$	Asymptotic equivalence	Both $f = O_{\mathbf{p}}(g)$ and $g = O_{\mathbf{p}}(f)$ are true for $x \in E$.
$f(x) \sim g(x)$ as $x \to x_0$	Equivalence	$\lim_{x\to x_0} \frac{f(x)}{g(x)} = 1$.
Symbol	Name	Definition

We remark that $f(x) = \Omega(g(x))$ is the negation of $f(x) = o(g(x))$ and that in the definition of $f(x) = O_{\mathbf{p}}(g(x))$, the constant c is called an *implied constant*. We sometimes omit the parameter list **p** in case it is clear from context. It should also be observed that we allow x_0 to be infinite in the above definitions.

Unless otherwise specified, the dash ′ in the summation sign $\sum'_{A \le m \le B}$ means that in case any of the endpoints A, B is an integer, the corresponding term is to be halved. Similarly, $\sum'_{n \leqslant x}$ means that in case x is an integer, the term corresponding to it is to be halved. If the starting point of a sum is not indicated, it is supposed to start from 1, or from the first term of the relevant sequence over which the sum runs. When the running index is p, the sum bears on *primes* in the indicated domain. Thus, $\sum_{p \leqslant x} f(p)$ sums $f(p)$ for primes $p \le x$. Idem, mutatis mutandis, for products.

Intervals are denoted as follows: $[A, B]$, $]A, B]$, $[A, B[$, $]A, B[$ with obvious meanings. We allow equalities such as $[A, B] = \mathbb{R}$ when this is convenient, although this is strictly not possible.

Next we use standard notations such as $\mathcal{C}^k(I), L^p(\mathbb{R})$ for the space of k-times continuously differentiable functions on I, the space of Lebesgue p-integrable functions, and so forth.

We say that a function is *of bounded variation* on a not necessarily bounded interval if it is so in any compact subinterval thereof.

Other notations will be introduced as necessary; more terminology may be found from the index at the end of this memoir.

Introduction générale

> Mais afin de vous la faire mieux conceuoir, i'introduiray encore icy vn iardinier qui ſ'en ſert a compaſſer la broderie de quelque parterre.
>
> DESCARTES, *La Dioptrique*

Présentation

Rappelons que la fonction zêta de Riemann $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ (série qui converge absolument pour $\sigma > 1$) admet un prolongement méromorphe sur $\mathbb{C}$ avec comme seul pôle $s = 1$, et vérifie l'équation fonctionnelle $\zeta(s) = \Psi(s)\zeta(1-s)$ où $\Psi(s) = \frac{\Gamma(\frac{1}{2}(1-s))}{\Gamma(\frac{1}{2}s)}\pi^{s-\frac{1}{2}}$. Donc, en posant $Z(t) = \Psi(\frac{1}{2}+it)^{-\frac{1}{2}}\zeta(\frac{1}{2}+it)$, nous trouvons facilement que

1. $Z(t)$ est à valeurs réelles,
2. $Z(t)$ est une fonction paire : $Z(-t) = Z(t)$,
3. $|Z(t)| = |\zeta(\frac{1}{2}+it)|$.

Cette fonction $Z(t)$ que l'on connaît aujourd'hui sous le nom de *fonction de Hardy* a été introduite par Hardy (voir [42, ch. 2, § 1.3]). Bien que la fonction de Hardy ait été définie assez tôt dans l'histoire de la fonction zêta de Riemann, à ce jour, la seule monographie dédiée à cette fonction est celle de Ivić [36] qui date de 2013. Hardy et Littlewood emploient cette fonction principalement pour étudier la distribution des zéros de $\zeta(s)$ sur la droite critique $\Re s = \frac{1}{2}$. Une idée de Hardy pour montrer l'existence d'une infinité de zéros de $\zeta(s)$ sur la droite critique consiste en ceci : si l'on peut montrer que $|\int_T^{2T} Z(t)\,\mathrm{d}t| < \int_T^{2T} |Z(t)|\,\mathrm{d}t$, pour T assez grand, alors il s'ensuit que $Z(t)$ a un zéro entre T et $2T$, et *a fortiori* que $\zeta(s)$ s'annule une infinité de fois sur la droite critique. Cette ligne de raisonnement a été mise en œuvre pour la première fois par Landau [51], qui montre que $\int_T^{2T} Z(t)\,\mathrm{d}t \ll T^{\frac{7}{8}}$ alors que $\int_T^{2T} |Z(t)|\,\mathrm{d}t > 0.5T$.

La majoration $F(T) = \int_0^T Z(t)\,\mathrm{d}t = O(T^{7/8})$ de Landau a été laissée à l'abandon jusqu'à ce que Ivić [34] l'améliore récemment en montrant que

$$F(T) = O_\epsilon(T^{\frac{1}{4}+\epsilon}). \tag{0.1}$$

En basant ses intuitions sur le fait que le terme principal d'une intégrale exponentielle à poids représente souvent l'ordre de grandeur du poids, il a également conjecturé le comportement $F(T) = \Omega_\pm(T^{1/4})$. Ceci a été confirmé par Korolëv [48]. Korolëv a non seulement montré la conjecture de Ivić, mais il a aussi exhibé un comportement très inattendu de $F(T)$, à savoir que la fonction $F(T)/T^{1/4}$ est une fonction presque-périodique étagée de la partie fractionnaire de $\sqrt{T/(2\pi)}$ qui prend des valeurs positives et négatives.

Ce résultat a été redémontré par Jutila [39] via la transformée de Laplace, qu'il a déjà employée à plusieurs reprises, notamment dans [58], pour étudier les moments supérieurs de $\zeta(\frac{1}{2}+it)$, et des fonctions L attachées aux formes modulaires cuspidales.

Or, les fonctions L de Dirichlet elles aussi vérifient des équations fonctionnelles d'un type similaire ; il est donc tout à fait naturel de s'intéresser à leurs fonctions de Hardy respectives, et d'essayer de voir ce qui se passe pour ces fonctions. C'est ce que nous nous proposons de faire dans la première partie de cette thèse.

Dans la seconde partie, nous traitons certaines fonctions sommatoires des nombres premiers, à savoir les trois sommes

$$\sum_{p\le x} \frac{1}{p},\ \sum_{p\le x} \frac{\log p}{p} \ \text{ et } \ \sum_{n\le x} \frac{\Lambda(n)}{n}$$

et le produit eulérien

$$\prod_{p\leqslant x}\left(1+\frac{z}{p}\right)\quad (0<|z|<2)$$

en vue d'estimations explicites dans la lignée de Rosser-Schoenfeld [72]. Ces sommes ont fait l'objet d'études qualitatives de Landau [49, 51] et les premières estimations explicites ont été données par Rosser et Schoenfeld [71, 72, 73]. Plus récemment, Ramaré [65] a étudié la troisième somme et a obtenu des estimations très précises dont nous reprenons les idées en ajoutant nos propres idées pour étudier ces fonctions.

Présentation des fonctions L de Dirichlet

Étant donné un caractère primitif χ mod q ($q>1$; voir § 1.2), la série de Dirichlet $L(s,\chi)=\sum_{n=1}^{\infty}\frac{\chi(n)}{n^s}$ converge pour $\Re s>0$ et définit une fonction holomorphe dans cette région qui se prolonge analytiquement sur $\mathbb{C}$ où elle vérifie l'équation fonctionnelle

$$\xi(s,\chi)=\mathfrak{w}(\chi)\xi(1-s,\overline{\chi})\tag{0.2}$$

où $\xi(s,\chi)$ est la *fonction L complétée* définie par $\xi(s,\chi)=(\frac{\pi}{q})^{-\frac{1}{2}(s+\mathfrak{a})}\Gamma(\frac{1}{2}(s+\mathfrak{a}))L(s,\chi)$, la quantité $\mathfrak{a}$ étant le porteur de signe de χ : $\mathfrak{a}=\frac{1-\chi(-1)}{2}$, et $\mathfrak{w}=\frac{\tau(\chi)}{i^{\mathfrak{a}}\sqrt{q}}$. Ici, $\tau(\chi)$ est une somme de Gauss dont la détermination explicite reste inconnue sauf pour certains cas, notamment pour q impair et sans facteurs carrés. Néanmoins, on sait que $\tau(\chi)$ est toujours de valeur absolue $\sqrt{q}$. Voir § 1.3 pour des remarques plus complètes.

Ces fonctions $L(s,\chi)$, ainsi que la notation L, ont été introduites par Dirichlet [13], qui les emploie pour démontrer son théorème célèbre sur les nombres premiers en progressions arithmétiques (voir également [14], [15], [16], [17]). Il les a même inventées dans ce but précis. Les fonctions $L(s,\chi)$ sont une généralisation naturelle de la fonction zêta de Riemann. Leur étude a été systématiquement entreprise par Landau [49], [50], [51]. Ce sont des objets centraux de la théorie analytique des nombres qui ont donné lieu à de nombreuses généralisations dans plusieurs directions, notamment les fonctions L de Hecke et de Artin.

Voir par exemple [18, 24].

Le comportement de ces fonctions est plutôt mystérieux dans la *bande critique* (c'est ainsi que l'on appelle la bande $0 \leq \Re s \leq 1$). Les questions les plus intéressantes s'y rattachant étant : où se situent les zéros de $L(s, \chi)$? Quel est l'ordre de grandeur de $L(s, \chi)$ dans cette bande ? Avant de détailler nos résultats, décrivons plus avant le monde des fonctions L.

[A]. La fonction $\xi(s, \chi)$ est une fonction entière d'ordre 1, donc admet un développement en *produit de Hadamard*

$$\xi(s, \chi) = e^{A_\chi + B_\chi s} \prod_{\rho_\chi} \left(1 - \frac{s}{\rho_\chi}\right) e^{\frac{s}{\rho_\chi}}$$

où A_χ, B_χ sont des constantes et ρ_χ parcourt les zéros de $\xi(s, \chi)$, qui sont, grâce à l'équation fonctionnelle (0.2), les zéros de $L(s, \chi)$ dans la bande critique. De plus, la somme $\sum_{\rho_\chi} 1/\rho_\chi$ diverge alors que la somme $\sum_{\rho_\chi} 1/\rho_\chi^{1+\epsilon}$ converge pour tout $\epsilon > 0$ (voir [53] par exemple) ; donc la fonction $L(s, \chi)$ a une infinité de zéros dans la bande critique. En plus, Hardy [29] a montré qu'une infinité de zéros de $\zeta(s)$ sont sur la *droite critique* $\Re s = \frac{1}{2}$. L'hypothèse que *tous* les zéros de $\zeta(s)$ sont sur cette droite est connue sous le nom d'*Hypothèse de Riemann* et l'hypothèse correspondante pour les $L(s, \chi)$ l'*Hypothèse de Riemann généralisée*. Il est à noter que van de Lune, te Riele et Winter [81] et Platt [62] ont calculé les premiers milliards ($1\,500\,000\,001$ pour [81]) de zéros de $\zeta(s)$ et ils ont montré que ces zéros sont tous simples et sont sur la droite critique. On sait désormais, grâce à Selberg [74], qu'une proportion positive des zéros de $\zeta(s)$ est en effet sur la droite critique. Cette proportion a été améliorée, notamment par Levinson [54] (« plus d'un tiers » des zéros sont sur la droite critique) et Conrey [7] (« plus de deux cinquièmes »). C'est un théorème classique de Bohr et Landau [3] que tous sauf une proportion infiniment petite sont dans une distance δ de la droite critique, pour tout $\delta > 0$.

Plus précisément, notons $N(T, \chi)$ le nombre de zéros de $L(s, \chi)$ dans le rectangle $0 < \sigma < 1, -T \leq t \leq T$ et $N_\delta(T, \chi)$ (resp. $N_0(T, \chi)$) le nombre de zéros de $L(s, \chi)$ avec $\frac{1}{2} - \delta < \sigma < \frac{1}{2} + \delta, -T \leq t \leq T$ (resp. $\sigma = \frac{1}{2}$), les valeurs correspondantes pour $\zeta(s)$ étant notées sans le χ, et en limitant le rectangle au demi-plan supérieur, puisque $\zeta(\overline{s}) = \overline{\zeta(s)}$. Alors le résultat classique de Selberg énonce que

$$\liminf_{T\to\infty} \frac{N_0(T+H) - N_0(T)}{N(T+H) - N(T)} > 0 \tag{0.3}$$

pour $H \geq T^{\frac{1}{2}+\epsilon}$, et le résultat de Bohr et Landau énonce que

$$\lim_{T\to\infty} \frac{N_\delta(T,\chi)}{N(T,\chi)} = 1$$

pour tout $\delta > 0$. Ici, l'on permet $q = 1$, auquel cas $L(s,\chi) = \zeta(s)$. Karatsuba [43] a amélioré le résultat (0.3) de Selberg en montrant que

$$N_0^o(T+H) - N_0^o(T) \gg H \log T$$

pour $H \geq T^{\frac{27}{82}+\epsilon}$, où $N_0^o(T)$ compte le nombre de zéros d'ordres impairs de $\zeta(\frac{1}{2} + it)$ dans $0 < t \leq T$. On dispose maintenant de résultats plus raffinés, notamment celui de Steuding [76] qui a montré que pour $H \geq T^{0,552}$, on a

$$\liminf_{T\to\infty} \frac{N_0^{(1)}(T+H) - N_0^{(1)}(T)}{N(T+H) - N(T)} > 0$$

où $N_0^{(1)}(T)$ compte le nombre des zéros *simples* de $\zeta(\frac{1}{2} + it)$ dans $0 < t \leq T$.

[B]. Ensuite, la fonction $L(s,\chi)$ admet une représentation en *produit d'Euler*

$$L(s,\chi) = \prod_p \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}$$

qui converge absolument pour $\Re s > 1$; on en déduit facilement que $L(s,\chi) \neq 0$ dans cette région, faisant de $\Re s > 1$ une *région sans zéros* pour $L(s,\chi)$. Dans cette direction, Hadamard [28] et de la Vallée Poussin [8] ont montré que $\zeta(s)$ ne s'annule pas sur la droite $\Re s = 1$, un fait que l'on connaît désormais équivalent à leur *Théorème des nombres premiers*, à savoir $\lim_{x\to\infty} \frac{\pi(x)\log(x)}{x} = 1$; en effet, de la Vallée Poussin [8, 1ère partie, chap. II,(38)] montre que $\zeta(s)$ n'a pas de zéro

$\rho = \beta + i\gamma$ vérifiant

$$\beta \geq 1 - \frac{1}{R \log |\gamma|} \tag{0.4}$$

où $R = 34{,}82$. Ainsi, la région $\sigma \geq 1 - {}^{1}/_{R \log |t|}$ est une région sans zéros pour $\zeta(s)$. La constante R a été améliorée plusieurs fois, dont l'une des valeurs les plus petites et l'une des plus récentes est celle de Kadiri [40] : $R = 5{,}696\,93$ (voir aussi [57], $R = 5{,}573\,412$). Dans cette lignée de pensée, on a également les résultats de Vinogradov [82] et Korobov [46], qui donnent une région sans zéros plus étendue, ayant la forme

$$\sigma \geq 1 - \frac{c}{\left(\log^{\frac{2}{3}} |t|\right) \left(\log \log^{\frac{1}{3}} |t|\right)},$$

également avec c explicite, cf. [23].

[C]. Quant à l'ordre de grandeur de $L(s, \chi)$, la convergence absolue de la série montre que $L(s, \chi) \ll 1$ dans $\Re s \geq 1 + \epsilon$, et donc, grâce à l'équation fonctionnelle et la formule de Stirling, $L(s, \chi) \ll t^{\frac{1}{2} - \sigma + \varepsilon}$ pour $\sigma < 0$, pour tout $\varepsilon > 0$. Ainsi, si l'on note

$$\mu(\sigma, \chi) = \limsup_{t \to \infty} \frac{\log |L(\sigma + it, \chi)|}{\log t}, \ \mu(\sigma) = \limsup_{t \to \infty} \frac{\log |\zeta(\sigma + it)|}{\log t}$$

alors, $\mu(\sigma, \chi) = 0$ pour $\sigma \geq 1$ et $\mu(\sigma, \chi) = \frac{1}{2} - \sigma$ pour $\sigma \leq 0$. En effet, $\mu(\sigma, \chi)$ est une fonction convexe donc continue de σ (voir [78, § 9.41]) d'où la validité pour $\sigma = 1$ et $\sigma = 0$. Le problème le plus difficile est de trouver l'ordre de grandeur de $L(s, \chi)$ dans la bande critique, c'est-à-dire la valeur de $\mu(\sigma, \chi)$ pour $0 < \sigma < 1$. L'*Hypothèse de Lindelöf* énonce que

$$\mu(\sigma, \chi) = \begin{cases} \frac{1}{2} - \sigma, & \sigma \leq \frac{1}{2}, \\ 0, & \sigma \geq \frac{1}{2}. \end{cases}$$

Grâce à la convexité de μ, on voit facilement que $\mu(\sigma, \chi) = \mu\left((1 - \sigma) \times 0 + \sigma \times 1, \chi\right) \leq (1 - \sigma)\mu(0, \chi) + \sigma\mu(1, \chi) = \frac{1}{2} - \frac{1}{2}\sigma$ pour $0 < \sigma < 1$; en particulier, $\mu(\frac{1}{2}, \chi) \leq \frac{1}{4}$, c'est-à-dire

$$L(\tfrac{1}{2}+it,\chi) \ll |t|^{\frac{1}{4}+\varepsilon} \quad (|t| \geq 2)$$

alors que l'Hypothèse de Lindelöf prédit

$$L(\tfrac{1}{2}+it,\chi) \ll |t|^{\varepsilon} \tag{0.5}$$

pour tout $\varepsilon > 0$. La majoration (0.5) dans le cas $q = 1$ (c-à-d, $\zeta(s)$) est équivalente à la majoration de la somme zêta

$$\sum_{n \leq x} n^{it} \ll \sqrt{x} t^{\varepsilon} \quad (1 \leq x \leq \sqrt{|t|}),$$

voir [44, V § 6] par exemple. Les meilleures majorations connues aujourd'hui sont celles de Kolesnik ($\zeta(\frac{1}{2}+it) \ll t^{\frac{35}{216}+\varepsilon}$, [45]), Bombieri-Iwaniec ($\zeta(\frac{1}{2}+it) \ll t^{\frac{9}{56}+\varepsilon}$, [4]), Huxley ($\zeta(\frac{1}{2}+it) \ll t^{\frac{32}{205}+\varepsilon}$, [31]) etc.

On remarque que l'Hypothèse de Lindelöf est une conséquence directe de l'Hypothèse de Riemann, celle-ci impliquant les deux majorations

$$\zeta(s) \ll t^{\varepsilon},$$
$$1/\zeta(s) \ll t^{\varepsilon}$$

pour tout $\sigma > \frac{1}{2}$, voir [35, § 1.9].

Fonction de Hardy de $L(s,\chi)$: Exposé de la méthode

L'équation fonctionnelle (0.2) s'écrit également sous la forme $L(s,\chi) = \Psi(s,\chi)L(1-s,\chi)$ avec $\Psi(s,\chi) = \mathfrak{w}(\chi)\left(\frac{\pi}{q}\right)^{s-\frac{1}{2}} \frac{\Gamma(\frac{1}{2}(1-s+\mathfrak{a}))}{\Gamma(\frac{1}{2}(s+\mathfrak{a}))}$, et l'on voit facilement que la *fonction de Hardy* définie par

$$Z(t,\chi) = \Psi(\tfrac{1}{2}+it,\chi)^{-\frac{1}{2}} L(\tfrac{1}{2}+it,\chi) \quad (t \in \mathbb{R})$$

est une fonction paire à valeurs réelles, et que $|Z(t,\chi)| = \left|L(\frac{1}{2}+it,\chi)\right|$. Donc, les zéros de $Z(t,\chi)$ correspondent exactement à ceux de $L(s,\chi)$ sur la droite critique. Il est donc de grand intérêt d'étudier le comportement de cette fonction.

Par exemple, l'Hypothèse de Lindelöf est équivalente à la majoration

$$\frac{1}{T}\int_0^T Z^{2k}(t)\,\mathrm{d}t = O(T^{\epsilon}), \quad (k = 1, 2, ...)$$

où $Z(t)$ est la fonction de Hardy pour $\zeta(s)$.

Dans la première partie de cette thèse, on va s'intéresser principalement à la primitive

$$F(T,\chi) = \int_0^T Z(t,\chi)\,\mathrm{d}t$$

qui suit les variations de $Z(t,\chi)$. Voici quelques questions d'intérêt par rapport à cette fonction $F(T,\chi)$:

1. Quel est l'ordre de grandeur de $F(T,\chi)$? Par exemple, a-t-on $F(T,\chi) \ll T^{\frac{1}{4}}$? Ou $F(T,\chi) \asymp T^{\frac{1}{4}}$?

2. Que peut-on dire sur le graphe de cette fonction ?

3. Pour quelles valeurs de $H > 0$ est-il vrai que

$$\liminf_{T\to\infty} \left(F(T+H,\chi) - F(T,\chi)\right) > 0\,?$$

On va s'intéresser principalement aux deux premières questions. Pour répondre à la première question, on va d'abord montrer une formule de type Atkinson, ayant l'allure suivante :

$$F(T,\chi) = S_1(T,\chi) + S_2(T,\chi) + O\left(\log^{5/4} T\right) \tag{0.6}$$

où S_1 et S_2 sont deux sommes finies, de longueurs à peu près $\sqrt{T}$, de termes généraux ressemblant respectivement à n^{-1} et $n^{-\frac{1}{2}+iT}$, voir Théorème 21. Ensuite, en utilisant la majoration de van der Corput $S_2(T,\chi) \ll T^{\frac{1}{6}} \log T$, et en exploitant les propriétés de $S_1(T,\chi)$, on va montrer une approximation de $F(T,\chi)$ par une fonction très simple, de la forme suivante :

$$F(T,\chi) = T^{\frac{1}{4}}\Big(\text{fonction périodique de } \vartheta\Big) + \text{termes d'erreur}$$

$$\text{avec } 0 \le \vartheta < q \text{ et } \sqrt{\frac{qT}{2\pi}} = qL + \vartheta, L \in \mathbb{N}.$$

Dans les pages qui suivent, nous allons discuter de la méthode utilisée pour démontrer (0.6), c'est-à-dire le Théorème 21.

Esquisse de l'idée

Nous allons utiliser la transformée de Laplace. Le but est le suivant : calculer la transformée de Laplace $\check{Z}(p,\chi)$ de $Z(t,\chi)$, et ensuite essayer de récupérer, grâce à la formule d'inversion, la fonction $Z(t,\chi)$ en forme d'une série ou d'une somme plus ou moins simple qui s'adonne à des traitements habituels, telle la méthode du point col, de van der Corput etc. (C'est à noter que $\check{F}(p,\chi) = \check{Z}(p,\chi)/p$ donc trouver $\check{Z}(p,\chi)$ est équivalent à trouver $\check{F}(p,\chi)$).

Or, calculer la transformée de Laplace de $Z(t,\chi)$ est difficile. L'idée de Jutila est la suivante : trouver une fonction proche de $Z(t,\chi)$ dont la trasformée de Laplace se calcule plus facilement. À cette fin, on construit une fonction holomorphe $H(s,\chi)$ telle que $H(s,\chi) = 1 + O\left(1/s\right)$ et $H'(s,\chi) = O\left(1/s^2\right)$ lorsque $|s| \to \infty$ dans une bande verticale fixe, de telle sorte que la transformée de Laplace de $Q(t,\chi) = H(\frac{1}{2}+it,\chi)Z(t,\chi)$ soit facile à calculer. En effet, on a

$$\check{Q}(p,\chi) = \kappa_1 e^{-ip}\sum_{n=1}^{\infty} n^{\frac{1}{2}}\chi(n)\exp(-\frac{\pi}{q}n^2 ie^{-2ip}) + \lambda(p,\chi),$$

où κ_1 est une constante et $\lambda(p,\chi) \ll 1/\left(|p|+1\right)$ est analytique dans $|\Re p| < \frac{\pi}{2}$, voir la Proposition 29. En plus, la transformée de Laplace de $Z(t,\chi)$ et celle de $Q(t,\chi)$ sont proches l'une de l'autre au sens de Théorème 30. Par la formule d'inversion de Laplace on a donc

$$F(T,\chi) = 2\Re\left(\frac{1}{2\pi i}\int_a^{a+i\infty}\frac{\check{Q}(p,\chi)}{p}e^{pT}\,\mathrm{d}p\right) + O\big((\log T)^{\frac{5}{4}}\big), \quad (a>0).$$

On choisit $a = 1/T$. Quand on injecte le développement en série de $\check{Q}(p,\chi)$ dans cette équation, un problème se pose : on a, pour $p = a + it$, $\left|\exp(-\frac{\pi}{q}n^2ie^{-2ip})\right| = \exp\left(-\frac{\pi}{q}n^2e^{2t}\sin 2a\right)$ ce qui est grand pour t très petit. Il faut donc traiter cette situation ; l'un des points clefs de l'approche de Jutila est de montrer que $\check{Q}(p,\chi) \ll 1$ pour $\Im p$ petit, disons $0 \leq t \ll 1/\left(\sqrt{T}\log T\right)$. C'est une étape très délicate de cette approche. En effet, pour montrer que $\check{Q}(p,\chi) \ll 1$ pour $0 \leq t \ll 1/\left(\sqrt{T}\log T\right)$, on montre que $\sum_{n=1}^{\infty}\chi(n)\psi(n)w(n)\exp\left(-\frac{\pi n^2 i}{q}\right) \ll N^{-\frac{1}{10}}$ où $w(t)$ est une fonction lisse à support dans $[N, 2N]$ avec $1 \ll N \ll \sqrt{T\log T}$ telle que $w^{(j)} \ll N^{-j}$ pour $j = 0, 1, ..., k$ (où k est assez grand) et $\psi(n) = \sqrt{n}\exp\left((1 - e^{-2ip})\frac{\pi}{q}n^2 i\right)$. Pour le cas de $\zeta(s)$, on peut utiliser le fait que $\sum_{n=m}^{m+1}(-1)^n = 0$ pour montrer, grâce à la formule de Taylor d'ordre 2, que la somme $\sum_{n=1}^{\infty}(-1)^n w(n)\psi(n)$ est de même ordre de grandeur que la somme sur la dérivée deuxième de $w\psi$, qui est petite. Or pour un caractère général χ, il n'est pas toujours vrai que $\sum_{r=-q}^{q}\chi(r)\exp\left(-\frac{\pi r^2 i}{q}\right) = 0$, ce qui est l'analogue du fait que $\sum_{n=m}^{m+1}(-1)^n = 0$ pour le cas de $\zeta(s)$. Nous avons réussi à adapter et modifier cette étape pour les caractères pairs de conducteurs impairs et les deux caractères impairs mod 4 et mod 8.

On peut alors omettre cet intervalle, et montrer que

$$F(T,\chi) = \frac{e}{\pi}\Re\Bigg(\kappa_1 e^{-ia}\sum_{n=1}^{\infty}\sqrt{n}\chi(n)\times \int_0^{\infty}\exp\left(t - \frac{\pi}{q}n^2e^{2t}ie^{-2ia} + itT\right)\frac{w(t)}{a+it}\,\mathrm{d}t\Bigg) + O\left((\log T)^{\frac{5}{4}}\right).$$

où $w(t)$ est une fonction lisse telle que $w(t) = 0$ pour $t \leq \frac{1}{\sqrt{T}\log T}$, et $w(t) = 1$ pour $t \geq \frac{2}{\sqrt{T}\log T}$ et telle que $0 \leq w(t) \leq 1$ sur $\mathbb{R}$. Ensuite, pour simplifier cette expression, on peut ignorer le facteur e^{-ia} (c'est-à-dire le remplacer par 1) et remplacer e^{-2ia} par $1 - 2ia$, avec une erreur $\ll \log T$ (voir § 2.7 sur *p.* 66). Ceci donne

$$F(T,\chi) = \frac{e}{\pi}\Re\Bigg(\kappa_1\sum_{n=1}^{\infty}\sqrt{n}\chi(n)\times \int_0^{\infty}\exp\left(t - 2a\frac{\pi}{q}n^2e^{2t} - \frac{\pi}{q}n^2e^{2t}i + itT\right)\frac{w(t)}{a+it}\,\mathrm{d}t\Bigg) + O\left((\log T)^{\frac{5}{4}}\right).$$

Le changement de variable $u = e^{2t} - 1$ transforme cette expression en :

$$F(T,\chi) = \frac{e}{2\pi}\Re\Bigg(\kappa_1 \sum_{n=1}^{\infty} \sqrt{n}\chi(n)e^{-\frac{\pi}{q}n^2 i}\times \\ \times \int_0^{\infty} \frac{\exp\big(-2a\frac{\pi}{q}n^2(1+u) - \frac{\pi}{q}n^2 iu + \frac{iT}{2}\log(1+u)\big)}{\sqrt{1+u}} \\ \times \frac{w(\frac{1}{2}\log(1+u))}{a+\frac{i}{2}\log(1+u)}\,\mathrm{d}u + \mathcal{O}\left((\log T)^{\frac{5}{4}}\right).$$

Il reste donc à étudier cette expression. Appelons cette somme I et la $n^{\text{ième}}$ intégrale $J_n = \int_0^{\infty} g_n(t)\exp\left(f_n(t)\right)\,\mathrm{d}t$ où g_n et f_n sont réelles. Maintenant, le comportement de chaque terme de cette somme dépend du point col $y_n = \frac{qT}{2\pi n^2} - 1$. Il est donc naturel de diviser la somme en plusieurs parties selon le positionnement de ce point col par rapport à l'intervalle d'intégration. Par exemple, pour $n \geq \sqrt{N''}$ (avec N, N' et N'' comme dans l'énoncé du Théorème 21), l'intégrale J_n n'a pas de point col, etc. La somme I est donc divisée en trois parties I_1, I_2 et I_3 :

$$I = \underbrace{I_1}_{\sqrt{N'}<n<\sqrt{N''}} + \underbrace{I_2}_{1\leq n<\sqrt{N'}} + \underbrace{I_3}_{n\geq\sqrt{N''}}.$$

Pour les intégrales dans I_3, le point col $y_n = \frac{qT}{2\pi n^2} - 1 \leq \frac{qT}{2\pi N''} - 1 < 0$ est hors de l'intervalle d'intégration $\mathbb{R}^+$. Donc, par intégration par parties, on peut voir facilement que

$$I_3 \ll 1.$$

Pour traiter les termes de I_2, on utilise un lemme de Atkinson pour la méthode du point col (voir Théorème 20) ; il vient

$$I_2 = \text{constante} \times S_2(T,\chi) + \text{erreur}.$$

C'est le traitement de I_1 qui est le plus délicat. Une application directe du lemme de Atkinson ne produit pas un bon résultat, c'est-à-dire, des termes faciles à comprendre. Il faut donc transformer d'abord les termes avant de procéder à une application de la méthode du point col (tel le lemme de Atkinson). L'idée majeure est de rentrer la somme dans l'intégrale comme suit :

$$\begin{aligned} I_1 &= \sum_{\sqrt{N'}<n<\sqrt{N''}} \sqrt{n}\chi(n)e^{-\frac{\pi}{q}n^2 i} \int_0^\infty g_n(t)\exp\left(f_n(t)\right)\,\mathrm{d}t \\ &= \int_0^\infty \sum_{\sqrt{N'}<n<\sqrt{N''}} \sqrt{n}\chi(n)e^{-\frac{\pi}{q}n^2 i} g_n(t)\exp\left(f_n(t)\right)\,\mathrm{d}t \end{aligned}$$

et ensuite on applique la formule de sommation de Poisson (voir Lemme 5) à la somme à l'intérieur de l'intégrale. Maintenant, en appliquant le lemme de Atkinson à chaque terme, on obtient une expression simple :

$$I_1 = \text{constante} \times S_1(T, \chi) + \text{erreur}.$$

Ceci donne le Théorème 21. Faire apparaître un comportement asymptotique est facile à partir de cette formule. Notons toutefois que ce dernier traitement ne ressemble à celui de Jutila que dans les très grandes lignes.

Sommes explicites

Nombre de mathématiciens s'intéressent à une détermination d'une somme portant sur les nombres premiers, telles les sommes $\sum_{p\leqslant x} 1, \sum_{p\leqslant x} \frac{1}{p}$ etc. Par exemple, Gauss a conjecturé dans sa jeunesse que la première somme vérifie

$$\pi(x) = \sum_{p\leqslant x} 1 \sim \frac{x}{\log x},$$

ce qui a été confirmé par Hadamard et de la Vallée Poussin, connu aujourd'hui sous le nom du *Théorème des nombres premiers.* On sait désormais que ce résultat est équivalent à la non annulation de $\zeta(s)$ sur la droite $\sigma = 1$. La connexion entre les nombres premiers et la fonction $\zeta(s)$ est clairement illustrée par ce fait. On connaît désormais, grâce à Dusart [19], que

$$\frac{x}{\log x}\left(1 + \frac{1}{\log x} + \frac{1{,}8}{\log^2 x}\right) \leq \pi(x) \leq \frac{x}{\log x}\left(1 + \frac{1}{\log x} + \frac{2{,}51}{\log^2 x}\right)$$

pour $x \geqslant 355\,991$.

Pour la deuxième somme, c'est certainement Legendre qui s'y intéresse le premier parmi les mathématiciens les plus connus. Il a formulé un résultat, sans en fournir une preuve rigoureuse, de la forme

$$\sum_{p \leqslant x} \frac{1}{p} = \log(\log x - 0{,}083\,66) + C \tag{0.7}$$

avec C une constante (voir [52]). Il est connu depuis Euler que la somme $\sum_p \frac{1}{p}$ diverge, mais c'est grâce à Mertens [56] qu'on connaît le bon résultat pour (0.7), à savoir

$$\lim_{x \to \infty} \left(\sum_{p \leqslant x} \frac{1}{p} - \log\log x - B \right) = 0$$

où $B = 0{,}261\,497\,212\,847\,643...$ est la constante de Meissel-Mertens. Pourtant, la détermination explicite de l'erreur dans l'approximation $\sum_{p \leqslant x} \frac{1}{p} \approx \log\log x - B$ reste longtemps inconnue jusqu'à ce que Rosser et Schoenfeld [72], en exploitant les régions explicites sans zéros pour $\zeta(s)$, montrent que

$$\log\log x + B - \frac{1}{2\log^2 x} \leq \sum_{p \leqslant x} \frac{1}{p} \leq \log\log x + B + \frac{1}{\log^2 x}. \tag{0.8}$$

Une fois de plus, on voit une application des résultats connus sur $\zeta(s)$ pour déterminer une somme portant sur les nombres premiers.

Dans la deuxième partie de cette thèse, nous allons nous intéresser à une amélioration de l'inégalité (0.8) et une détermination explicite des sommes $\sum_{p \leqslant x} \frac{\log p}{p}$, $\sum_{n \leqslant x} \frac{\Lambda(n)}{n}$ et des produits eulériens

$$\prod_{p \leqslant x} \left(1 + \frac{z}{p}\right) \quad (0 < |z| < 2).$$

La somme $\sum_{n \leqslant x} \frac{\Lambda(n)}{n}$ a été étudiée récemment par Ramaré [65] ; nous allons reprendre ses idées en ajoutant nos propres idées pour déterminer ces sommes et produits.

Exposé de la méthode

Voici l'idée principale pour déterminer, à titre d'exemple, la somme $\lambda(x) = \sum_{p\leqslant x} 1/p$.

Par la formule d'intégration par parties de l'intégrale de Stieltjes, on a

$$\begin{aligned}\lambda(x) &= \int_{2^-}^{x} \frac{\mathrm{d}\vartheta(t)}{t\log t} = \frac{\vartheta(x)}{x\log x} + \int_{2}^{x} \frac{1+\log t}{t^2\log^2 t}\vartheta(t)\,\mathrm{d}t \\ &= \log\log x + B + \frac{\vartheta(x)-x}{x\log x} - \int_{x}^{\infty} \frac{1+\log t}{t^2\log^2 t}(\vartheta(t)-t)\,\mathrm{d}t \qquad (0.9)\end{aligned}$$

où $\vartheta(x)$ est la fonction ϑ de Tchebychev. Il s'agit donc d'exploiter les encadrements connus pour la différence $\vartheta(t) - t$. En effet, on a le résultat de Rosser-Schoenfeld

$$0 \leq \psi(t) - \vartheta(t) \leq 1{,}0012\sqrt{t} + 3t^{\frac{1}{3}} \quad (t > 0),$$

donc, on peut utiliser le résultat de Ramaré-Saouter (voir Lemme 13) dans l'intégrale (0.9) pour obtenir une expression de la forme

$$\lambda(x) = \log\log x + B + \frac{\vartheta(x)-x}{x\log x} + O^*\left(\text{termes explicites}\right).$$

En réalité, les « termes explicites » contiennent les expressions

$$J(x) = \sum_{\rho} \frac{x^{\rho-1}}{\rho(\rho-1)} \text{ et } S_2(x) = \sum_{\rho} \frac{x^{\beta-1}}{|\gamma|^3}$$

où les sommes portent sur les zéros non triviaux $\rho = \beta + i\gamma$ de la fonction zêta de Riemann $\zeta(s)$. Grâce à Odlyzko [59], on a une liste des premiers zéros de $\zeta(s)$ qu'on peut utiliser pour donner des approximations très précises des sommes $J(x)$ et $S_2(x)$, les termes d'erreur étant majorés à l'aide d'une région explicite sans zéros pour $\zeta(s)$, donnée par Kadiri [40]. Ensuite, on traite la différence $\vartheta(x)-x$ à l'aide des résultats de Dusart [20]. De cette façon, on obtient le résultat,

$$\sum_{p\leqslant x} \frac{1}{p} = \log\log x + B + O^*\left(\frac{1}{\log^3 x}\right)$$

pour $x \geqslant 24\,284$, que nous citons ici à titre d'exemple.

Annexe : Prolongement analytique de $L(s,\chi)$ sur $\mathbb{C}$

Voici une manière simple de voir le prolongement analytique de $L(s,\chi)$ sur le plan complexe. Il est facile de voir que

$$L(s,\chi) = q^{-s} \sum_{r=1}^{q-1} \chi(r)\zeta(s, \frac{r}{q})$$

où la fonction

$$\zeta(s,x) = \sum_{n=0}^{\infty} \frac{1}{(n+x)^s} \quad (\Re s > 1, 0 < x \leq 1)$$

s'appelle la *fonction zêta de Hurwitz*. Soit $m \geq 1$ un entier. Par la formule de sommation d'Euler-Maclaurin (voir Lemme 4) appliquée à la fonction $f(t) = \frac{1}{(t+x)^s}$, on obtient

$$\sum_{n\leqslant y} \frac{1}{(n+x)^s} = \frac{s+x}{s-1} + \frac{(x+y)^{1-s}}{1-s} - \sum_{k=0}^{m-1} \frac{\Gamma_k(s)}{(k+1)!}\left(\frac{B_{k+1}(y)}{(x+y)^{s+k}} - \frac{B_{k+1}}{(1+x)^{s+k}}\right) - \frac{\Gamma_{m+1}(s)}{(m+1)!}\int_1^y \frac{B_{m+1}(t)}{(t+x)^{s+m+1}}\,\mathrm{d}t$$

où les B_n sont les nombres et fonctions de Bernoulli (voir § 1.4) et où

$$\Gamma_0(s) \equiv 1,\ \Gamma_k(s) = s(s+1)\cdots(s+k-1), \quad (k \geq 1).$$

Il s'ensuit que, pour $\Re s > 1$,

$$\zeta(s,x) = x^{-s} + \frac{s+x}{s-1} + \sum_{k=0}^{m-1} \frac{B_{k+1}}{(k+1)!}\frac{\Gamma_k(s)}{(1+x)^{s+k}} - \frac{\Gamma_{m+1}(s)}{(m+1)!}\int_1^{\infty} \frac{B_{m+1}(t)}{(t+x)^{s+m+1}}\,\mathrm{d}t. \qquad (0.10)$$

Le membre de droite étant défini pour tout $s \neq 1$ dans $\Re s > -m$, cette équation donne un prolongement méromorphe de $\zeta(s,x)$ dans cette région, le point $s = 1$

étant un pôle simple, et on obtient ainsi un prolongement méromorphe de $\zeta(s,x)$ sur tout le plan complexe, l'entier m étant arbitraire. Ainsi, la fonction $L(s,\chi)$ se prolonge sur tout le plan complexe.

Grâce à l'équation (0.10), on voit facilement, par exemple, que

$$L(0,\chi) = -\frac{1}{q}\sum_{r=1}^{q} r\chi(r)$$

ce qui vaut 0 lorsque χ est un caractère pair. Donc, $s=0$ est un zéro *trivial* de $L(s,\chi)$ lorsque χ est un caractère pair, contrairement à $\zeta(s)$, où l'on a $\zeta(0) = -\frac{1}{2}$.

Première partie

Matériel préparatoire

Background material

Chapter 1

Basic tools

> Je n'ai fait celle-ci plus longue que parce que je n'ai pas eu le loifir de la faire plus courte.
>
> PASCAL

1.1 Introduction

In this chapter, we give a very short introduction to the theory of Dirichlet L-functions. The basic lemmas that we will need in the subsequent pages are stated. Most of the material presented here is standard and the proofs may be found in the books mentioned in the general references. The narration is coherent; most of the statements we make along the passage are well-known results, so we do not give reference for each statement.

1.2 Dirichlet characters

A *character* of a finite group G is a homomorphism $f : G \to \mathbb{C}^\times$ of G to the multiplicative group $\mathbb{C}^\times$ of nonzero complex numbers. It is obvious that f maps G to the group $\mathbb{U}$ of all roots of unity; in fact, of G is of order n, then the image of G is contained in the group $\mathbb{U}_n$ of n^{th} roots of unity. The characters of

G themselves, under the natural binary operation, form a group $\hat{G}$, called the *character group* of G, and the two groups G and $\hat{G}$ are isomorphic for G abelian.

For a positive integer q, we denote by $\mathbb{G}_q$ the finite group of integers with respect to multiplication modulo q; it is of order $\varphi(q)$, where φ denotes the Euler totient. To each element $f \in \hat{\mathbb{G}}_q$ corresponds in a natural manner (by defining it to be zero at integers not relatively prime to q) a unique completely multiplicative function $\chi_f : \mathbb{Z} \to \mathbb{C}$, called a *Dirichlet character* to the modulus q, often abbreviated $\bmod q$. Thus, a Dirichlet character $\chi \bmod q$ is a completely multiplicative periodic function of period dividing q, vanishing at integers not relatively prime to q. The complex conjugate $\overline{\chi}$ of χ is also a Dirichlet character to the same modulus. We sometimes simply call them *characters*, dropping the qualifier *Dirichlet*. The character χ_e corresponding to the unit element $e \in \hat{\mathbb{G}}_q$ is called the *principal* character. Other characters are *nonprincipal*.

If d is a divisor of q, a character $\chi_1 \bmod d$ *induces* a character $\chi \bmod q$ by $\chi = \chi_1\chi_e$, where χ_e is the principal character $\bmod q$; d is then called an *induced* modulus for χ. Characters $\bmod q$ which arise in this way are called *imprimitive*, all other characters are *primitive*. There is a smallest induced modulus, see [75]; such a smallest induced modulus is called the *conductor* of the character. Thus, a primitive character cannot be factorised as a product of the principal character and a character of lower modulus. It is obvious that 1 is always an induced modulus for the principal character, and that every nonprincipal character modulo a prime is primitive.

For every integer $q \geq 3$, there is at least one primitive character $\bmod q$ if and only if $q \not\equiv 2 \pmod 4$.

A character χ is *even* if $\chi(-1) = 1$, otherwise it is *odd*. We have a special symbol to serve as a bookkeeper for the parity of χ, namely

$$\mathfrak{a} \equiv \mathfrak{a}(\chi) = \frac{1 - \chi(-1)}{2} \tag{1.1}$$

which equals 0 if χ is even, and equals 1 if χ is odd. We trivially have $\mathfrak{a}(\chi) = \mathfrak{a}(\overline{\chi})$.

Here are some examples of characters:

1. The character χ_q to the modulus q given by the Jacobi symbol $\chi_q = \left(\frac{\cdot}{q}\right)$ where $q \equiv 1 \pmod 4$ and q is squarefree. All these characters are even and

primitive.

2. When $q \equiv -1 \pmod 4$ and q is squarefree, the character $\left(\frac{\cdot}{q}\right)$ is odd and primitive.
3. The character χ_4 to the modulus 4, given by $\chi_4(-1) = -1$ is odd and primitive.
4. The character χ_8 to the modulus 8 given by $\chi_8(-1) = -1 = \chi(-3)$ is odd and primitive; the character χ_8' $\bmod 8$ given by $\chi_8'(-1) = 1$, $\chi_8'(\pm 3) = -1$ is even and primitive.

1.3 Character sums

Given a character $\chi \bmod q$ and an integer m, the sum

$$\tau(m,\chi) \stackrel{\text{def}}{=} \sum_{a \bmod q} \chi(a) e\left(\frac{am}{q}\right)$$

is called a (linear) *Gauss sum* corresponding to the character χ. One writes $\tau(\chi) = \tau(1,\chi)$; we easily see that

$$\tau(\overline{\chi}) = \chi(-1)\overline{\tau(\chi)}. \tag{1.2}$$

For χ primitive, we always have $\tau(m,\chi) = \overline{\chi}(m)\tau(\chi)$, and this is true of all m if and only if χ is primitive; also, for χ primitive,

$$|\tau(\chi)| = \sqrt{q} \tag{1.3}$$

and the quantity

$$\mathfrak{w} \equiv \mathfrak{w}(\chi) = \frac{\tau(\chi)}{i^{\mathfrak{a}}\sqrt{q}} \tag{1.4}$$

is called the *root number* of χ. It is of modulus 1. However, we do not know in general the argument of $\tau(\chi)$; all we know is that $\frac{\tau(\chi)}{\sqrt{q}}$ are asymptotically equidistributed on the unit circle as χ ranges over all primitive characters mod

q and as q tends to infinity over the primes (Deligne's Equidistribution Theorem, [9]), see for example Theorem 21.5 and Theorem 21.6 of [37]. But if χ is a real character and if q is odd and squarefree, we can say the following:

Theorem (Gauss). *Let χ be a real primitive character modulo q, where q is odd and squarefree. Then, we have*

$$\tau(\chi) = \frac{1+i}{1+i^q}\sqrt{q}.$$

Thus, this Theorem of Gauss tells us that the Gauss sum $\tau(\chi)$ is simply $\sqrt{q}$ when q is squarefree and $q \equiv 1 \bmod 4$.

Next, we have the less classical *quadratic* Gauss sums, namely,

$$\tau(m, n, \chi) \stackrel{\text{def}}{=} \sum_{r \bmod 2q} \chi(r) e\left(\tfrac{mr+nr^2}{2q}\right)$$

where χ is a character $\bmod q$. We will mostly need the two cases $n = \pm 1$. We easily see that $\tau(m, n, \chi) = 0$ if $(-1)^{m+nq} = -1$, that is, if nq and m are of different parity; otherwise we simply have

$$\tau(m, n, \chi) = 2\sum_{r=1}^{q} \chi(r) e^{\frac{mr+nr^2}{q}\pi i}.$$

We have

Lemma 1. *If χ is an odd primitive character or an even primitive character of odd conductor, then*

$$\sum_{r=-q}^{q} \chi(r) e^{-\frac{\pi r^2}{q} i} = 0. \tag{1.5}$$

Moreover, if χ is even*, we also have*

$$\sum_{r=-q}^{q} r\chi(r) e^{-\frac{\pi r^2}{q} i} = 0. \tag{1.6}$$

Proof. The equation (1.5) is trivial for χ odd and (1.6) is trivial for χ even. We prove (1.5) for χ even characters of odd conductor; we have

$$\sum_{r=-q}^{q} \chi(r)e^{-\frac{\pi r^2}{q}i} = 2\sum_{r=1}^{q} \chi(r)e^{-\frac{\pi r^2}{q}i}.$$

Writing $z = \sum_{r=1}^{q} \chi(r)e^{-\frac{\pi r^2}{q}i}$ and observing that $-z = \sum_{r=1}^{q} \chi(-r)e^{-\frac{\pi r^2}{q}i} = z$, we obtain $z = 0$. Thus (1.5) is true for any primitive character, even or odd. □

It should be remarked that we have the explicit determination $\sum_{r=1}^{q} e^{-\frac{2\pi r^2}{q}i} = \frac{1+i^q}{1+i}\sqrt{q}$ due to Gauss (see [68, ch. 14 § 3.2]).

1.4 Analytic tools

In this section, we state some basic lemmas that we will need in the sequel. The results are not proved, since they are either elementary or are easily found in most textbooks on Analytic Number Theory.

The first lemma is the important Abel's partial summation formula, namely

Lemma 2 (Abel's partial summation formula)**.** *Let a_n, b_n be two complex sequences and let $A(m,n) = \sum_{m<k\leq n} a_k$. Then,*

$$\sum_{m<k\leq n} a_k b_k = A(m,n)b_N + \sum_{k=m}^{n-1} A(m,k)(b_k - b_{k+1}).$$

Next, we have the partial and Euler-Maclaurin summation formulas:

Lemma 3 (Partial summation)**.** *Let f be continuously differentiable on $[x,y]$ and let $a(n)$ be a complex sequence. Write $A(t,u) = \sum_{t<n\leq u} a(n)$. Then*

$$\sum_{x<n\leq y} a(n)f(n) = A(x,y)f(y) - \int_x^y A(x,t)f'(t)\,\mathrm{d}t. \tag{1.7}$$

Lemma 4 (Euler-Maclaurin summation formula). *Let $f \in \mathcal{C}^{m+1}[x,y]$, then*

$$\sum_{x<n\leq y} f(n) = \sum_{k=0}^{m-1} \frac{(-1)^{k+1}}{(k+1)!}\Delta_{k+1}(f^{(k)}) + \frac{(-1)^m}{(m+1)!} I_{m+1}(f^{(m+1)}) + I_0(f),$$

where $\Delta_n(g) = g(y)B_n(y) - g(x)B_n(x)$, $I_n(g) = \int_x^y g(t)B_n(t)\,\mathrm{d}t$, the B_n being the usual Bernoulli functions.

We recall that the *Bernoulli polynomials* are defined recursively by

$$\begin{aligned} b_0(t) &\equiv 1, \\ b_n'(t) &= nb_{n-1}(t), \quad (n \geq 1) \\ \int_0^1 b_n(t)\,\mathrm{d}t &= 0, \qquad\qquad (n \geq 1). \end{aligned}$$

We then define the *Bernoulli functions* by $B_n(x) = b_n(\{x\})$. They are thus periodic functions of period dividing 1. The *Bernoulli numbers* are $B_n \overset{\text{def}}{=} B_n(0)$ and it is clear that $B_n(0) = B_n(1)$ for $n \geq 1$.

We recall that the Fourier transform of a function $f \in L^1(\mathbb{R})$ is defined by

$$\hat{f}(y) = \int_{\mathbb{R}} f(x)e(-xy)\,\mathrm{d}x.$$

We have the following forms of Poisson summation formula:

Lemma 5 (Poisson summation formula). *Suppose f is continuous and of bounded variation on a not necessarily bounded interval $[A,B] \subset \mathbb{R}$. Then,*

$$\sideset{}{'}\sum_{A\leq m\leq B} f(m) = \sum_{n\in\mathbb{Z}} \int_A^B f(x)e(nx)\,\mathrm{d}x. \tag{1.8}$$

Also, for any primitive character χ of conductor q,

$$\sideset{}{'}\sum_{A\leq m\leq B} f(m)\chi(m)e^{\pm\frac{\pi m^2}{q}i} = \frac{1}{2q}\sum_{r=1}^{2q} \tau_{\pm}(r,\chi)\sum_{n\in\mathbb{Z}} \int_A^B f(t)e(-(n+\frac{r}{2q})t)\,\mathrm{d}t \tag{1.9}$$

where $\tau_{\pm}(n,\chi) = \sum_{r \bmod 2q} \chi(r) e(\frac{nr \pm r^2}{2q})$. *In particular, we have*

$$\sum_{A \le m \le B}{}' (-1)^m f(m) = \sum_{n \in \mathbb{Z}} \int_A^B f(t) e\big((n + \frac{1}{2})t\big)\,\mathrm{d}t. \tag{1.10}$$

The dash $'$ *in the summation sign means that if any of the endpoints is an integer, the corresponding term is to be halved.*

Proof. The equation (1.8) is the usual form of Poisson summation formula, see [6, Proposition 2.2.16]; the other two are derived from the first one by considering $f(qx + r)$ in place of $f(x)$ and doing appropriate sums. See also [39, Lemma 5] for a proof of (1.10). □

Note that in case $[A, B] = \mathbb{R}$, the integrals on the right of (1.8), (1.9) and (1.10) are respectively $\hat{f}(n), \hat{f}(n + \frac{r}{2q})$ and $\hat{f}(n + \frac{1}{2})$.

It is often very useful to have a truncated version of Poisson summation, so we quote [32, eq. (10.2)] which gives the truncated Poisson summation formula with a good error term:

Theorem 6 (Truncated Poisson summation formula)**.** *Let* f *be differentiable with* $|f'(x)| \le A$ *on the interval* $I = [M, M + L]$ *where* L, M *are integers. Then,*

$$\sum_{m \in I}{}' f(m) = \sum_{|n| \le N} \int_I f(t) e\,(nt)\,\mathrm{d}t + O\left(\frac{AL \log N}{N}\right).$$

Next, we denote by $\check{f}(p)$ the Laplace transform of a function $f(t)$ defined on the positive ray; thus

$$\check{f}(p) = \int_0^\infty f(t) e^{-pt}\,\mathrm{d}t \quad (p \in \mathbb{C}) \tag{1.11}$$

whenever the integral on the right exists. We recall that if the integral does not diverge everywhere or converge everywhere, then there is a real number σ_c, called the *abscissa of convergence*, such that the integral (1.11) converges for $\Re p > \sigma_c$ and diverges for $\Re p < \sigma_c$; in such a situation, we have

Lemma 7 (Laplace inversion formula)**.** *Let* $f(t)$ *and* $\check{f}(p)$ *be related by* (1.11) *where* $\check{f}(p)$ *exists for* $\Re p > \sigma_c$. *Then, the function* $\check{f}(p)$ *is analytic in the half-plane*

$\Re p > \sigma_c$ *and*

$$f(t) = \frac{1}{2\pi i}\int_{(a)} \check{f}(p)e^{pt}\,\mathrm{d}p \quad (t > 0) \tag{1.12}$$

where $a > \sigma_c, a > 0$. *In general, suppose* $\check{f}(p)$ *is meromorphic in some half-plane and that* $\check{f}(p)$ *and* $f(t)$ *are related by* (1.11) *for* p *in the half-plane of meromorphy; then* (1.12) *holds for* a *exceeding the real parts of all the singularities of* $\check{f}(p)$.

We next state for reference some properties of Euler's Gamma function $\Gamma(s)$ and the digamma function $\frac{\Gamma'}{\Gamma}$. See also [1, § 6] or [26, § 8.36]

Lemma 8 (Stirling's formula)**.** *For* $0 < \epsilon < 1$, *we have*

$$\Gamma(z+a) = \sqrt{2\pi}e^{-z}z^{z+a-\frac{1}{2}}\big(1 + O(\frac{1}{|z|})\big)$$

uniformly in $-\pi + \epsilon \leq \arg z \leq \pi - \epsilon$ *and* a *varying in any fixed compact set in* $\mathbb{C}$.

Lemma 9. *We have the* **Cahen-Mellin formula**

$$e^{-z} = \frac{1}{2\pi i}\int_{(c)} \Gamma(s)z^{-s}\,\mathrm{d}s$$

valid for all $c > 0$ *and all* z *with* $\Re z > 0$, z^{-s} *being on the principal branch.*

Lemma 10. *The digamma function satisfies the functional equation*

$$\frac{\Gamma'}{\Gamma}(1-z) = \frac{\Gamma'}{\Gamma}(z) + \pi\cot\pi z \quad (\forall z \in \mathbb{C}) \tag{1.13}$$

and the asymptotic

$$\frac{\Gamma'}{\Gamma}(z) = \log z - \frac{1}{2z} + O(\frac{1}{|z|^2}) \quad (z \to \infty \textit{ in } |\arg z| < \pi). \tag{1.14}$$

1.5 Dirichlet series

We give a brief review of the general theory of Dirichlet series. A *Dirichlet series* is a series of the form

$$F(s) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n s} \quad (s \in \mathbb{C}) \tag{1.15}$$

where (λ_n) is a strictly increasing sequence of nonnegative real numbers tending to $+\infty$ and $a_n \in \mathbb{C}$. The case $\lambda_n = \log n$ is the most important case and the name "Dirichlet series" is often reserved for it. If the series $F(s)$ converges for some $s = s_0$, then the series (1.15) converges uniformly in the closed sector $\Re(s - s_0) \geq 0, |\arg(s - s_0)| \leq \theta$ for any fixed $0 < \theta < \frac{\pi}{2}$, and thus defines a holomorphic function in the half-plane $\Re(s - s_0) \geq 0$, and we have

$$F^{(k)}(s) = \sum_{n=1}^{\infty} (-\lambda_n)^k a_n e^{-\lambda_n s} \quad (k \geq 0)$$

therein. If the series (1.15) does not converge everywhere or diverge everywhere, the lower bound of all $\sigma = \Re s$ for which the series (1.15) converges is called the *abscissa* of convergence of the Dirichlet series, often denoted σ_c; we have the formula

$$\sigma_c = \limsup_{n \to \infty} \frac{1}{\lambda_n} \log \left| \sum_{k=1}^{n} a_k \right|$$

in case this limit is nonzero. Similarly, we define abscissa of absolute convergence σ_a etc and we have analogous formulas for them. See [63, § 4] for more details.

We take from now on $\lambda_n = \log n$, so the series is simply

$$F(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s};$$

we assume that $F(s)$ has finite abscissae of convergence σ_c and of absolute convergence σ_a, so $F(s)$ is holomorphic in $\Re s > \sigma_c$. Then we have the following formula for the partial sums of the series in terms of the function $F(s)$:

Theorem 11 (Perron's formula). *Let $x > 0, c > 0$. Then for $\sigma > \sigma_a - c$,*

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s+z)\frac{x^z}{z}\,\mathrm{d}z = {\sum_{n\le x}}' \frac{a_n}{n^s}$$

where the dash ′ on the summation sign means that the last term is to be halved in case x is an integer. In particular, if $c > \sigma_a$, we have

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(z)\frac{x^z}{z}\,\mathrm{d}z = {\sum_{n\le x}}' a_n. \tag{1.16}$$

1.6 Dirichlet L-functions

We now turn to the case which interests us most, namely, the case $\lambda_n = \log n$ and $a_n = \chi(n)$ for a fixed character $\chi \bmod q$. In this case, we write

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} \quad (s \in \mathbb{C}); \tag{1.17}$$

the series (1.17) is called a Dirichlet *L-series* and the resulting function a Dirichlet *L-function*. In case $q = 1$, the series (1.17) is denoted by $\zeta(s)$ and is the prototype of all L-functions; it is the well-known *Riemann zeta function:*

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad (\sigma > 1).$$

If χ is nonprincipal, the abscissa of convergence of (1.17) is obviously $\sigma_c = 0$ and the abscissa of absolute convergence is $\sigma_a = 1$. By our remark on general Dirichlet series, it follows that $L(s,\chi)$ is holomorphic in the positive half-plane $\Re s > 0$. Moreover, we have the absolutely convergent *Euler product* representation

$$L(s,\chi) = \prod_p \left(1 - \frac{\chi(p)}{p^s}\right)^{-1} \tag{1.18}$$

for $\sigma > 1$. By logarithmic differentiation of (1.18), we have the identity

$$\frac{L'}{L}(s,\chi) = -\sum_{n=1}^{\infty} \frac{\chi(n)\Lambda(n)}{n^s} \quad (\sigma > 1). \tag{1.19}$$

Any Dirichlet L-function has a meromorphic continuation to the whole complex plane and the resulting function is entire if and only if the character is not principal; for the principal character $\chi_e \bmod q$, the corresponding L-function $L(s,\chi_e)$ has a pole at $s = 1$ with residue $\varphi(q)/q$.

If $\chi \bmod q$ is induced by $\chi_1 \bmod d|q$, we have

$$L(s,\chi) = L(s,\chi_1) \prod_{p|q} \left(1 - \frac{\chi_1(p)}{p^s}\right)$$

for all $s \in \mathbb{C}$, and $L(s,\chi)$ is thus not very different from $L(s,\chi_1)$; that is why we often limit our studies to primitive characters. Note in particular that

$$L(s,\chi_e) = \zeta(s) \prod_{p|q} \left(1 - p^{-s}\right)$$

and $L(s,\chi_e)$ is thus not much different from the Riemann zeta function.

Since the Euler product (1.18) converges absolutely for $\sigma > 1$, it easily follows that $L(s,\chi) \neq 0$ in the region $\Re s > 1$. The nonvanishing of $\zeta(s)$ on the line $\Re s = 1$ is equivalent to the Prime Number Theorem (PNT for short)

$$\lim_{x \to \infty} \frac{\pi(x) \log x}{x} = 1 \tag{1.20}$$

of Hadamard and de la Vallée Poussin.

If $\chi \bmod q$ is primitive, then $L(s,\chi)$ satisfies the functional equation

$$L(s,\chi) = \Psi(s,\chi) L(1-s,\overline{\chi}) \tag{1.21}$$

with

$$\Psi(s,\chi) = \mathfrak{w}(\chi)\left(\frac{\pi}{q}\right)^{s-\frac{1}{2}} \frac{\Gamma(\frac{1}{2}(1-s+\mathfrak{a}))}{\Gamma(\frac{1}{2}(s+\mathfrak{a}))} \tag{1.22}$$

$$= \frac{q^{-s}\Gamma(1-s)i}{(2\pi)^{1-s}}\left\{-e^{\frac{\pi s i}{2}} + \chi(-1)e^{-\frac{\pi s i}{2}}\right\}\tau(\chi). \tag{1.23}$$

In terms of the *completed L-function*

$$\xi(s,\chi) = (\frac{\pi}{q})^{-\frac{1}{2}(s+\mathfrak{a})}\Gamma(\frac{1}{2}(s+\mathfrak{a}))L(s,\chi),$$

the functional equation (1.21) gets rewritten as

$$\xi(s,\chi) = \mathfrak{w}(\chi)\xi(1-s,\overline{\chi}). \tag{1.24}$$

Moreover, $\xi(s,\chi)$ is an entire function of order 1, and hence by the general theory of entire functions of finite order (see for example [53]), there exist two constants A_χ, B_χ such that the *Hadamard product* formula

$$\xi(s,\chi) = e^{A_\chi + B_\chi s}\prod_{\rho_\chi}\left(1-\frac{s}{\rho_\chi}\right)e^{\frac{s}{\rho_\chi}}$$

holds for all $s \in \mathbb{C}$; the product runs through the zeros ρ_χ of $\xi(s,\chi)$. The sum $\sum \frac{1}{\rho_\chi}$ diverges whereas the sum $\sum \frac{1}{\rho_\chi^{1+\epsilon}}$ converges for any $\epsilon > 0$. By (1.24), it is obvious that every zero of $\xi(s,\chi)$ is a zero of $L(s,\chi)$ in the *critical strip* $0 \le \Re s \le 1$; such zeros are called *nontrivial.* All other zeros are *trivial.* The divergence of $\sum \frac{1}{\rho_\chi}$ then means the existence of infinitely many nontrivial zeros.

For the Riemann zeta function, the completed function takes the form

$$\xi(s) = \frac{1}{2}\pi^{-\frac{s}{2}}s(s-1)\Gamma(\frac{s}{2})\zeta(s),$$

where the factors s and $s-1$ are introduced to cancel out the pole of $\Gamma(s/2)$ at $s=0$, and the pole of $\zeta(s)$ at $s=1$ respectively, which was not necessary in case of $L(s,\chi)$ with χ primitive. This makes $\xi(s)$ entire; the functional equation for

$\zeta(s)$ has then the simple form

$$\xi(s) = \xi(1-s).$$

The zeros are symmetrically distributed with respect to the critical line as well as the real axis, which is not the case for general L-functions.

1.7 Zeros of L-functions

By (1.24) again, the zeros of $L(s,\chi)$ are symmetrically distributed with respect to the *critical line* $\Re s = \frac{1}{2}$. The hypothesis that all nontrivial zeros of $L(s,\chi)$ lie on the critical line is known as the *Generalised Riemann Hypothesis,* GRH for short. The same hypothesis for $\zeta(s)$ is the *Riemann Hypothesis,* RH for short. It is a classical theorem of Hardy that there are indeed infinitely many zeros on the critical line.

The number of zeros $\rho = \beta + i\gamma$ of $\zeta(s)$ with $0 \le \beta \le 1$, $0 < \gamma \le T$ is denoted by $N(T)$, those with $\beta = \frac{1}{2}$ is denoted by $N_0(T)$ and those with $\beta \ge \sigma$ by $N(\sigma, T)$. Note that $\zeta(s)$ does not vanish in $0 \le s \le 1$, as is easily seen from the equation

$$(1-2^{1-s})\zeta(s) = 1 - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + \cdots .$$

In contrast to the Riemann zeta function, the L-function $L(s,\chi)$ has a trivial zero at $s = 0$ in case χ is even. We have the explicit form of the *Riemann-von Mangoldt formula* for $N(T)$ (see [55, Satz (3.)], [67, Lemma 1], and [80] for a sharper result):

Lemma 12. *For* $T \ge 1\,000$*,*

$$N(T) = N^*(T) + O^*(0.67 \log \frac{T}{2\pi}), \tag{1.25}$$

where

$$N^*(T) = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + \frac{7}{8}.$$

We have mentioned earlier that the nonvanishing of $\zeta(s)$ on the line $\Re s = 1$ is equivalent to the PNT; de la Vallée Poussin in fact proves that $\zeta(s)$ has no zero $\rho = \beta + i\gamma$ with

$$\beta \geq 1 - \frac{1}{R \log |\gamma|}, \tag{1.26}$$

where $R = 34.82$ (see [8, 1ère partie, chap. II, (38)], also [41]). Thus, the region

$$\sigma \geq 1 - \frac{1}{R \log |t|}$$

is a *zero-free region,* known as the de la Vallée Poussin form of zero-free region. Smaller values of R are known, for example, thanks to [40], one may take $R = 5.696\,93$, and in [57], it is proved that one may take $R = 5.573\,412$. Stronger zero-free regions are also known, such as the Vinogradov-Korobov form (cf. [46, 82])

$$\sigma \geq 1 - \frac{c}{\left(\log^{\frac{2}{3}} |t|\right) \left(\log \log^{\frac{1}{3}} |t|\right)},$$

also with explicit c (cf. [23]).

1.8 The explicit formula for ψ

The so-called explicit formula for the Chebyshev function

$$\psi(x) = \sum_{n \leq x} \Lambda(n)$$

stems from an application of Perron's formula (1.16) to the logarithmic derivative (1.19) of $\zeta(s)$. It takes the form

$$\psi_0(x) = x - \sum_{\rho} \frac{x^\rho}{\rho} - \log 2\pi - \frac{1}{2} \log\left(1 - x^{-2}\right) \tag{1.27}$$

for $x > 1$, where $\psi_0(x) = \sum'_{n \leq x} \Lambda(n)$, with the same summation convention as in (1.16). The sum $\sum_\rho \frac{x^\rho}{\rho}$ runs through the nontrivial zeros ρ of $\zeta(s)$, and its value should be taken in terms of the Cauchy principal value of the series, that is

$$\sum_{\rho} \frac{x^\rho}{\rho} \overset{\text{def}}{=} \lim_{T\to\infty} \sum_{|\rho|<T} \frac{x^\rho}{\rho}.$$

A related useful formula which may be derived from the explicit formula (1.27) is [67, Lemma 4]

Lemma 13. *Let* $g \in \mathcal{C}^1[a,b]$ *with* $2 \le a < b < \infty$. *Then*

$$\int_a^b \psi(t) g(t)\,\mathrm{d}t = \int_a^b t g(t)\,\mathrm{d}t - \sum_\rho \int_a^b \frac{t^\rho}{\rho} g(t)\,\mathrm{d}t - \int_a^b \left(\log \frac{2\pi\sqrt{t^2-1}}{t}\right) g(t)\,\mathrm{d}t.$$

1.9 Hardy's Z-function

For a primitive character $\chi \bmod q$, analogously to the case $q = 1$, we propose to define *Hardy's Z-function* corresponding to the Dirichlet L-function $L(s,\chi)$ by

$$Z(t,\chi) \overset{\text{def}}{=} \Psi(\tfrac12 + it, \chi)^{-\frac12} L(\tfrac12 + it, \chi)$$

for real t.

Multiplying $\Psi(s,\chi)$ and $\Psi(1-s,\overline{\chi})$ and using (1.2) and (1.3), we obtain

$$\Psi(s,\chi)\Psi(1-s,\overline{\chi}) = 1 \quad (s \in \mathbb{C});$$

from this it follows easily that $Z(t,\chi)$ is an even function which is real for real t and that $|Z(t,\chi)| = \left|L(\tfrac12 + it, \chi)\right|$. Thus the zeros of $Z(t,\chi)$ correspond exactly to the zeros of $L(s,\chi)$ on the critical line. Moreover, the expression (1.22) easily gives

$$\Psi(\tfrac12 + it, \chi) = \mathfrak{w}(\chi) \left(\frac{\pi}{q}\right)^{it} \frac{\left|\Gamma(\frac12(\frac12 + it + \mathfrak{a}))\right|^2}{\Gamma(\frac12(\frac12 + it + \mathfrak{a}))^2}$$

so

$$\Psi(\tfrac{1}{2}+it,\chi)^{-\frac{1}{2}} = \frac{1}{\sqrt{\mathfrak{w}(\chi)}}\left(\frac{\pi}{q}\right)^{-\frac{it}{2}} \frac{\Gamma\big(\frac{1}{2}(\frac{1}{2}+it+\mathfrak{a})\big)}{\big|\Gamma\big(\frac{1}{2}(\frac{1}{2}+it+\mathfrak{a})\big)\big|} \underset{\text{def}}{=} e^{i\theta(t,\chi)}$$

where $\theta(t,\chi)$ is the so-called *Riemann-Siegel* theta function. Thus, in terms of this function, Hardy's function has the form

$$Z(t,\chi) = e^{i\theta(t,\chi)} L(\tfrac{1}{2}+it,\chi). \tag{1.28}$$

By a simple application of Stirling's formula, we have the expression

$$\theta(t,\chi) = \frac{t}{2}\log\frac{qt}{2\pi} - \frac{t}{2} - \frac{\pi}{2}\left(\frac{1}{4}-\frac{\mathfrak{a}}{2}\right) + \frac{i}{2}\log\mathfrak{w} + \Delta(t,\chi)$$

where

$$\begin{aligned}\Delta(t,\chi) = {} & \frac{t}{4}\log\left(1+\frac{4\mathfrak{a}^2+4\mathfrak{a}+1}{4t^2}\right) + \left(\frac{1}{4}-\frac{\mathfrak{a}}{2}\right)\arctan\left(\frac{2\mathfrak{a}+1}{2t}\right) \\ & + \frac{t}{2}\int_0^\infty \frac{B_1(u)\,\mathrm{d}u}{(u+\frac{\mathfrak{a}}{2}+\frac{1}{4})^2+\frac{t^2}{4}},\end{aligned}$$

$B_1(u) = \{u\} - \frac{1}{2}$ being the first Bernoulli function. Note that $\Delta(t,\chi) = O(1/t)$.

The equation (1.28) gives

$$L(\tfrac{1}{2}+it,\chi) = Z(t,\chi)\Big(\cos\theta(t,\chi) - i\sin\theta(t,\chi)\Big)$$

so $L(\frac{1}{2}+it,\chi)$ takes real values exactly when $\theta(t,\chi) = n\pi$. Given $n \geq -1$, the unique solutions $g_n > 7$ to the equation

$$\theta(g_n,\chi) = n\pi$$

are known as the *Gram points*. Note that at the Gram point g_n,

$$L(\tfrac{1}{2}+ig_n,\chi) = (-1)^n Z(g_n,\chi).$$

1.10 Exponential sums and integrals

An *exponential sum* is a sum of the form

$$S(M,N;f)=\sum_{M<n\leq N}e\left(f(n)\right)$$

where f is a real function. Such sums arise naturally in many contexts in Analytic Number Theory, for example, the *Lindelöf Hypothesis* in zeta function theory, which is the conjecture that $\zeta(\frac{1}{2}+it)\ll|t|^{\varepsilon}$ for $|t|\geq 2$, is equivalent to the assertion (see [44, V § 6])

$$\sum_{n\leq x}n^{it}\ll\sqrt{x}\,|t|^{\varepsilon}\quad(1\leq x\leq\sqrt{|t|}). \tag{1.29}$$

Note that $n^{it}=e\left(f(n)\right)$ with $f(n)=\frac{t}{2\pi}\log n$. Thus, the sum on the left of (1.29) is an exponential sum, called a *zeta sum*. In general, by partial summation, a sum of the form

$$\sum_{M<n\leq N}g(n)e\left(f(n)\right)$$

with $g(n)$ real and smooth can be reduced to an exponential sum. Similarly, an *exponential integral* is an integral of the form

$$\int_a^b g(t)e\left(f(t)\right)\,\mathrm{d}t$$

with f,g real. Exponential sums and integrals are closely related, and a good estimate for one of them often gives a good estimate for the other. For example, we have the following lemma from the van der Corput method ([79, Lemma 4.7] or [27, Lemma 3.5]):

Lemma 14. *Suppose f has two continuous derivatives on $I=]a,b]$ and that f' is decreasing on $]a,b[$ with $\alpha=f'(a),\beta=f'(b)$. Then*

$$\sum_{m\in I}e\left(f(m)\right)=\sum_{\alpha-\eta<n<\beta+\eta}\int_I e\left(f(t)-nt\right)\,\mathrm{d}t+O\left(\log(\alpha-\beta+2)\right)$$

where η is any positive constant less than 1.

We will content ourselves with giving the most well-known estimates that we will need here. The first is

Theorem 15 (Kusmin-Landau)**.** *If f is continuously differentiable on $I =]a, b]$ (where $a, b \in \mathbb{Z}$) with monotonic derivative f' satisfying $\|f'(x)\| \geq \lambda > 0$ on I, then*

$$\sum_{n \in I} e\left(f(n)\right) \ll \frac{1}{\lambda}.$$

See [27, Theorem 2.1] for a proof. A similar estimate for the corresponding exponential integral is

Theorem 16 (First derivative test, [79, Lemma 4.2])**.** *Suppose f is differentiable with monotonic derivative f' satisfying $f'(x) \geq \lambda > 0$ (or $f'(x) \leq -\lambda < 0$) on the interval $[a, b]$, then*

$$\int_a^b e\left(f(t)\right) \mathrm{d}t \ll \frac{1}{\lambda}.$$

In the following two theorems, we take $I =]a, b]$ with a and b not necessarily integers and $b - a \geq 2$; $h \geq 1$ is a real parameter. The function f is real and is supposed to be defined in a neighbourhood of I.

Theorem 17 (Second derivative test, [79, Theorem 5.9])**.** *Suppose f is twice differentiable with $0 < \lambda \leq f''(x) \leq h\lambda$ (or $0 < \lambda \leq -f''(x) \leq h\lambda$) throughout the interval $[a, b]$, then*

$$\sum_{n \in I} e\left(f(n)\right) = O\left(h(b-a)\sqrt{\lambda}\right) + O\left(\frac{1}{\sqrt{\lambda}}\right).$$

Theorem 18 (Third derivative test, [79, Theorem 5.11])**.** *Suppose f is thrice continuously differentiable with $0 < \lambda \leq f'''(x) \leq h\lambda$ (or $0 < \lambda \leq -f'''(x) \leq h\lambda$) throughout the interval $[a, b]$, then*

$$\sum_{n \in I} e\left(f(n)\right) = O\left(\sqrt{h}(b-a)\lambda^{\frac{1}{6}}\right) + O\left(\sqrt{b-a}\lambda^{-\frac{1}{6}}\right).$$

See [69, 70] for better results which apply in particular situations.

We next state the following result of the van der Corput method as a lemma (see [27, Theorem 2.9]):

Lemma 19. *Let $\nu \geq 0, N \geq 1$ be integers and $Q = 2^\nu$, $I =]a, b] \subset]N, 2N]$ with a, b integers. Suppose that $f \in \mathcal{C}^{\nu+2}(I; \mathbb{R})$ and that there is a constant $F > 0$ satisfying*

$$f^{(j)}(x) \asymp FN^{-j} \quad (x \in I) \tag{1.30}$$

for $j = 1, 2, ..., \nu + 2$. Then

$$\sum_{n \in I} e(f(n)) \ll \left(\left(\frac{F}{N^{\nu+2}} \right)^{\frac{1}{4Q-2}} + F^{-1} \right) N$$

where the implied constant depends only on the implied constants in (1.30).

Finally, we have the following result for exponential integrals (see[35, Theorem 2.2]). This is a saddle-point type of result with additional assumptions on the phase f.

Theorem 20 (Atkinson)**.** *Let $a < b$ be two real numbers and let f, g be two complex functions satisfying the following conditions:*

1. *$f(x)$ is real for $a \leq x \leq b$ and $f''(x) < 0$. Moreover, for a certain real number k, $f' + k$ has a zero in $[a, b]$, say $f'(x_0) + k = 0$.*
2. *There exists a continuously differentiable function $\mu : [a, b] \to \mathbb{R}^+$ such that for each $x \in [a, b]$, both f and g are holomorphic in the disk $|z - x| \leq \mu(x)$.*
3. *There are two positive functions F and G on $[a, b]$ such that*

$$f'(z) \ll F(x)\mu(x)^{-1}, |f''(x)| \geq F(x)\mu(x)^{-2},$$
$$g(z) \ll G(x)$$

   for each $x \in [a, b]$ and all z with $|z - x| \leq \mu(x)$.

Then, we have

$$\int_a^b g(t)e(f(t)+kt)\,\mathrm{d}t = \frac{g(x_0)}{\sqrt{|f''(x_0)|}}e(f(x_0)+kx_0-\frac{1}{8}) + O(\frac{G(x_0)\mu(x_0)}{F(x_0)^{\frac{3}{2}}})$$
$$+ O\biggl(\int_a^b G(t)\exp(-c\,|k|\,\mu(t)-cF(t))(1+|\mu'(t)|)\,\mathrm{d}t\biggr)$$
$$+ O\biggl(\frac{G(a)}{|f'(a)+k|+\sqrt{|f''(a)|}}\biggr) + O\biggl(\frac{G(b)}{|f'(b)+k|+\sqrt{|f''(b)|}}\biggr).$$

Here, c denotes a general positive constant. If, instead of $f''(x) < 0$, we have $f''(x) > 0$ and $f''(x) \ll F(x)\mu(x)^{-2}$, then we have the same equation except that $-\frac{1}{8}$ in the main term is replaced by $+\frac{1}{8}$.

Part II

Fonctions de Hardy des fonctions L de Dirichlet

Hardy's functions for Dirichlet L-functions

Chapter 2

Hardy's functions for Dirichlet L-functions

2.1 Introduction, history and results

Let us consider a Dirichlet L-*series*

$$L(s,\chi)=\sum_{n=1}^{\infty}\frac{\chi(n)}{n^s}, \tag{2.1}$$

corresponding to a primitive character χ of conductor q; the series converges in $\Re s>0$ provided $q>1$, and therein defines an analytic function, called a *Dirichlet L-function*, having an analytic continuation to the whole complex plane wherein it satisfies the functional equation

$$L(s,\chi)=\Psi(s,\chi)L(1-s,\overline{\chi}). \tag{2.2}$$

These functions, including the notation L, were introduced by Dirichlet in [13], who employed them to prove his celebrated theorem on primes in arithmetic progressions (see also [15, 16, 17]). The studies of these L-functions were first systematically undertaken by Landau [49, 51]; they are now a central object in Analytic Number Theory and have seen many generalisations, including the L-functions of Hecke and Artin etc (see for example [18, 24]).

Given the functions in (2.2), the *Hardy's Z-function* corresponding to the Dirichlet L-function is defined by

$$Z(t,\chi) = \Psi(\tfrac{1}{2}+it,\chi)^{-\frac{1}{2}} L(\tfrac{1}{2}+it,\chi)$$

for real t. It is easily seen from (2.2) that $Z(t,\chi)$ is real for real t, and moreover $|Z(t,\chi)| = \left|L(\tfrac{1}{2}+it,\chi)\right|$ and thus the zeros of $Z(t,\chi)$ correspond to the zeros of $L(s,\chi)$ on the critical line. To study the variation of this function, we look at the integrals

$$F(T,\chi) = \int_0^T Z(t,\chi)\,\mathrm{d}t$$

for $T \geq 0$.

The behaviour of these functions is rather mysterious, but recently Korolëv [48] and Jutila [39] exhibited a very remarkable property of the function $F(T)$ corresponding to the Riemann zeta function, which this article proposes to extend to the case of L-functions.

Hardy's function $Z(t)$ corresponding to the Riemann zeta function $\zeta(s)$ was studied by Hardy and Littlewood as a means for studying the distribution of zeros of $\zeta(s)$ on the critical line $\Re s = \frac{1}{2}$. They suggested an alternative method for proving the existence of infinitely many zeros of $\zeta(s)$ on the critical line (this fact was earlier proved by Hardy [29]), namely: if one can prove the inequality

$$\left|\int_T^{2T} Z(t)\,\mathrm{d}t\right| < \int_T^{2T} |Z(t)|\,\mathrm{d}t \tag{2.3}$$

for sufficiently large T, then this would imply that $Z(t)$ has a zero in the interval $[T,2T]$ and *a fortiori* that the zeta function vanishes infinitely many times as it walks up the critical line. This argument for proving Hardy's theorem, as it is now known, was first carried out in [51], where Landau proves that the right member of (2.3) is $> 0.5T$ whereas the left member is $\ll T^{7/8}$. As an improvement of this classical estimate $F(T) = O(T^{7/8})$ of Landau, Ivić [34] showed that

$$F(T) = O_\epsilon(T^{\frac{1}{4}+\epsilon}), \quad (\forall \epsilon > 0) \tag{2.4}$$

conjecturing the behaviour $F(T) = \Omega_\pm(T^{\frac{1}{4}})$. This conjecture was recently proved by Korolëv [48], also removing the ϵ in (2.4) (see also [47]). He also proved that $F(T)$ can be asymptotically approximated by a very irregular periodic function $\mathfrak{K}(\vartheta)$ of the fractional part ϑ of $\sqrt{T/(2\pi)}$. Both Ivić and Korolëv used variants of the Riemann-Siegel formula for $Z(t)$.

More recently, Jutila [39] gave an alternative proof of these, dispensing with the Riemann-Siegel formula and removing a certain nonuniformity existing in the approximation, instead proving an Atkinson-like formula for $Z(t)$ by means of the Laplace transform. In [39], there are two error terms in the approximation; in [38], a new version of the asymptotic approximation is proved in which one of the error terms is in the form of an integral.

We now state the result for the case of Dirichlet L-functions.

Theorem 21. *Assume that χ is an even primitive character of odd conductor or one of the odd quadratic characters* $\bmod 4$ *and* $\bmod 8$. *Then we have*

$$F(T,\chi) = 2\Re S_1(T,\chi) + 2\Re S_2(T,\chi) + O((\log T)^{5/4}), \tag{2.5}$$

with

$$S_1(T,\chi) = \kappa^* \left(\frac{qT}{2\pi}\right)^{\frac{1}{4}} \sum_{r=0}^{2q-1} \tau^*(r,\chi) \sum_{\substack{1\le n\le 2\sqrt{N} \\ n\equiv -r[2q]}} \frac{(-1)^{\frac{n^2-r^2}{8q}}}{n} \,\mathrm{e}\left(T, \frac{n^2}{4q}\right) \exp\left(\frac{i}{2}\,\mathrm{f}(T,\frac{n^2}{4q}) - \frac{3\pi i}{8}\right) \tag{2.6}$$

and

$$S_2(T,\chi) = \kappa_* \left(\frac{qT}{2\pi e}\right)^{\frac{iT}{2}} \sum_{n<\sqrt{N'}} \frac{\chi(n)}{\log(\frac{qT}{2\pi n^2})} n^{-\frac{1}{2}-iT} \tag{2.7}$$

where N is a parameter chosen so that $N \asymp T$, and $N' = \frac{qT}{2\pi} + \frac{N}{2} - \sqrt{\frac{N^2}{4} + \frac{qNT}{2\pi}}$

and where e *and* f *are defined respectively by* [1]

$$\mathsf{e}(T,\ell) = (1+\frac{\pi\ell}{2T})^{-\frac{1}{4}}\left\{\sqrt{\frac{\pi\ell}{2T}}\,\mathrm{argsinh}^{-1}\left(\sqrt{\frac{\pi\ell}{2T}}\right)\right\},$$

$$\mathsf{f}(T,\ell) = 2T\,\mathrm{argsinh}\left(\sqrt{\frac{\pi\ell}{2T}}\right) + \sqrt{(\pi\ell)^2+2\pi\ell T} - \frac{\pi}{4}.$$

Also, the coefficients $\tau^*(r,\chi)$ *are defined by* $\tau^*(r,\chi) = e^{-\frac{\pi r^2}{8q}i}\sum_{a \bmod 2q}\chi(a)e(\frac{ar-a^2}{2q})$*, and* $\tau^*(r,\chi)=0$ *if* $q+r$ *is odd;* κ^*,κ_* *are constants depending on the character* χ *and the modulus* q (*see* (2.21)*,* (2.67) *and* (2.68)).

Remark. We observe that the factor $e^{-\frac{\pi r^2}{8q}i}(-1)^{\frac{n^2-r^2}{8q}} = e^{\frac{\pi n^2}{8q}i}$ occurring inside $S_1(T,\chi)$ is independent of r, so the sum is not ambiguous.

Note that although we cannot prove that $L(\frac{1}{2}+it,\chi)$, or equivalently $Z(t,\chi)$, is $O_\epsilon\left((1+|t|)^\epsilon\right)$ (the Lindelöf Hypothesis), but we dispose here of an error term for $\int_0^T Z(t,\chi)\,\mathrm{d}t$ of size $O\left((\log T)^{\frac{5}{4}}\right)$.

One part of the extension of the Korolëv-Jutila result to the case of L-functions is quite automatic, but another part is new. The proof follows that of Jutila [39], which is quite original and rather mysterious; this article also details some points of the argument brushed off as obvious by Jutila. The Laplace transform was earlier applied by him in [58, § 12] to study the fourth power moment of $\left|\zeta(\frac{1}{2}+it)\right|$ and to L-functions attached to holomorphic and non-holomorphic cusp forms.

2.2 A more explicit approximation for $F(T,\chi)$

Although the formula (2.5) is a fairly simple approximation of the function $F(T,\chi)$, it is difficult to read the true behaviour of $F(T,\chi)$ from it. We therefore give a less precise, but more comprehensible approximation for it. This will also exhibit a rather strange phenomenon extending the one obtained by Jutila.

First of all, since the term $S_2(T,\chi)$ can be easily reduced to a zeta sum by partial summation, an immediate corollary of the theorem is

1. See (2.70) and (2.71) for some properties of $\mathsf{e}(T,\ell)$ and $\mathsf{f}(T,\ell)$.

Corollary 22. *We have*

$$F(T,\chi) = 2\Re S_1(T,\chi) + O\big(T^{\frac{1}{6}}\log T\big).$$

We observe that the error term is still almost as good as that for $Z(t,\chi)$. Proof of the corollary is given in § 2.8 for completeness. We will prove that the function $S_1(T,\chi)$ can be approximated by a fairly simple logarithmic function. This limiting function corresponds to the function $\mathfrak{K}(\vartheta)$ of Korolëv [48] and to the step function $K(x)$ of Jutila [39].

Let us define a function $K_{q,r}(x)$ in the following way. When q and r are even, first set

$$\epsilon_{q,r} = \frac{1-(-1)^{\frac{q-r}{2}}}{2} = \begin{cases} 0, & \text{if } q \equiv r \pmod 4, \\ 1, & \text{otherwise,} \end{cases}$$

and then

$$K_{q,r}(x) = -\frac{1}{2q} e\big(\frac{r\epsilon_{q,r}}{4q}\big) \sum_{k=1}^{2q} e\left(\frac{rk}{2q}\right) \log\big(1 - e(x + \frac{2k+\epsilon_{q,r}}{4q})\big) \tag{2.8}$$

provided $x + \frac{2k+\epsilon_{q,r}}{4q}$ is not an integer for any of the k's. We first state a theorem for primitive characters of even conductor describing the behaviour at generic points.

Theorem 23. *Suppose χ is an odd character modulo $q = 4$ or $q = 8$. Write $\sqrt{qT/(2\pi)} = qL + \vartheta$ where $L \in \mathbb{N}$ and $0 \le \vartheta < q$. Also let $\vartheta_0 = \min\{\,|\vartheta - \frac{n}{2}| : n \in \mathbb{Z}\}$. When $\vartheta_0 \neq 0$, we have*

$$F(T,\chi) = 2\Re\Bigg\{\kappa^* \left(\frac{qT}{2\pi}\right)^{\frac{1}{4}} \sum_{r=0}^{2q-1} \tau^*(r,\chi) K_{q,r}\big(\frac{\vartheta}{2q}\big)\Bigg\} + O\left(\min\left\{T^{\frac{1}{8}}\vartheta_0^{-1}, T^{\frac{1}{4}}\right\}\right) + O\left(T^{\frac{1}{6}}\log T\right). \tag{2.9}$$

We next investigate the behaviour at the special points defined by $\vartheta_0 = 0$, the notation being as in Theorem 23.

Theorem 24. *When $\vartheta_0 = 0$, we have*

$$F(T,\chi) = \lambda_e(\vartheta)\left(\frac{qT}{2\pi}\right)^{\frac{1}{4}}\log T + \mu_e(\vartheta)\left(\frac{qT}{2\pi}\right)^{\frac{1}{4}} + O\left(T^{\frac{1}{6}}\log T\right) \tag{2.10}$$

where

$$\lambda_e(\vartheta) = \tfrac{1}{6q}\Re\left(-i\kappa^*\sum_{r=0}^{2q-1}\tau^*(r,\chi)e\left(-\tfrac{r\vartheta}{2q}\right)\right)$$

and

$$\mu_e(\vartheta) = 2\Re\left(-i\kappa^*\sum_{r=0}^{2q-1}\tau^*(r,\chi)e\left(-\tfrac{r\vartheta}{2q}\right)\gamma(1,-\tfrac{r}{2q})\right).$$

The quantities $\gamma(\varrho,\theta)$ are defined in (2.81) *on page 88.*

When both q and r are odd, put

$$\begin{aligned} K_{q,r}(x) \quad = \quad & -\frac{1}{4q}e\left(\tfrac{r}{8q}\right)\sum_{k=0}^{2q-1}(1-\delta_{q,r}(-1)^k i)e\left(\frac{kr}{4q}\right)\times \\ & \left\{\log\left(1-e(x+\frac{2k+1}{8q})\right) - \log\left(1+e(x+\frac{2k+1}{8q})\right)\right\} \end{aligned}$$

provided $x + \frac{2k+1}{8q} \notin \frac{1}{2}\mathbb{Z}$ for any of the k's, where $\delta_{q,r} = (-1)^{\frac{q-r}{2}}$. The corresponding results for odd conductors are given in the theorems which follow.

Theorem 25. *Let χ be an even character of odd conductor q. Let us write $\sqrt{qT/2\pi} = qL + \vartheta$ with $L \in \mathbb{N}$ and $0 \le \vartheta < q$. Let $\vartheta_0 = \min\{\left|\vartheta - \frac{2k+1}{4}\right| : k = 0, ..., 2q-1\}$. For $\vartheta_0 \ne 0$, we have*

$$\begin{aligned} F(T,\chi) = 2\Re\Big\{\kappa^*\left(\frac{qT}{2\pi}\right)^{\frac{1}{4}}\sum_{r=0}^{2q-1}\tau^*(r,\chi)K_{q,r}(\frac{\vartheta}{2q})\Big\} \\ + O\left(T^{\frac{1}{6}}\log T\right) + O\left(\min\{T^{\frac{1}{4}}, T^{\frac{1}{8}}\vartheta_0^{-1}\}\right). \end{aligned} \tag{2.11}$$

Here is a theorem describing the behaviour at the special points $\vartheta_0 = 0$.

Theorem 26. *For $\vartheta_0 = 0$ with $\vartheta = \frac{2j+1}{4}$, we have*

$$F(T,\chi) = (-1)^L\lambda_o(\vartheta)\left(\frac{qT}{2\pi}\right)^{\frac{1}{4}}\log T + (-1)^L\mu_o(\vartheta)\left(\frac{qT}{2\pi}\right)^{\frac{1}{4}} + O\left(T^{\frac{1}{6}}\log T\right) \tag{2.12}$$

where

$$\lambda_o(\vartheta) = \tfrac{1}{12q}\Re\left(-i\kappa^* \sum_{r=0}^{2q-1} \tau^*(r,\chi)e\left(-\tfrac{r\vartheta}{2q}\right)\left(1+\delta_{q,r}(-1)^j i\right)\right)$$

and

$$\mu_o(\vartheta) = \frac{1}{12q}\Re\left(-i\kappa^* \sum_{r=0}^{2q-1} \tau^*(r,\chi)\gamma_j(2,r)\right).$$

Here,

$$\gamma_j(\varrho,r) = \frac{1}{2q}e\left(-\frac{r\vartheta}{2q}\right)\left\{\gamma(\varrho,-\frac{r}{2q}) + (-1)^j i\delta_{q,r}\gamma(\varrho,1-\frac{r}{2q})\right\}$$

with $\gamma(\varrho,\theta)$ *as above.*

2.3 Omega results

The equations (2.10) and (2.12) provide omega results for the function $F(T,\chi)$, namely

Corollary 27. *Let* χ *be an even primitive character modulo an odd prime power* $q \leq 100$ *then*

$$F(T,\chi) = \Omega\left(T^{\frac{1}{4}}\right).$$

In fact, a Pari/GP computation shows that, for such characters, $\lambda_o\left(\frac{2j+1}{4}\right) = 0$ for all j of interest.

Corollary 28. *If* $\lambda_e(\vartheta)$ *(resp.* $\lambda_o(\vartheta)$ *in case of* q *odd) is nonzero for some* $\vartheta = \frac{n}{2}$ *(resp.* $\vartheta = \frac{2j+1}{4}$*), then the function* $F(T,\chi)$ *satisfies*

$$F(T,\chi) = \Omega\left(T^{\frac{1}{4}}\log T\right).$$

Remark. In fact, it appears that λ_o and λ_e always vanish at the interesting points, but we are in the process of proving it.

The Omega result trivially implies that $Z(t, \chi)$ has infinitely many zeros on the critical line $\Re s = \frac{1}{2}$.

The analogue of $Z(t)$ for the Selberg class of L-functions is briefly discussed in [36, Ch. 3]. The adaptation of $Z(t)$ to the Selberg class of L-functions, especially to Dirichlet L-functions, is evident; the functional equation for these functions dictates that Hardy's function is always real-valued on $\mathbb{R}$.

Further notations

Besides the notations we listed earlier, we introduce further notations for this chapter.

The symbol $\tau(\chi)$ denotes a linear Gauss sum and is defined in § 1.3; also, $\tau^*(r, \chi), \tau_*(r, \chi)$ are quadratic Gauss sums defined in the statement of Theorem 21. For the symbols κ, κ^* and κ_*, see (2.21), (2.67) and (2.68). For the symbols $\gamma(\varrho, \theta)$ and $\gamma_j(\varrho, r)$, see (2.81) and (2.88).

2.4 Outline of proof of Theorem 21

The idea is the following. We would like to find the Laplace transform of $F(T, \chi)$ in such a manner that the inversion formula would enable us to retrieve the function $F(T, \chi)$ in the form of a more or less simple series. How this is accomplished is explained in the ensuing paragraphs. The results stated are proved in later sections.

The function $Z(t, \chi)$ itself does *not* admit of a simple Laplace transform, so we look for a function which *does*, and which differs very little from $Z(t, \chi)$. For this, we first explicitly construct an analytic function $H(s, \chi)$ which has the properties $H(s, \chi) = 1 + O(|s|^{-1})$ and $H'(s, \chi) = O(|s|^{-2})$ as $|s| \to \infty$ in any fixed closed strip of finite width inside $\Re s > 0$. We then define the function we want by $Q(t, \chi) = H(\frac{1}{2} + it, \chi) Z(t, \chi)$. As desired, the function $Q(t, \chi)$ so defined has a simple Laplace transform:

Proposition 29. *Let χ be any primitive character* $\bmod q$. *For $0 < \Re p < \frac{\pi}{2}$,*

$$\check{Q}(p,\chi) = \kappa_1 e^{-ip} \sum_{n=1}^{\infty} n^{\frac{1}{2}} \chi(n) \exp(-\frac{\pi}{q} n^2 i e^{-2ip}) + \lambda(p,\chi), \tag{2.13}$$

where $\lambda(p,\chi)$ has an analytic extension to the strip $|\Re p| < \frac{\pi}{2}$ and satisfies

$$|\lambda(p,\chi)| \ll \frac{1}{|p|+1} \tag{2.14}$$

therein. Here, $\kappa_1 = 2\pi i \kappa q^{-\frac{1}{4}} e^{\frac{\pi i}{8}}$ is a constant, see (2.21).

(The proof is given at the beginning of § 2.6). Furthermore, the Laplace transform $\check{Z}(p,\chi)$ of $Z(t,\chi)$ differs very little from $\check{Q}(p,\chi)$ in the following sense:

Theorem 30. *For any $0 < \vartheta < 1$, we have*

$$\check{Z}(p,\chi) - \check{Q}(p,\chi) \ll \frac{1}{|p|} (\log \tfrac{1}{\Re p})^{\frac{9}{4}} \tag{2.15}$$

uniformly in $0 < \Re p \le \vartheta$.

(See page 55 for the proof). Given these, the Laplace transform of $F(T,\chi)$ being equal to $\check{F}(p,\chi) = \frac{\check{Z}(p,\chi)}{p}$ for $\Re p > 0$, the Laplace inversion formula gives

$$F(T,\chi) = 2\Re \left(\frac{1}{2\pi i} \int_a^{a+i\infty} \frac{\check{Q}(p,\chi)}{p} e^{pT} \, \mathrm{d}p \right) + O\big((\log T)^{\frac{5}{4}}\big), \quad (a > 0) \tag{2.16}$$

by (2.15). We choose $a = \frac{1}{T}$. Now, we have to deal with points close to the real axis, that is, those $p = a + it$ for which t is small. In fact, the exponential factor in the series expansion (2.13) has absolute value $\exp\big(-\frac{\pi}{q} n^2 e^{2t} \sin 2a\big)$ at $p = a + it$, which is big when t is close to 0 (note that $\sin 2a$ is close to $\frac{2}{T}$, which is very small for big T). Jutila's trick to deal with this situation is to show that $\check{Q}(p,\chi) \ll 1$ when $p = \frac{1}{T} + it$ and $0 \le t \ll \frac{1}{\sqrt{T}\log T}$. In this way, we can omit an interval of the form $[0, \frac{c}{\sqrt{T}\log T}]$ in the integral (2.16), so that (2.13) in turn leads to a series of the form

$$F(T,\chi)=\frac{e}{\pi}\Re\Bigg(\kappa_1 e^{-ia}\sum_{n=1}^{\infty}\sqrt{n}\chi(n)\times \int_0^{\infty}\exp\big(t-\frac{\pi}{q}n^2e^{2t}ie^{-2ia}+itT\big)\frac{w(t)}{a+it}\,\mathrm{d}t\Bigg)+O\left((\log T)^{\frac{5}{4}}\right). \tag{2.17}$$

where $w(t)$ is a smooth function with $w(t)=0$ for $t\le\frac{1}{\sqrt{T}\log T}$, $w(t)=1$ for $t\ge\frac{2}{\sqrt{T}\log T}$ and such that $0\le w(t)\le 1$ on $\mathbb{R}$. Moreover, the factor e^{-ia} can be ignored and the number e^{-2ia} inside the exponential in (2.17) can be replaced by $1-2ia$ with an error $\ll\log T$ (see **Step III** and **Step IV** of the proof, page 66). Thus we obtain

$$F(T,\chi)=\frac{e}{\pi}\Re\Bigg(\kappa_1\sum_{n=1}^{\infty}\sqrt{n}\chi(n)\times \int_0^{\infty}\exp\big(t-2a\frac{\pi}{q}n^2e^{2t}-\frac{\pi}{q}n^2e^{2t}i+itT\big)\frac{w(t)}{a+it}\,\mathrm{d}t\Bigg)+O\left((\log T)^{\frac{5}{4}}\right). \tag{2.18}$$

We further simplify this by the change of variable $u=e^{2t}-1$:

$$F(T,\chi)=\frac{e}{2\pi}\Re\Bigg(\kappa_1\sum_{n=1}^{\infty}\sqrt{n}\chi(n)e^{-\frac{\pi}{q}n^2i}\times \times\int_0^{\infty}\frac{\exp\big(-2a\frac{\pi}{q}n^2(1+u)-\frac{\pi}{q}n^2iu+\frac{iT}{2}\log(1+u)\big)}{\sqrt{1+u}} \times\frac{w(\frac{1}{2}\log(1+u))}{a+\frac{i}{2}\log(1+u)}\,\mathrm{d}u+O\left((\log T)^{\frac{5}{4}}\right). \tag{2.19}$$

At this point, to extract information out of the integrals in the sum, we have to check for *saddle-points*, that is, points at which the derivative of the function inside the oscillating exponential factor $\exp\big(-\frac{\pi}{q}n^2iu+\frac{iT}{2}\log(1+u)\big)$ vanishes; this is the point $u=\frac{qT}{2\pi n^2}-1$, which depends upon n. Accordingly, to deal with the various locations of this with respect to the interval of integration, the sum (2.19) is divided into three parts, namely with n in the intervals $[\sqrt{N'},\sqrt{N''}]$, $[1,\sqrt{N'}[$ and $[\sqrt{N''},\infty[$ respectively (N',N'' are as in the statement of Theorem 21). In fact, for n in the third interval, the corresponding integral in (2.19) has no saddle point, so integration by parts shows that these integrals have a small overall contribution to $F(T,\chi)$. For n in either one of the first two intervals, the corresponding integral has a saddle-point. For n in the interval $[1,\sqrt{N'}]$, a direct application of Atkinson's lemma on saddle-point method (see Theorem 20) to each term gives the term $S_2(T,\chi)$ immediately, which is a fairly easily recognisable term (zeta sum). Most of the difficulty is concentrated in the interval $[\sqrt{N'},\sqrt{N''}]$, and a direct application of saddle-point methods does not lead to good results. So we treat the sum over the whole interval as one term; applying Poisson summation formula (1.9) to the whole sum over this interval, we get new terms to which we can now apply results from saddle-point methods. In this way, we get the term $S_1(T,\chi)$. The details are given in several steps in § 2.7.

2.5 Auxiliary lemmas

In this section, besides stating several lemmas that will be used later, we construct the function $H(s,\chi)$ alluded to earlier and prove the properties stated thereof. We now define the function $H(s,\chi)$ as follows. First recall that

$$\begin{aligned}\Psi(s,\chi) &= \mathfrak{w}(\chi)\left(\frac{\pi}{q}\right)^{s-\frac{1}{2}}\frac{\Gamma(\frac{1}{2}(1-s+\mathfrak{a}))}{\Gamma(\frac{1}{2}(s+\mathfrak{a}))}\\ &= \frac{q^{-s}\Gamma(1-s)i}{(2\pi)^{1-s}}\left\{-e^{\frac{\pi si}{2}}+\chi(-1)e^{-\frac{\pi si}{2}}\right\}\tau(\chi)\end{aligned} \tag{2.20}$$

where the constant $\mathfrak{a}$ and the *root number* $\mathfrak{w}$ are classical quantities defined respectively in (1.1) and (1.4). Now write

$$\kappa = \kappa(\chi) = \sqrt{2/\mathfrak{w}}\,e^{\frac{\pi i}{4}(\mathfrak{a}-2)}q^{-\frac{1}{4}} \tag{2.21}$$

where for the square root of a complex number $z = |z|\,e^{i\theta}$ with $-\pi < \theta \leq \pi$, we take $\sqrt{z} = \sqrt{|z|}e^{\frac{i\theta}{2}}$. Then

$$H(s,\chi) = \pi^{-\frac{1}{4}}(\tfrac{\pi}{q})^{-\frac{s}{2}}\kappa\sin\tfrac{\pi s}{4}\Psi(s,\chi)^{\frac{1}{2}}\Gamma(\tfrac{1}{4}+\tfrac{s}{2}). \tag{2.22}$$

We prove the asymptotic properties of $H(s,\chi)$ mentioned earlier.

Lemma 31. *In any closed vertical strip of finite width within the half-plane $\Re s > 0$, we have*

$$H(s) = 1 + O(1/|s|) \tag{2.23}$$

and

$$H'(s,\chi) \ll 1/|s|^2 \tag{2.24}$$

uniformly.

Proof. Stirling's formula gives us $\Gamma(s) = \sqrt{2\pi}e^{-it}e^{(s-\frac{1}{2})(\log t+i\frac{\pi}{2})}\big(1 + O(\frac{1}{t})\big)$ uniformly, with σ varying, say in the band $-\infty < \alpha \le \sigma \le \beta < +\infty$. Using this, we have

$$\Gamma(1-s)^{\frac{1}{2}}\Gamma(\frac{1}{4}+\frac{s}{2}) = (2\pi)^{\frac{3}{4}}2^{\frac{1}{4}-\frac{s}{2}}e^{\frac{\pi s i}{2}-\frac{\pi i}{4}}\big(1+O(\frac{1}{t})\big).$$

Now, expanding $H(s,\chi)$ using the expression (2.20) for $\Psi(s,\chi)$ and using the fact that $\sin\frac{\pi s}{4} = \frac{1}{2}e^{-\frac{\pi s i}{4}+\frac{\pi i}{2}} + O(e^{-\frac{\pi t}{4}})$ and $-e^{\frac{\pi s i}{2}} = O(e^{-\frac{\pi t}{2}})$, we immediately obtain (2.23). To prove (2.24), taking the logarithmic derivative of H and using the two properties of the digamma function stated above, we have

$$\begin{aligned}\tfrac{H'}{H}(s,\chi) &= \tfrac{1}{2}\log q - \tfrac{1}{2}\log\pi + \tfrac{\pi}{4}\frac{\cos\frac{\pi s}{4}}{\sin\frac{\pi s}{4}}\\ &+ \tfrac{1}{2}\log\tfrac{\pi}{q} - \tfrac{1}{4}\tfrac{\Gamma'}{\Gamma}(\tfrac{1}{2}(1-s+\mathfrak{a})) - \tfrac{1}{4}\tfrac{\Gamma'}{\Gamma}(\tfrac{1}{2}(s+\mathfrak{a}))\\ &+ \tfrac{1}{2}\tfrac{\Gamma'}{\Gamma}(\tfrac{s}{2}+\tfrac{1}{4})\\ &= \tfrac{1}{4}\log\tfrac{(2s+1)^2}{4(1+s-\mathfrak{a})(s+\mathfrak{a})} + \tfrac{1-4\mathfrak{a}+4\mathfrak{a}^2}{4(2s+1)(1+s-\mathfrak{a})(s+\mathfrak{a})} + O(\tfrac{1}{|s|^2}) = O(\tfrac{1}{|s|^2}).\end{aligned}$$

The first equality comes from direct differentiation, the second equality from (1.13) and (1.14), and the last equality from the fact that $(2s+1)^2/(4s(s+1)) = 1 + O(1/|s|^2)$. Thus $\frac{H'}{H}(s,\chi) \ll |s|^{-2}$; this then gives (2.24). □

We next need results on the moments of Z and its derivatives (see [64]):

Lemma 32. *We have*

$$\frac{1}{T}\int_T^{2T} \left|Z^{(\nu)}(t,\chi)\right| \,\mathrm{d}t \ll (\log T)^{\frac{1}{4}+\nu}$$

for $\nu \geq 0$ *and all* $T > 1$.

Using this last lemma, we can prove

Proposition 33. *Let* $0 < \vartheta < 1$. *Then for* $\nu = 0, 1$,

$$\int_0^\infty \frac{\left|Z^{(\nu)}(t,\chi)\right|}{(1+t)^{2-\nu}} e^{-xt}\,\mathrm{d}t \ll (\log\frac{1}{x})^{\frac{5}{4}+\nu}$$

uniformly in $0 < x \leq \vartheta$.

Proof. We have

$$\begin{aligned}\int_0^\infty \frac{\left|Z^{(\nu)}(t,\chi)\right|}{(1+t)^{2-\nu}} e^{-xt}\,\mathrm{d}t \leq \int_0^1 \frac{\left|Z^{(\nu)}(t,\chi)\right|}{(1+t)^{2-\nu}} e^{-xt}\,\mathrm{d}t \\ + \sum_{k=0}^\infty \frac{e^{-2^k x}}{(1+2^k)^{2-\nu}} \int_{2^k}^{2^{k+1}} \left|Z^{(\nu)}(t,\chi)\right| \,\mathrm{d}t.\end{aligned}$$

Lemma 32 gives

$$\begin{aligned}\sum_{k=0}^\infty \frac{e^{-2^k x}}{(1+2^k)^{2-\nu}} \int_{2^k}^{2^{k+1}} \left|Z^{(\nu)}(t,\chi)\right| \,\mathrm{d}t &\ll \sum_{k=0}^\infty \frac{e^{-2^k x}}{(1+2^k)^{2-\nu}} 2^k (\log 2^k)^{\frac{1}{4}+\nu} \\ &\ll \sum_{k=1}^\infty \frac{e^{-2^k x}}{(1+2^k)^{1-\nu}} k^{\frac{1}{4}+\nu}.\end{aligned}$$

We then use the estimates (2.93) and (2.94) in the appendix. □

2.6 The Laplace transform of $Q(t,\chi)$

In this section, we prove Proposition 29 and Theorem 30. We also prove a key lemma that will allow us to simplify some integrals in later sections.

Proof of Proposition 29. To evaluate the integral, let us consider the function

$$J(p,\chi)=\int_{(\frac{1}{2})}\frac{H(s,\chi)L(s,\chi)\Psi(1-s,\overline{\chi})^{\frac{1}{2}}e^{is(p-\frac{\pi}{4})}}{2i\sin\frac{\pi s}{4}}\,\mathrm{d}s$$

defined in the strip $0<\Re p<\frac{\pi}{2}$. We first evaluate this integral explicitly. Since the character χ is assumed to be primitive and hence nonprincipal (as $q>1$), the L-series is convergent for $\Re s>0$; moreover, the integral is exponentially convergent in this strip, so we can do term-by-term integration to obtain

$$\begin{aligned}J(p,\chi)&=-\frac{1}{2}i\kappa\pi^{-\frac{1}{4}}\sum_{n=1}^{\infty}\chi(n)\int_{(\frac{1}{2})}\left(\frac{\pi}{q}n^2ie^{-2ip}\right)^{-\frac{s}{2}}\Gamma(\frac{1}{4}+\frac{s}{2})\,\mathrm{d}s\\&=2\pi\kappa q^{-\frac{1}{4}}e^{\frac{\pi i}{8}}e^{-\frac{ip}{2}}\sum_{n=1}^{\infty}\chi(n)n^{\frac{1}{2}}\exp(-\frac{\pi}{q}n^2ie^{-2ip}),\end{aligned}$$

where we have used the Cahen-Mellin formula.

On the other hand,

$$\begin{aligned}J(p,\chi)&=\int_{-\infty}^{\infty}\frac{H(\frac{1}{2}+it,\chi)L(\frac{1}{2}+it,\chi)\Psi(\frac{1}{2}+it,\chi)^{-\frac{1}{2}}e^{i(\frac{1}{2}+it)(p-\frac{\pi}{4})}}{e^{\frac{\pi}{4}(\frac{1}{2}+it)i}-e^{-\frac{\pi}{4}(\frac{1}{2}+it)i}}i\,\mathrm{d}t\\&=ie^{\frac{ip}{2}}\big(\lambda(p,\chi)-\check{Q}(p,\chi)\big)\end{aligned}\tag{2.25}$$

where

$$\lambda(p,\chi)=-e^{-\frac{\pi i}{4}}\int_0^{\infty}g(t)e^{-(p+\frac{\pi}{2})t}\,\mathrm{d}t+\int_0^{\infty}h(t)e^{-(\frac{\pi}{2}-p)t}\,\mathrm{d}t\tag{2.26}$$

with

$$g(t)=\frac{e^{\frac{\pi t}{4}}}{e^{\frac{\pi t}{4}}-e^{-\frac{\pi t}{4}+\frac{\pi i}{4}}}Q(t,\chi),\quad h(t)=\frac{e^{\frac{\pi t}{4}}}{e^{\frac{\pi i}{4}}e^{\frac{\pi t}{4}}-e^{-\frac{\pi t}{4}}}\overline{Q(t,\overline{\chi})}.$$

It is easily seen that the first integral in (2.26) converges for $\Re p>-\frac{\pi}{2}$ and the second integral converges for $\Re p<\frac{\pi}{2}$. Hence, $\lambda(p,\chi)$ is analytic in the strip $|\Re p|<\frac{\pi}{2}$. It only remains to prove (2.14). It suffices to prove that both integrals in (2.26) are $\ll\frac{1}{|p|+1}$. Integrating the first integral by parts with respect to the factor $g(t)$ shows that $\int_0^\infty g(t)e^{-(p+\frac{\pi}{2})t}\,\mathrm{d}t \ll \frac{1}{|p|+1}$. Similarly, the second integral $\ll \frac{1}{|p|+1}$ and the proof of the lemma is complete. It is (2.25) which gives (2.13). □

We have evaluated $\check{Q}(p,\chi)$ in the preceding Proposition. We now show how it approximates $\check{Z}(p,\chi)$, as stated in Theorem 30:

Proof of Theorem 30 . We look at the difference

$$\check{Z}(p,\chi) - \check{Q}(p,\chi) = \int_0^\infty Z(t,\chi)\left(H(\tfrac{1}{2}+it,\chi)-1\right)e^{-pt}\,\mathrm{d}t.$$

Integrating by parts with respect to e^{-pt} we obtain

$$\begin{aligned}\check{Z}(p,\chi) - \check{Q}(p,\chi) = \frac{Z(0,\chi)H(\frac{1}{2},\chi)}{p} &+ \frac{1}{p}\int_0^\infty Z'(t,\chi)\left(H(\tfrac{1}{2}+it,\chi)-1\right)e^{-pt}\,\mathrm{d}t \\ &+ \frac{i}{p}\int_0^\infty Z(t,\chi)H'(\tfrac{1}{2}+it,\chi)e^{-pt}\,\mathrm{d}t.\end{aligned}$$

Now, using the asymptotics (2.23) and (2.24), we have

$$\int_0^\infty Z'(t,\chi)\left(H(\tfrac{1}{2}+it,\chi)-1\right)e^{-pt}\,\mathrm{d}t \ll \int_0^\infty \frac{|Z'(t,\chi)|}{1+t}e^{-xt}\,\mathrm{d}t$$

and

$$\int_0^\infty Z(t,\chi)H'(\tfrac{1}{2}+it,\chi)e^{-pt}\,\mathrm{d}t \ll \int_0^\infty \frac{|Z(t,\chi)|}{(1+t)^2}e^{-xt}\,\mathrm{d}t$$

where we have written $x = \Re p$. We now appeal to Proposition 33 to get the result straight away. □

We now reach a turning point in the paper. Whatever we have proved of Z and Q are valid for *all* primitive characters of *any* conductor q. However, we prove the following lemma (Lemma 34) only for certain primitive characters. We remark that there are infinitely many even primitive characters, such as the characters χ_q given by the Jacobi symbol $\chi_q = \left(\frac{\cdot}{q}\right)$ where q is squarefree and $q \equiv 1 \pmod 4$.

In general, (1.6) is not true of odd characters, as is shown by the character $\chi_8 \bmod 8$ given by $\chi_8(\pm 1) = \pm 1 = \chi_8(\pm 3)$. Let χ_4 denote the character $\bmod\, 4$

given by $\chi_4(\pm 1) = \pm 1$.

Lemma 34. *Let χ be an even primitive character of odd conductor or one of the odd characters χ_4 and χ_8. Let $p = a + iu$ where $a = \frac{1}{T}$ and T is sufficiently large. Then for $0 \le u \ll \frac{1}{\sqrt{T}\log T}$ we have*

$$\check{Q}(p, \chi) \ll 1.$$

Proof. By Proposition 29, it suffices to prove that

$$\sum_{n=1}^{\infty} n^{\frac{1}{2}} \chi(n) \exp(-\frac{\pi}{q} n^2 i e^{-2ip}) \ll 1 \tag{2.27}$$

for p as in the statement of the lemma. With $p = a + iu$ one has

$$\exp(-\tfrac{\pi}{q} n^2 i e^{-2ip}) \quad = \quad \exp(-i\tfrac{\pi}{q} n^2 e^{2u} \cos 2a) \exp(-\tfrac{\pi}{q} n^2 e^{2u} \sin 2a).$$

Since $a = \frac{1}{T}$ and $0 \le u \ll \frac{1}{\sqrt{T}\log T}$, we have the estimates $\sin 2a \ge \frac{4}{\pi T}$, $e^{2u} \ge 1$ and hence

$$\exp(-\tfrac{\pi}{q} n^2 e^{2u} \sin 2a) \quad \le \quad \exp(-\tfrac{4}{qT} n^2).$$

Therefore, for $N_0 \ge 0$,

$$\begin{aligned} \sum_{n \ge N_0} \left| n^{\frac{1}{2}} \chi(n) \exp(-\tfrac{\pi}{q} n^2 i e^{-2ip}) \right| \quad &\le \quad \sum_{n \ge N_0} n^{\frac{1}{2}} \exp(-\tfrac{\pi}{q} n^2 e^{2u} \sin 2a) \\ &\le \quad \int_{N_0}^{\infty} \sqrt{t} \exp(-\tfrac{4}{qT} t^2) \, \mathrm{d}t \\ &\ll \quad \frac{e^{-\frac{1}{2} b^2 N_0^2}}{b\sqrt{b}}, \text{ where } b = \frac{2}{\sqrt{qT}}. \end{aligned}$$

Setting $N_0^2 = \frac{qT}{4} \log T$, we have $N_0 \asymp \sqrt{T \log T}$ and $\frac{e^{-\frac{1}{2} b^2 N_0^2}}{b\sqrt{b}} \le 1$. Therefore, with this N_0, we see that

$$\sum_{n \ge N_0} \left| n^{\frac{1}{2}} \chi(n) \exp(-\tfrac{\pi}{q} n^2 i e^{-2ip}) \right| \ll 1.$$

Hence it is enough to show that

$$\sum_{n\le N_0} n^{\frac12}\chi(n)\exp(-\frac{\pi}{q}n^2ie^{-2ip}) \ll 1$$

for $N_0 \ll \sqrt{T\log T}$. Let us write $\psi(x)=\sqrt{x}\exp\left((1-e^{-2ip})\frac{\pi}{q}x^2i\right)$ so that

$$\psi(n)\exp(-\tfrac{\pi}{q}n^2i)=\sqrt{n}\exp(-\frac{\pi}{q}n^2ie^{-2ip})$$

and

$$\psi(x)\ll\sqrt{N}\exp(-\frac{4}{qT}N^2)$$

for $x\in[N,2N]$. Now, let us take a partition of unity $w_1,w_2,w_3,\dots$ subordinate to the cover $[2^k,2^{k+1}], k\ge 0$ of $[1,N_0]$ where N_0 is some power of 2, as may be arranged easily. Let us suppose further that $w_n^{(\nu)}(t)\ll 2^{-\nu n}$ for sufficiently many derivatives $w_n^{(\nu)}$ of w_n. We are going to show that

$$\sum_{k=1}^{\infty} w_n(k)\chi(k)\exp(-\frac{\pi}{q}k^2i)\psi(k)\ll 2^{-\frac{n}{10}}. \tag{2.28}$$

For if (2.28) were proved, we would have

$$\sum_{n\le\log_2 N_0}\sum_{k=1}^{\infty} w_n(k)\chi(k)\exp(-\frac{\pi}{q}k^2i)\psi(k) \ll \sum_{n=1}^{N_0}\frac{1}{2^{\frac{n}{10}}}\le\sum_{n=1}^{\infty}\frac{1}{2^{\frac{n}{10}}}\ll 1$$

which gives (2.27).

We first do this for smaller u, namely for $0\le u\ll 2^{-\frac{9n}{5}}$. We treat the three cases separately:

Case 1. **Even χ of odd conductor**. We apply the quadratic Taylor formula to the function $f(x)=w_n(x)\psi(x)$ at $x=2qk\pm r$ as follows: for $r=1,\dots,q$, there exist $x_{\pm r}$ with $2qk<x_r<2qk+r$, $2qk-r<x_{-r}<2qk$ such that

$$f(2qk\pm r)=f(2qk)\pm rf'(2qk)+\tfrac12 r^2f''(x_{\pm r}).$$

This gives, by Lemma 1,

$$\sum_{r=-q}^{q} f(2qk+r)\chi(2qk+r)e^{-\frac{\pi}{q}(2qk+r)^2 i} = O_k(f'') \tag{2.29}$$

for even characters χ.

Case 2. **The character** χ_4. For the character χ_4 an analogue of (2.29) holds as follows:

$$\begin{aligned} f(4k-1) &= f(4k+1) - 2f'(4k+1) + \tfrac{1}{2}f''(t_k), \\ f(4k+3) &= f(4k+1) + 2f'(4k+1) + \tfrac{1}{2}f''(u_k), \end{aligned}$$

for some $4k-1 < t_k < 4k+1 < u_k < 4k+3$, so adding these two equations, we have

$$\begin{aligned} -\tfrac{1}{2}f(4k-1) + f(4k+1) - \tfrac{1}{2}f(4k+3) &= -\tfrac{1}{4}\big(f''(t_k) + f''(u_k)\big) \\ &= O_k(f''). \end{aligned} \tag{2.30}$$

Case 3. **The character** χ_8. This can be done almost exactly as for χ_4:

$$\begin{aligned} f(8k-1) &= f(8k+1) - 2f'(8k+1) + \tfrac{1}{2}f''(v_k), \\ f(8k+3) &= f(8k+1) + 2f'(8k+1) + \tfrac{1}{2}f''(y_k), \\ f(8k+7) &= f(8k+5) + 2f'(8k+5) + \tfrac{1}{2}f''(a_k), \\ f(8k+3) &= f(8k+5) - 2f'(8k+5) + \tfrac{1}{2}f''(b_k) \end{aligned}$$

for a_k, b_k, v_k, y_k in appropriate intervals. Adding all the equations, we obtain

$$\tfrac{1}{2}f(8k-1) - f(8k+1) + f(8k+3) - f(8k+5) + \tfrac{1}{2}f(8k+7) = O(f''). \tag{2.31}$$

Summing over k (after multiplication by appropriate factors if necessary), the left hand side of each of the equations (2.29), (2.30) and (2.31) gives the sum (2.28). Hence it is enough to prove that

$$f'' \ll 2^{-\frac{11n}{10}} \tag{2.32}$$

uniformly in $[2^n, 2^{n+1}]$. Now, $f''(x) = w_n''(x)\psi(x) + 2w_n'(x)\psi'(x) + w_n(x)\psi''(x)$. We note that

$$\psi'(x) = \alpha(x)\psi(x) \tag{2.33}$$

where $\alpha(x) = \frac{1}{2x} + \frac{2\pi}{q}xi(1 - e^{-2ip})$; so $\psi''(x) = (\alpha'(x) + \alpha(x)^2)\psi(x)$. For $x \in [2^n, 2^{n+1}] = [N, 2N]$, $w_n^{(j)}(x) \ll 2^{-nj}$, so

$$\psi'(x) \ll \frac{\exp(-\frac{4N^2}{qT})}{\sqrt{N}}(1 + N^2u)$$

and

$$\psi''(x) \ll \frac{\exp(-\frac{4N^2}{qT})}{\sqrt{N}}\left(Nu + N^3u^2 + \frac{1}{N}\right).$$

Hence, for $x \in [2^n, 2^{n+1}] = [N, 2N]$ we have

$$\begin{aligned} f''(x) &\ll N^{-2}\sqrt{N}\exp(-\tfrac{N^2}{T}) + N^{-1}\frac{\exp(-\frac{N^2}{T})}{\sqrt{N}}(1 + N^2u) \\ &+ \frac{\exp(-\frac{4N^2}{qT})}{\sqrt{N}}\left(Nu + N^3u^2 + \frac{1}{N}\right) \\ &\ll \sqrt{N}(N^{-\frac{9}{5}} + N^2N^{-\frac{18}{5}} + N^{-2}) \ll N^{-\frac{11}{10}}. \end{aligned}$$

Thus (2.32) is proved for u small, namely $0 \le u \ll N^{-\frac{9}{5}}$.

For bigger u, namely for $N^{-\frac{9}{5}} \le u \ll \frac{1}{\sqrt{T}\log T}$, we use the Poisson summation formula to the series $S_r \stackrel{\text{def}}{=} \sum_{k=0}^{\infty} w_n(qk + r)\exp(-\frac{\pi}{q}(qk + r)^2 i)\psi(qk + r)$ separately for $r = 1, ..., q$. First note that

$$S_r = e^{-\frac{\pi}{q}r^2 i}\sum_{k=0}^{\infty}(-1)^{qk}w_n(qk + r)\psi(qk + r);$$

hence, for q even, we simply have

$$\begin{aligned} S_r &= e^{-\frac{\pi r^2}{q}i} \sum_{k=0}^{\infty} w_n(qk+r)\psi(qk+r) \\ &= e^{-\frac{\pi r^2}{q}i} \sum_{m\in\mathbb{Z}} \int_{-\infty}^{\infty} w_n(qt+r)\psi(qt+r)e(-mt)\,\mathrm{d}t \\ &= \frac{e^{-\frac{\pi r^2}{q}i}}{q} \sum_{m\in\mathbb{Z}} e\left(\frac{mr}{q}\right) \int_N^{2N} g(x)e\left(f_m(x)\right)\,\mathrm{d}x \end{aligned} \tag{2.34}$$

where $g(x) = w_n(x)\sqrt{x}\exp(-\frac{\pi}{q}x^2e^{2u}\sin 2a)$ and $f_m(x) = -\frac{mx}{q}+(1-e^{2u}\cos 2a)\frac{1}{2q}x^2$. For q odd, $(-1)^{qk} = (-1)^k$ so we apply (1.10) to S_r and get

$$S_r = \frac{e^{-\frac{\pi r^2}{q}i}}{q} \sum_{m\in\mathbb{Z}} e\left(\frac{r}{q}(m+\frac{1}{2})\right) \int_N^{2N} g(x)e\left(f_m(x) - \frac{1}{2q}x\right)\,\mathrm{d}x. \tag{2.35}$$

We estimate the integral in (2.34) as follows; the treatment of (2.35) being the same *mutatis mutandis*, we only consider (2.34).

Case 1. $m \neq 0$. The integral in (2.34) is $I = \int_N^{2N} f(x)e\left(-\frac{mx}{q}\right)\,\mathrm{d}x$. Doing integration by parts twice with respect to $e\left(-\frac{mx}{q}\right)$ gives

$$I = \int_N^{2N} g(x)\exp(2\pi i f_m(x))\,\mathrm{d}x = -\frac{q^2}{4\pi^2 m^2}\int_N^{2N} f''(x)e\left(-\frac{mx}{q}\right)\,\mathrm{d}x. \tag{2.36}$$

Now

$$f''(x) = w_n''(x)\psi(x) + 2w_n'(x)\psi'(x) + w_n(x)\psi''(x)$$

and

$$\begin{aligned} \psi''(x) &= \left(\alpha'(x) + \alpha(x)^2\right)\psi(x) \\ &= \left(-\tfrac{1}{4x^2} + \frac{4\pi i}{q}(1-e^{-2ip}) - \tfrac{4\pi^2}{q^2}(1-e^{-2ip})^2x^2\right)\psi(x); \end{aligned}$$

substituting these into (2.36), we obtain

$$\left(1+\frac{q}{\pi m^2}i(1-e^{-2ip})\right)I$$
$$=-\frac{q}{4\pi^2m^2}\left(\int_N^{2N} w_n''(x)\psi(x)e\left(-\frac{mx}{q}\right)\mathrm{d}x+2\int_N^{2N} w_n'(x)\psi'(x)e\left(-\frac{mx}{q}\right)\mathrm{d}x\right.$$
$$\left.-\frac{1}{4}\int_N^{2N}\frac{w_n(x)}{x^2}\psi(x)e\left(-\frac{mx}{q}\right)\mathrm{d}x-\frac{4\pi^2}{q^2}(1-e^{-2ip})^2I_2'\right) \quad (2.37)$$

where $I_2'=\int_N^{2N} w_n(x)x^2\psi(x)e\left(-\frac{mx}{q}\right)\mathrm{d}x$. Let us treat each term separately. Since $\psi(x)\ll\sqrt{N}\exp(-\frac{N^2}{T})$, the first and third integrals are trivially $\ll N^{-\frac{1}{2}}$. Using (2.33), we may write the second integral as

$$2\int_N^{2N} w_n'(x)\psi'(x)e\left(-\frac{mx}{q}\right)\mathrm{d}x=\int_N^{2N}\frac{w_n'(x)\psi(x)}{x}e\left(-\frac{mx}{q}\right)\mathrm{d}x$$
$$+\frac{4\pi i}{q}(1-e^{-2ip})\int_N^{2N} w_n'(x)x\psi(x)e\left(-\frac{mx}{q}\right)\mathrm{d}x. \quad (2.38)$$

Here it is clear that the first integral is again $\ll N^{-\frac{1}{2}}$. As for the second, integrating by parts only once with respect to the factor $e\left(f_m(x)\right)$, we easily see that it is $\ll N^{\frac{5}{2}}/T$. Therefore, the contribution of the second term in (2.38) is $\ll uN^{\frac{5}{2}}/T\ll N^{-\frac{1}{10}}$.

It remains to prove that the fourth term in (2.37) has a contribution $\ll N^{-\frac{1}{10}}$. For this, it is enough to show that $I_2'\ll\frac{N^{\frac{11}{2}}}{T^2}$, because then the contribution of the fourth term would be $\ll u^2\frac{N^{11/2}}{T^2}\ll N^{-\frac{1}{10}}$. Repeated integration by parts with respect to the factor $e\left(f_m(x)\right)$ gives

$$I_2'=\frac{1}{2\pi i}\int_N^{2N}\frac{w_n(x)x^2\sqrt{x}\exp\left(-\frac{\pi}{q}x^2e^{2u}\sin 2a\right)}{f_m'(x)}\mathrm{d}\Big(e\big(f_m(x)\big)\Big)$$
$$=-\frac{1}{2\pi i}\int_N^{2N}\left(\frac{V_0(x)}{f_m'(x)}\right)'e\big(f_m(x)\big)\mathrm{d}x=-\frac{1}{4\pi^2}\int_N^{2N}V_2(x)e\big(f_m(x)\big)\mathrm{d}x$$

where $V_0(x)=w_n(x)x^2\sqrt{x}\exp(-\frac{\pi}{q}x^2e^{2u}\sin 2a)$ and $V_j(x)=(\frac{V_{j-1}(x)}{f_m'(x)})'$

for $j = 1, 2$. Thus,

$$V_j(x) = \frac{V'_{j-1}(x)}{f'_m(x)} - f''_m(x)\frac{V_{j-1}(x)}{f'_m(x)^2}.$$

It is easily verified that $V_0(x) \ll N^{\frac{5}{2}}$, $V'_0(x) \ll N^{\frac{7}{2}}/T$ so that

$$V_1(x) \ll \tfrac{N^{\frac{7}{2}}}{T} + uN^{\frac{5}{2}} \ll \tfrac{N^{\frac{7}{2}}}{T}.$$

Similarly, $V'_1(x) \ll \frac{N^{\frac{9}{2}}}{T^2}$ so $V_2(x) \ll T^{-2}N^{\frac{9}{2}} + uT^{-1}N^{\frac{7}{2}} \ll T^{-2}N^{\frac{9}{2}}$. It follows that $I'_2 \ll T^{-2}N^{\frac{11}{2}}$ and the proof is complete.

Case 2. $m = 0$. This case can be treated similarly as we have treated I'_2. Here, we integrate by parts repeatedly with respect to $\exp\left((1-e^{-2ip})\frac{\pi}{q}x^2 i\right)$ as follows:

$$\begin{aligned}
\int_N^{2N} w_n(x)\psi(x)\,\mathrm{d}x &= \frac{q}{2\pi i(1-e^{-2ip})}\int_N^{2N}\frac{w_n(x)}{\sqrt{x}}\,\mathrm{d}\exp((1-e^{-2ip})\frac{\pi}{q}x^2 i) \\
&= -\frac{q}{2\pi i(1-e^{-2ip})}\int_N^{2N} U_0(x)\exp((1-e^{-2ip})\frac{\pi}{q}x^2 i)\,\mathrm{d}x \\
&= (-1)^{j+1}\left(\tfrac{q}{2\pi i(1-e^{-2ip})}\right)^{j+1}\int_N^{2N} U_j(x)\exp\left((1-e^{-2ip})\frac{\pi}{q}x^2 i\right)\mathrm{d}x
\end{aligned}$$

where $U_0(x) = (\frac{w_n(x)}{\sqrt{x}})', U_{j+1}(x) = (\frac{U_j(x)}{x})', j \geq 0$. We may easily prove by induction that

$$U_j(x) \ll \tfrac{1}{N^{\frac{3}{2}+2j}}, \quad U'_j(x) \ll \tfrac{1}{N^{\frac{3}{2}+2j+1}}, \quad (j \geq 0).$$

Hence,

$$\begin{aligned}
\int_N^{2N} w_n(x)\psi(x)\,\mathrm{d}x &\ll u^{-j-1}N(N^{-\frac{3}{2}-2j}) \\
&\ll N^{\frac{9}{5}(j+1)}N^{-2j-\frac{1}{2}} = N^{-\frac{j}{5}+\frac{13}{10}} \ll N^{-\frac{1}{10}}
\end{aligned}$$

as soon as $j \geq 7$.

Thus, the contribution of each term is $\ll N^{-\frac{1}{10}}$. It follows that

$$S_r \ll N^{-\frac{1}{10}} + \sum_{m \in \mathbb{Z} \setminus 0} \frac{1}{m^2} N^{-\frac{1}{10}} \ll N^{-\frac{1}{10}}.$$

This completes the proof of the lemma. □

2.7 Proof of the main theorem

We now prove Theorem 21. The proof is divided into several steps.

First of all, the Laplace transform of $F(T,\chi) = \int_0^T Z(t,\chi)\,\mathrm{d}t$ is $\check{F}(p,\chi) = \frac{\check{Z}(p,\chi)}{p}$ (integrating by parts). Therefore, the Laplace inversion formula gives

$$F(T,\chi) = \frac{1}{2\pi i} \int_{(a)} \frac{\check{Z}(p,\chi)}{p} e^{pT}\,\mathrm{d}p \tag{2.39}$$

for $a > 0$. We choose henceforward $a = 1/T$.

Step I. Expressing F in terms of $\check{Q}$

We have

$$F(T,\chi) = 2\Re \left(\frac{1}{2\pi i} \int_a^{a+i\infty} \frac{\check{Q}(p,\chi)}{p} e^{pT}\,\mathrm{d}p \right) + O\left((\log T)^{\frac{5}{4}} \right). \tag{2.40}$$

Proof. From (2.39)

$$F(T,\chi) = 2\Re \left(\frac{1}{2\pi i} \int_a^{a+i\infty} \frac{\check{Q}(p,\chi)}{p} e^{pT}\,\mathrm{d}p \right) + F_0(T,\chi) \tag{2.41}$$

with $F_0(T,\chi) = 2\Re \left(\frac{1}{2\pi i} \int_a^{a+i\infty} \frac{\check{Z}(p,\chi) - \check{Q}(p,\chi)}{p} e^{pT}\,\mathrm{d}p \right)$. We will show that $F_0(T,\chi) \ll (\log T)^{5/4}$.

We consider the integral

$$G_0(T,\chi) = \frac{1}{2\pi i}\int\limits_a^{a+i\infty} \frac{\check{Z}(p,\chi)-\check{Q}(p,\chi)}{p} e^{pT}\,\mathrm{d}p$$

for $0 < a \leq \vartheta < 1$. Let us separate the integral into two u-ranges, $[0,\log T]$ and $]\log T,\infty[$. Theorem 30 gives immediately that the integral over the range $u > \log T$ is $\ll (\log T)^{5/4}$. It now remains to estimate the integral over the range 0 to $\log T$, which we write in the form

$$\int_{a+i0}^{a+i\log T}\left(\Big(\int_0^{T/2}+\int_{T/2}^{\infty}\Big) Z(t,\chi)\big(1-H(\tfrac12+it,\chi)\big)e^{-pt}\,\mathrm{d}t\right)\frac{e^{pT}}{p}\,\mathrm{d}p. \tag{2.42}$$

Now, the inner integral over $t \geq \frac{T}{2}$ is

$$\begin{aligned}\int_{T/2}^{\infty} Z(t,\chi)\big(1-H(\tfrac12+it,\chi)\big)e^{-pt}\,\mathrm{d}t &\ll \int_{T/2}^{\infty}\frac{|Z(t,\chi)|}{t+1}e^{-t\Re p}\,\mathrm{d}t\\ &\leq \sum_{k=0}^{\infty}\frac{e^{-2^{k-1}T\Re p}}{2^{k-1}T+1}\int_{2^{k-1}T}^{2^kT}|Z(t,\chi)|\,\mathrm{d}t.\end{aligned}$$

Using Lemma 32 and Proposition 33, we immediately see that this is $\ll (\log T)^{1/4}$. Hence the inner integral over $t \geq \frac{T}{2}$ has a contribution $\ll (\log T)^{\frac14}\log\log T$ to $G_0(T,\chi)$.

Let us next estimate the integral over $0 \leq t \leq \frac{T}{2}$. Integrating by parts once with respect to the factor $e^{(T-t)p}$, we easily see that $\int_{a+i0}^{a+i\log T}\frac{e^{(T-t)p}}{p}\,\mathrm{d}p \ll 1$. It follows immediately that the integral over $0 \leq t \leq \frac{T}{2}$ is $\ll \int_0^{T/2}\frac{|Z(t,\chi)|}{t+1}\,\mathrm{d}t$. To estimate this last integral, let us choose n such that $2^{n-1} < T \leq 2^n$ (i.e., $n-1 < {}^{\log T}\!/_{\log 2} \leq n$) and let us look at the integral $\int_0^{2^{n-1}}\frac{|Z(t,\chi)|}{t+1}\,\mathrm{d}t$. Note that this is no smaller than the integral over $0 \leq t \leq \frac{T}{2}$ that we are about to estimate. We express this integral as a sum of the integrals $\int_0^1, \int_{2^k}^{2^{k+1}}, k = 0,1,...,n-2$ and see that the k^{th} summand $\int_{2^k}^{2^{k+1}}$ is $\ll (\log 2^k)^{1/4}$ by Lemma 32. Hence their sum is

$\ll \sum_{k=1}^{n-1} k^{\frac{1}{4}}$ which is in turn $\ll (\log T)^{5/4}$. That is,

$$\int_0^{T/2} \frac{|Z(t,\chi)|}{t+1}\,\mathrm{d}t \ll (\log T)^{\frac{5}{4}}.$$

The result follows, that is, we have $F_0(T,\chi) \ll (\log T)^{5/4}$. □

Step II. Removing the tiny ordinates

We have, with $u_0 = \frac{1}{\sqrt{T}\log T}$*,*

$$F(T,\chi) = 2\Re\Bigg(i\kappa q^{-\frac{1}{4}} e^{1+\frac{\pi i}{8}-ia} \sum_{n=1}^{\infty} \sqrt{n}\chi(n) \times \int_0^{\infty} \exp\left(t - \frac{\pi}{q} n^2 e^{2t} i e^{-2ia} + itT\right) \frac{w(t)}{a+it}\,\mathrm{d}t\Bigg) + O\left((\log T)^{\frac{5}{4}}\right), \quad (2.43)$$

where w *is a smooth weight function satisfying* $0 < w(t) < 1$ *for* $u_0 < t < 2u_0$ *and*

$$w(t) = \begin{cases} 0, & \text{if } 0 \le t < u_0, \\ 1 & \text{if } t \ge 2u_0 \end{cases}$$

and which moreover satisfies $w^{(j)}(t) \le u_0^{-j}$ *for sufficiently many derivatives. This means that we can omit the interval* $0 \le \Im p \le u_0$ *in (2.40).*

Proof. If $0 < u \ll \frac{1}{\sqrt{T}\log T}$ is fixed, then Lemma 34 gives

$$\int_{a+i0}^{a+iu} \frac{\check{Q}(p,\chi)}{p} e^{pT}\,\mathrm{d}p \ll \int_0^u \frac{\mathrm{d}t}{|a+it|} \ll \log T.$$

So we may take a smooth weight function w as in the statement, and then (2.40) gives

$$F(T,\chi) = F_1(T,\chi) + O\left((\log T)^{\frac{5}{4}}\right)$$

with

$$F_1(T,\chi) = 2e\Re\left(\frac{1}{2\pi}\int_0^{\infty} \frac{\check{Q}(a+it,\chi)}{a+it} w(t) e^{itT}\,\mathrm{d}t\right). \quad (2.44)$$

Using Proposition 29 and the fact that

$$\int_0^\infty \lambda(a+it,\chi)(a+it)^{-1}w(t)e^{itT} \ll \int_0^\infty \frac{|\lambda(a+it,\chi)|}{|a+it|}w(t)\,\mathrm{d}t \ll \log T$$

we immediately obtain (2.43). □

In the next two steps, we seek to further simplify the above expression without affecting the error term.

Step III (Simplification 1). Replacing e^{-2ia} by $1-2ia$

We have

$$F(T,\chi) = 2\Re\Bigg(i\kappa q^{-\frac{1}{4}} e^{1+\frac{\pi i}{8}-ia} \sum_{n=1}^{\infty} \sqrt{n}\chi(n)\times \int_0^\infty \exp\left(t - 2a\frac{\pi}{q}n^2e^{2t} - \frac{\pi}{q}n^2e^{2t}i + itT\right)\frac{w(t)}{a+it}\,\mathrm{d}t\Bigg) + O\left((\log T)^{\frac{5}{4}}\right). \tag{2.45}$$

Proof. We observe that approximating e^{-2ia} in (2.43) by $1-2ia$ amounts to a shift of variable of $\ll 1/T^2$. Then the error incurred in the above sum is

$$\sum_{n=1}^{\infty}\sqrt{n}\chi(n)\int_0^\infty \Bigg\{\exp\left(t-\frac{\pi}{q}n^2e^{2t}ie^{-2ia}+itT\right) - \exp\left(t-\frac{\pi}{q}n^2e^{2t}i(1-2ia)+itT\right)\Bigg\}\frac{w(t)}{a+it}\,\mathrm{d}t \ll \frac{1}{T^2}\sum_{n=1}^{\infty} n^{\frac{5}{2}} \times \int_0^\infty \exp\left(3t-\frac{\pi}{q}n^2e^{2t}\sin 2a\right)\frac{\mathrm{d}t}{|a+it|}. \tag{2.46}$$

We divide the interval of integration in (2.46) into two parts, namely $[0,c]$ and $[c,\infty[$, where $c>1$ is to be chosen later. Using the fact that

$$\exp(-\frac{\pi}{q}n^2e^{2t}\sin 2a) \leq (\tfrac{4}{qT}n^2e^{2t})^{-2},$$

we immediately see that the integral over $[c,\infty[$ is $\ll c^{-1}n^{-4}T^2e^{-c}$. Also, the integral over $[0,c]$ is easily seen to be $\ll e^{3c-\frac{4}{qT}n^2}\log(Tc)$. Hence, separating the sum to $n \le \sqrt{qT\log T}$ and $n > \sqrt{qT\log T}$ and choosing $c = \log T$, we see that the overall contribution of the terms on the right hand side of (2.46) is $\ll T^{-\frac{1}{4}}(\log T)^{5/4}$. Thus we may rewrite (2.43) as (2.45). □

Step IV (Simplification 2). Replacing e^{-ia} in front of (2.45) by 1

We have

$$F(T,\chi) = 2\Re\Bigg(i\kappa q^{-\frac{1}{4}} e^{1+\frac{\pi i}{8}} \sum_{n=1}^{\infty} \sqrt{n}\chi(n) \times \qquad (2.47)$$
$$\int_0^{\infty} \exp\left(t - 2a\frac{\pi}{q}n^2e^{2t} - \frac{\pi}{q}n^2e^{2t}i + itT\right)\frac{w(t)}{a+it}\,\mathrm{d}t\Bigg) + O\left((\log T)^{\frac{5}{4}}\right).$$

Proof. Again in (2.45), we omit the factor e^{-ia} in front; the error incurred is then

$$\ll \frac{1}{T}\sum_{n=1}^{\infty}\sqrt{n}\int_0^{\infty}\exp(t - 2a\frac{\pi}{q}n^2e^{2t})\frac{\mathrm{d}t}{|a+it|}.$$

Separating the interval of integration again into $t \le \log T$ and $\log T \le t$ and proceeding similarly as in the preceding step, we see that the error is $\ll T^{-\frac{1}{4}}(\log T)^{\frac{5}{4}}$. Thus (2.45) finally reduces to the desired form (2.47). □

Step V. Decomposing the sum

The substitution $u = e^{2t} - 1$ in (2.47) gives

$$\sum_{n=1}^{\infty}\sqrt{n}\chi(n) \times \int_0^{\infty}\exp\left(t - 2a\frac{\pi}{q}n^2e^{2t} - \frac{\pi}{q}n^2e^{2t}i + itT\right)\frac{w(t)}{a+it}\,\mathrm{d}t$$
$$= \frac{1}{2}\sum_{n=1}^{\infty}\sqrt{n}\chi(n)e^{-\frac{\pi}{q}n^2i}\int_0^{\infty}\frac{\exp\left(-2a\frac{\pi}{q}n^2(1+u) - \frac{\pi}{q}n^2iu + \frac{iT}{2}\log(1+u)\right)}{\sqrt{1+u}}\times$$
$$\frac{w(\frac{1}{2}\log(1+u))}{a+\frac{i}{2}\log(1+u)}\,\mathrm{d}u = \frac{1}{2}I, \qquad (2.48)$$

say. Let us write $I = I_1 + I_2 + I_3$, where the sums run respectively over the intervals $[\sqrt{N'}, \sqrt{N''}]$, $[1, \sqrt{N'}[$ and $[\sqrt{N''}, \infty[$ where

$$N' = \frac{qT}{2\pi} + \frac{N}{2} - \sqrt{\frac{N^2}{4} + \frac{qNT}{2\pi}}$$

with $N \asymp qT$ and $N'' = A\frac{qT}{2\pi}$, $A > 1$ being a constant. We see that $N' + \sqrt{NN'} = \frac{qT}{2\pi}$ and this will be used frequently in the sequel. We now estimate the sums I_1, I_2 and I_3 one by one.

2.7.1 Estimation of I_3

Let us write the integral in (2.48) as $\int_0^\infty g_n(t)e(f_n(t))\,\mathrm{d}t$ where $f_n(t) = -\frac{n^2}{2q}t + \frac{T}{4\pi}\log(1+t)$ and $g_n(t) = \frac{\exp(-b(1+t))w(\frac{1}{2}\log(1+t))}{\sqrt{1+t}(a+\frac{i}{2}\log(1+t))}$ with $b = {}^{(2\pi n^2)}/_{(qT)}$. Note that the derivative $f_n'(t) = -\frac{n^2}{2q} + \frac{T}{4\pi(1+t)}$ of f_n has no zero in $\mathbb{R}^+$ if $n > \sqrt{\frac{qT}{2\pi}}$. Therefore, the integrals in I_3 can be integrated by parts with respect to the factor $e(f_n(t))$, which we do as follows. We have

$$\int_0^\infty g_n(t)e(f_n(t))\,\mathrm{d}t = -\frac{1}{2\pi i}\int_0^\infty \left(\frac{g_n'(t)}{f_n'(t)} - f_n''(t)\frac{g_n(t)}{f_n'(t)^2}\right)e(f_n(t))\,\mathrm{d}t.$$

Now,

$$g_n(t) \ll \frac{\exp(-b(1+t))w(\frac{1}{2}\log(1+t))}{\sqrt{1+t}\log(1+t)}, \quad f_n'(t) \gg \frac{n^2}{2q}, \quad f_n''(t) = -\frac{T}{4\pi(1+t)^2},$$

so

$$\int_0^\infty f_n''(t)\frac{g_n(t)}{f_n'(t)^2}\exp(2\pi i f_n(t))\,\mathrm{d}t \ll \frac{T}{n^4}\int_{v_0}^\infty \frac{e^{-b(1+t)}}{(1+t)^{\frac{5}{2}}\log(1+t)}\,\mathrm{d}t, \qquad (2.49)$$

where $v_0 = e^{2u_0} - 1$. Using (2.92), we see that the contribution of these integrals to I_3 is $\ll T^{-\frac{1}{4}}\log T$.

It remains to estimate the integral $\xi = \int_0^\infty \frac{g_n'(t)}{f_n'(t)}e(f_n(t))\,\mathrm{d}t$. The derivative of g_n may be written as $g_n'(t) = h_n(t) - \alpha_n(t) g_n(t)$ where

$$h_n(t) = \frac{e^{-b(1+t)} w'(\frac{1}{2}\log(1+t))}{2(1+t)v(t)}, \quad \alpha_n(t) = b + \frac{1}{2(1+t)} + \frac{i}{2v(t)\sqrt{1+t}}$$

with $v(t) = \sqrt{1+t}(a + \frac{i}{2}\log(1+t))$. Thus, $\xi = J_1 - J_2$ where $J_1 = \int_0^\infty \frac{h_n(t)}{f_n'(t)} e(f_n(t))\,\mathrm{d}t$ and $J_2 = \int_0^\infty \frac{\alpha_n(t) g_n(t)}{f_n'(t)} e(f_n(t))\,\mathrm{d}t$. We estimate the two integrals one by one.

First J_1. Integrating by parts, we get

$$\begin{aligned} J_1 &= -\frac{1}{2\pi i}\int_0^\infty \Big(\frac{h_n'(t)}{f_n'(t)^2} - 2\frac{f_n''(t)}{f_n'(t)^3} h_n(t)\Big) e(f_n(t))\,\mathrm{d}t \\ &= -\frac{1}{2\pi i}(J_1' - 2J_1''), \text{ say.} \end{aligned}$$

It is easy to see that the second integral is

$$J_1'' \ll \frac{Tu_0^{-1}}{n^6}\int_{v_0}^\infty \frac{e^{-b(1+t)}}{(1+t)^{\frac{5}{2}}\log(1+t)}\,\mathrm{d}t.$$

Using (2.92), we get that the contribution of J_1'' to I_3 is $\ll T^{-\frac{3}{4}}(\log T)^2$. Similarly, one can prove that the contribution of J_1' to I_3 is $\ll T^{-\frac{1}{4}}(\log T)^3$.

As for J_2, proceeding similarly as with J_1, we see that similar integrals turn up, which we estimate one by one. The biggest contribution from the resulting integrals is $T^{-\frac{1}{4}}\log^3 T$. Thus $I_3 \ll T^{-\frac{1}{4}}\log^3 T$.

2.7.2 Evaluation of I_2

The saddle point of the n^{th} integral is $\frac{qT}{2\pi n^2} - 1$. Since $n < \sqrt{N'}$ for the terms in I_2, the saddle point is therefore $> \frac{qT}{2\pi N'} - 1 = 2y_0$, say (clearly $y_0 \ll 1$). We therefore divide the interval of integration into two parts, $[0, y_0]$ where no saddle point lies, and $[y_0, \infty[$ where the saddle-points of all the terms in I_2 lie.

On the first interval, we do integration by parts three times to get

$$\int_0^{y_0} g_n(t)e(f_n(t))\,\mathrm{d}t = \left(\frac{1}{2\pi i}\frac{g_n(y_0)}{f_n'(y_0)} - (\frac{1}{2\pi i})^2\frac{V_1(y_0)}{f_n'(y_0)} + (\frac{1}{2\pi i})^3\frac{V_2(y_0)}{f_n'(y_0)}\right)e(f_n(y_0))$$
$$- (\frac{1}{2\pi i})^3\int_0^{y_0} V_3(t)e(f_n(t))\,\mathrm{d}t$$

where $V_0(t) = g_n(t)$ and $V_{j+1}(t) = \left(\frac{V_j(t)}{f_n'(t)}\right)'$, $j = 0, 1, 2, \ldots$ Now, for $n < \sqrt{N'}$, it is easy to see that

$$\begin{aligned} V_0(y_0) &= g_n(y_0) \ll 1, V_1(y_0) \ll 1 \\ f_n'(y_0) &\asymp T; \\ V_1'(y_0) &\ll \log^2 T, \\ V_2(y_0) &\ll \frac{\log^2 T}{T}, \\ V_3'(t) &\ll \frac{u_0^{-4}}{T^3}\forall t \in [0, y_0]; \end{aligned}$$

the contribution of the first three terms to I_2 is therefore $\ll T^{-\frac{1}{2}}$, and the contribution of $\int_0^{y_0} V_3(t)e(f_n(t))\,\mathrm{d}t$ to I_2 is $\ll \sqrt{T}u_0^{-4}/T^3 = (\log^4 T)/\sqrt{T}$. Hence we may write

$$I_2 = \sum_{n<\sqrt{N'}} \sqrt{n}\chi(n)e^{-\frac{\pi i}{q}n^2}\int_{y_0}^{\infty} g_n(t)e(f_n(t))\,\mathrm{d}t + O\left(\tfrac{\log^4 T}{\sqrt{T}}\right).$$

To evaluate these integrals, we consider the integrals $\int_{y_0}^{\alpha} g_n(t)e(f_n(t))\,\mathrm{d}t$ where we will let $\alpha \to \infty$. We use Atkinson's lemma (see Theorem 20). This theorem is applicable to our case with $f(t) = \frac{T}{4\pi}\log(1+t), k = -\frac{n^2}{2q}, g(t) = g_n(t)$ together with $F(t) = T, \mu(t) = \frac{1}{2}t, G(t) = \frac{e^{-\frac{2\pi}{qT}n^2(1+t)}}{\sqrt{1+t}}$; note that $x_0 = \frac{qT}{2\pi n^2} - 1$ is then a zero of $f' + k$. The main term is thus

$$\frac{\sqrt{2q}}{ne}e^{\frac{\pi n^2}{q}i}\frac{\exp\left(i(\frac{T}{2}(\log(\frac{qT}{2\pi n^2}) - 1) - \frac{\pi}{4})\right)}{a + \frac{i}{2}\log(\frac{qT}{2\pi n^2})};$$

we may omit the term a in the denominator, thus incurring an error of $\ll \frac{1}{T}$. Hence the main term is

$$\frac{-2\sqrt{2q}}{ne\log(\frac{qT}{2\pi n^2})}e^{\frac{\pi n^2}{q}i}\exp\big(i(\tfrac{T}{2}(\log(\tfrac{qT}{2\pi n^2})-1)+\tfrac{\pi}{4})\big)+O(\frac{1}{T}).$$

We easily see that

$$\begin{aligned}\frac{G(x_0)\mu(x_0)}{F(x_0)^{\frac{3}{2}}} &\ll \frac{1}{nT},\\ \frac{G(y_0)}{|f'(y_0)+k|+\sqrt{|f''(y_0)|}} &\ll \frac{e^{-\frac{2\pi}{qT}n^2}}{T},\\ \frac{G(\alpha)}{|f'(\alpha)+k|+\sqrt{|f''(\alpha)|}} &\ll \sqrt{1+\alpha}e^{-\frac{2\pi}{qT}n^2(1+\alpha)},\\ \int_{y_0}^{\alpha} G(t)\exp\big(-c\,|k|\,\mu(t)-cF(t)\big)(1+|\mu'(t)|)\,\mathrm{d}t &\ll \frac{1}{n^2}\end{aligned}$$

so the error terms have a contribution $\ll T^{-\frac{1}{4}}$ to I_2, so that we finally get

$$I_2 = -2\sqrt{2q}e^{-1}\sum_{n<\sqrt{N'}}\frac{\chi(n)}{\sqrt{n}\log(\frac{qT}{2\pi n^2})}\exp\big\{i\big(\tfrac{T}{2}(\log(\tfrac{qT}{2\pi n^2})-1)+\tfrac{\pi}{4}\big)\big\} \qquad (2.50)$$
$$+O(T^{-\frac{1}{4}}).$$

2.7.3 Evaluation of I_1

We write

$$I_1 = \int_0^\infty \frac{(1+u)^{\frac{iT}{2}}w(\frac{1}{2}\log(1+u))S(u)}{\sqrt{1+u}(a+\frac{i}{2}\log(1+u))}\,\mathrm{d}u$$

with

$$S(u) = \sum_{\sqrt{N'}\le n\le\sqrt{N''}}\sqrt{n}\chi(n)e^{-\frac{\pi}{q}n^2 i}\exp\big(-\frac{2\pi}{qT}n^2(1+u)-\tfrac{\pi}{q}n^2 iu\big).$$

We recall that $w(\frac{1}{2}\log(1+u)) = 0$ for $u \leq e^{2u_0} - 1$. Let us take a big positive number c. We define

$$\tilde{w}(t) = \begin{cases} w(\frac{1}{2}\log(1+t)), & 0 \leq t \leq c-1, \\ \frac{f(c-t)}{f(t-c+1)+f(c-t)}, & t \geq c-1 \end{cases}$$

where $f(t) = e^{-\frac{1}{t}}$ for $t > 0$ and $f(t) = 0$ for $t \leq 0$ is a smooth function on $\mathbb{R}$. It is clear that $\tilde{w}(t) = 0$ for $t \geq c$ and that $0 < \tilde{w}(t) < 1$ for $c - 1 < t < c$. Moreover, $\tilde{w}$ is a smooth function. Let us put

$$\tilde{I}_1 = \int_0^\infty \frac{(1+u)^{\frac{iT}{2}}\tilde{w}(u)S(u)}{\sqrt{1+u}(a+\frac{i}{2}\log(1+u))}\,\mathrm{d}u.$$

Then,

$$\begin{aligned} I_1 - \tilde{I}_1 = &\int_{c-1}^{c} \frac{(1+u)^{\frac{iT}{2}}(1-\tilde{w}(u))S(u)}{\sqrt{1+u}(a+\frac{i}{2}\log(1+u))}\,\mathrm{d}u \\ &+ \int_c^\infty \frac{(1+u)^{\frac{iT}{2}}S(u)}{\sqrt{1+u}(a+\frac{i}{2}\log(1+u))}\,\mathrm{d}u \end{aligned} \tag{2.51}$$

We consider the integrals $\int \tilde{g}_n(u)e(f_n(u))\,\mathrm{d}u$ with $\tilde{g}_n(u) = \frac{e^{-\frac{2\pi}{qT}n^2(1+u)}(1-\tilde{w}(u))}{\sqrt{1+u}(a+\frac{i}{2}\log(1+u))}$. It is clear that this difference can be made as small as we want. In fact, we trivially have

$$\begin{aligned} \int_c^\infty \tilde{g}_n(u)e(f_n(u))\,\mathrm{d}u &\ll \frac{1}{\sqrt{c}\log c}\int_c^\infty e^{-\frac{2\pi}{qT}n^2(1+t)}\,\mathrm{d}t \\ &\ll \frac{T}{n^2\sqrt{c}\log c}e^{-\frac{2\pi}{qT}n^2 c}, \end{aligned}$$

so the second term in (2.51) is

$$\ll \frac{T}{\sqrt{c}\log c}\sum_{\sqrt{N'}\leq n\leq\sqrt{N''}} n^{-\frac{3}{2}}e^{-\frac{2\pi}{qT}n^2 c} \ll \frac{T^{\frac{3}{4}}}{\sqrt{c}\log c}e^{-\frac{2\pi}{qT}cN'};$$

so, for $c = \frac{qT}{\pi N'}\log T \asymp \log T$, we see that this is $\ll T^{-\frac{5}{4}}(\sqrt{\log T}\log\log T)^{-1}$. Similarly, the first term in (2.51) is $\ll T^{-\frac{5}{4}}(\sqrt{\log T}\log\log T)^{-1}$, so that $I_1 = \tilde{I}_1 + O(T^{-\frac{5}{4}}(\sqrt{\log T}\log\log T)^{-1})$. Let us write $I_1 = \tilde{I}_1 + O(\frac{1}{T})$ for simplicity.

We transform the sum $S(u)$ using Poisson summation. Applying (1.9) to $S(u)$ with $f(t) = \sqrt{t}\exp(-\frac{2\pi}{qT}t^2(1+u) - \frac{\pi}{q}t^2 iu)$ and choosing N' and N'' such that $\sqrt{N'}, \sqrt{N''} \notin \mathbb{N}$, we have

$$S(u) = \frac{1}{2q}\sum_{r=0}^{2q-1}\tau_-(r,\chi)\sum_{m\in\mathbb{Z}}\int_{\sqrt{N'}}^{\sqrt{N''}}\sqrt{t}e^{-\frac{2\pi}{qT}t^2(1+u)}e(-(m+\frac{r}{2q})t - \frac{u}{2q}t^2)\,\mathrm{d}t. \quad (2.52)$$

Let us consider each of the integrals $I_m(u) = \int_{\sqrt{N'}}^{\sqrt{N''}} g_u(t)e(f_{m,u}(t))\,\mathrm{d}t$, where $g_u(t) = \sqrt{t}e^{-\frac{2\pi}{qT}t^2(1+u)}$ and $f_{m,u}(t) = -(m+\frac{r}{2q})t - \frac{u}{2q}t^2$. We have $f'_{m,u}(t) = -(m+\frac{r}{2q}) - \frac{u}{q}t$, so for a fixed u, the integral has a saddle-point at $t = -\frac{1}{2u}(2qm+r)$; this saddle-point is inside the interval of integration iff $m \in [-N_2, -N_1]$ where $N_1 = (2q)^{-1}(2u\sqrt{N'}+r), N_2 = (2q)^{-1}(2u\sqrt{N''}+r)$. Therefore, there are no saddle-points if $|m| \geq c_1\sqrt{T}$, say, for a suitable constant $c_1 > 0$. For such m, we integrate by parts twice to get

$$\begin{aligned}
I_m(u) &= \frac{1}{2\pi i}\left(\frac{g_u(B)}{f'_{m,u}(B)}e(f_{m,u}(B)) - \frac{g_u(A)}{f'_{m,u}(A)}e(f_{m,u}(A))\right)\\
&- \frac{1}{(2\pi i)^2}\left(\frac{V_1(B)}{f'_{m,u}(B)}e(f_{m,u}(B)) - \frac{V_1(A)}{f'_{m,u}(A)}e(f_{m,u}(A))\right)\\
&+ \frac{1}{(2\pi i)^2}\int_A^B V_2(t)e(f_{m,u}(t))\,\mathrm{d}t
\end{aligned}$$

where $V_1(t) = (\frac{g_u(t)}{f'_{m,u}(t)})'$ and $V_2(t) = (\frac{V_1(t)}{f'_{m,u}(t)})'$. We may further modify N' and N'', if necessary, to satisfy $\{\sqrt{N'}\} = \frac{1}{2} = \{\sqrt{N''}\}$. We have then $e(f_{m,u}(B)) = (-1)^m e(-B\frac{r+Bu}{2q})$ and $e(f_{m,u}(A)) = (-1)^m e(-A\frac{r+Au}{2q})$. The first two terms in $I_m(u)$ therefore give alternating series of the form $\sum_{m\geq c\sqrt{T}}\frac{(-1)^m z}{m+\alpha}$ with z complex and α real. It follows that the contribution of the first two integrated terms to $S(u)$ is $\ll T^{-\frac{1}{4}}$. We also have $V_2(t) \ll e^{-\frac{2\pi}{qT}(1+u)N'}(m^{-2}T^{-\frac{3}{4}}u + m^{-3}T^{-\frac{1}{4}}u +$

$m^{-4}T^{\frac{1}{4}}u^2)$ and so the contribution of the last integral to $S(u)$ is

$$\ll \sum_{m\geq c\sqrt{T}} e^{-\frac{2\pi}{qT}(1+u)N'}(m^{-2}T^{-\frac{1}{4}}u + m^{-3}T^{\frac{1}{4}}u + m^{-4}T^{\frac{3}{4}}u^2)$$
$$\ll e^{-\frac{2\pi}{qT}(1+u)N'}T^{-\frac{3}{4}}(u+u^2).$$

Hence we can truncate the series (2.52) to $m \ll \sqrt{T}$ with an error of $\ll_u T^{-\frac{1}{4}}$. Thus,

$$S(u) = \frac{1}{2q}\sum_{r=0}^{2q-1}\tau_-(r,\chi)\sum_{N_1\leq m\leq N_2}\int_{\sqrt{N'}}^{\sqrt{N''}}\sqrt{t}e^{-\frac{2\pi}{qT}t^2(1+u)}e((m-\frac{r}{2q})t-\frac{u}{2q}t^2)\,\mathrm{d}t \tag{2.53}$$
$$+ O_u(T^{-\frac{1}{4}}).$$

Now, the integral $I_m(u)$ in this last equation has a saddle point at $x_m = \frac{q}{u}(m-\frac{r}{2q})$. Hence, Theorem 20 transforms the sum $S(u)$ to

$$\frac{e^{-\frac{\pi i}{4}}}{2u}\sum_{r=0}^{2q-1}\tau_-(r,\chi)\sum_{N_1\leq m\leq N_2}(m-\frac{r}{2q})^{\frac{1}{2}}\exp\left(-\frac{2\pi q}{u^2T}(1+u)(m-\frac{r}{2q})^2+\frac{\pi qi}{u}(m-\frac{r}{2q})^2\right) \tag{2.54}$$

with error terms to be discussed later (See **subsection 2.7.4**). Therefore, the main term in I_1 becomes

$$\frac{1}{2}e^{-\frac{\pi i}{4}}\sum_{r=0}^{2q-1}\tau_-(r,\chi)\sum_{\frac{r}{2q}\leq m\leq\frac{2c\sqrt{N''}+r}{2q}}(n-\frac{r}{2q})^{\frac{1}{2}}\int_{\frac{2qm-r}{2\sqrt{N''}}}^{\frac{2qm-r}{2\sqrt{N'}}}\frac{\tilde{w}(u)}{u\sqrt{1+u}(a+\frac{i}{2}\log(1+u))}\times$$
$$\exp\left(\frac{iT}{2}\log(1+u)-\frac{2\pi q}{u^2T}(1+u)(m-\frac{r}{2q})^2+\frac{\pi qi}{u}(m-\frac{r}{2q})^2\right)\mathrm{d}u$$

The substitution $y = {}^1\!/_u$ transforms this sum to

$$\frac{1}{2}e^{-\frac{\pi i}{4}}\sum_{r=0}^{2q-1}\tau_-(r,\chi)\sum_{\frac{r}{2q}\le m\le\frac{2c\sqrt{N''}+r}{2q}}(m-\frac{r}{2q})^{\frac{1}{2}}\int_{\frac{2\sqrt{N'}}{2qm-r}}^{\frac{2\sqrt{N''}}{2qm-r}}\frac{\tilde{w}({}^1\!/_y)}{\sqrt{y(1+y)}(a+\frac{i}{2}\log(1+\frac{1}{y}))}\times$$
$$\exp\left(\frac{iT}{2}\log(1+\frac{1}{y})-\frac{2\pi qy}{T}(1+y)(m-\frac{r}{2q})^2+\pi qyi(m-\frac{r}{2q})^2\right)\mathrm{d}y \quad (2.55)$$

Let us write the m^{th} integral above as $I_m(r)=\int_{A_1}^{B_1}\phi(t)e(\psi(t))\,\mathrm{d}t$ where ϕ and ψ are obvious (both are real functions). Now, the $I_m(r)$ has a saddle-point at y_0 with $y_0(y_0+1)=\frac{2qT}{\pi(2qm-r)^2}$(so $y_0=-\frac{1}{2}+\sqrt{\frac{1}{4}+\frac{T}{2\pi q(m-\frac{r}{2q})^2}}$). We note that

$$\psi(y_0)=\frac{T}{4\pi}\log(1+\frac{1}{y_0})+\frac{qy_0}{2}(m-\frac{r}{2q})^2$$

and

$$\log(1+\frac{1}{y_0})=2\operatorname{argsinh}\big(\sqrt{\frac{\pi q}{2T}}(m-\frac{r}{2q})\big),$$
$$\pi qy_0(m-\frac{r}{2q})^2=-\frac{\pi q}{2}(m-\frac{r}{2q})^2+\frac{1}{2}\sqrt{\pi^2q^2(m-\frac{r}{2q})^4+2\pi qT(m-\frac{r}{2q})^2};$$

hence $e(\psi(y_0))$ may be written as $(-1)^{\frac{m(qm-r)}{2}}e^{\frac{2q-r^2}{8q}\pi i}\exp\big(\frac{i}{2}\,\mathrm{f}(T,q(m-\frac{r}{2q})^2)\big)$ where

$$\mathrm{f}(T,\ell)=2T\operatorname{argsinh}\big(\sqrt{\frac{\pi\ell}{2T}}\big)+\sqrt{(\pi\ell)^2+2\pi\ell T}-\frac{\pi}{4}$$

Also, we easily calculate

$$\frac{\phi(y_0)}{\sqrt{\psi''(y_0)}}=e^{-1}\sqrt{\frac{2}{q}}(m-\frac{r}{2q})^{-1}\big(\frac{1}{4}+\frac{T}{2\pi q(m-\frac{r}{2q})^2}\big)^{-\frac{1}{4}}\times$$
$$\left(a+i\operatorname{argsinh}\big(\sqrt{\frac{\pi q}{2T}}(m-\frac{r}{2q})\big)\right)^{-1}.$$

Hence, applying Theorem 20, the main term of the integral $I_m(r)$

$$\frac{\phi(y_0)}{\sqrt{\psi''(y_0)}}e(\psi(y_0)+\frac{1}{8}) = (-1)^{\frac{m(qm-r)}{2}}\sqrt{\frac{2}{q}}\frac{e^{-1+\frac{3\pi i}{8}-\frac{\pi r^2}{8q}i}}{(m-\frac{r}{2q})}\Big(\frac{1}{4}+\frac{T}{2\pi q(m-\frac{r}{2q})^2}\Big)^{-\frac{1}{4}}\times$$
$$\Big(a+i\operatorname{argsinh}\big(\sqrt{\frac{\pi q}{2T}}(m-\frac{r}{2q})\big)\Big)^{-1}\exp\big(\frac{i}{2}\,\mathsf{f}(T,q(m-\tfrac{r}{2q})^2)\big).$$

Using this in (2.55) and noting that the condition ${}^{2\sqrt{N'}}/(2qm-r) \le y_0 \le {}^{2\sqrt{N''}}/(2qm-r)$ is equivalent to $(m-\frac{r}{2q})^2 \le \frac{N}{q^2}$, we get that the main term of I_1 is

$$\frac{1}{\sqrt{2q}}e^{-1+\frac{\pi i}{8}}\sum_{r=0}^{2q-1}\tau_-(r,\chi)e^{-\frac{\pi r^2}{8q}i}\sum_{0\le m-\frac{r}{2q}\le\frac{1}{q}\sqrt{N}}(-1)^{\frac{m(qm-r)}{2}}(m-\frac{r}{2q})^{-\frac{1}{2}}\Big(\frac{1}{4}+\frac{T}{2\pi q(m-\frac{r}{2q})^2}\Big)^{-\frac{1}{4}}\times$$
$$\Big(a+i\operatorname{argsinh}\big(\sqrt{\frac{\pi q}{2T}}(m-\frac{r}{2q})\big)\Big)^{-1}\exp\big(\frac{i}{2}\,\mathsf{f}(T,(m-\tfrac{r}{2q})^2)\big). \tag{2.56}$$

We may omit a with an error $\ll 1$. Also,

$$\Big(\frac{1}{4}+\frac{T}{2\pi q(m-\frac{r}{2q})^2}\Big)^{-\frac{1}{4}}\operatorname{argsinh}^{-1}\big(\sqrt{\frac{\pi q}{2T}}(m-\frac{r}{2q})\big) = \frac{2}{\sqrt{q}}\big(\frac{qT}{2\pi}\big)^{\frac{1}{4}}(m-\frac{r}{2q})^{-\frac{1}{2}}\,\mathsf{e}\,\big(T,q(m-\frac{r}{2q})^2\big)$$

where

$$\mathsf{e}(T,n) = \Big(1+\frac{\pi n}{2T}\Big)^{-\frac{1}{4}}\Big\{\sqrt{\frac{\pi n}{2T}}\operatorname{argsinh}^{-1}\big(\sqrt{\frac{\pi n}{2T}}\big)\Big\},$$

so that after the change of variable $n = 2qm-r$, the main term becomes

$$-2\sqrt{2}ie^{-1+\frac{\pi i}{8}}\big(\frac{qT}{2\pi}\big)^{\frac{1}{4}}\sum_{r=0}^{2q-1}\tau_-(r,\chi)e^{-\frac{\pi r^2}{8q}i}\sum_{\substack{n\le 2\sqrt{N}\\ n\equiv -r[2q]}}\frac{(-1)^{\frac{n^2-r^2}{8q}}}{n}\,\mathsf{e}\,\big(T,\frac{n^2}{4q}\big)\exp\big(\frac{i}{2}\,\mathsf{f}(T,\tfrac{n^2}{4q})\big) \tag{2.57}$$

Reintroducing the factor $i\kappa q^{-\frac{1}{4}}e^{1+\frac{\pi i}{8}}$, we get the expression for $S_1(T,\chi)$ as stated in the theorem. The error terms are discussed in the next section.

2.7.4 The error term for I_1

We now estimate the error incurred on passing from (2.53) to (2.54), following the method of proof of Theorem 20 as found, for example, in [35].

Consider the saddle-point $x_m \equiv x_m(u) = \frac{q}{u}(m - \frac{r}{2q})$ of the m^{th} integral; write $u(m) = \frac{2qm-r}{2\sqrt{N'}}$. Three cases arise: $x_m \leq \sqrt{N'}$ or $\sqrt{N'} < x_m < \sqrt{N''}$ or $\sqrt{N''} \leq x_m$. We note that

$$x_m(u) \leq \sqrt{N'} \iff u \geq u(m) \text{ and } x_m(u) > \sqrt{N'} \iff u < u(m). \tag{2.58}$$

Let us first suppose that the saddle-point $x_m \equiv x_m(u) = \frac{q}{u}(m - \frac{r}{2q})$ satisfies $\sqrt{N'} < x_m < \sqrt{N''}$; we will shortly deal with other cases. Write $\rho = 1 - i$. By Cauchy's theorem, we can replace the path of integration (of $I_m(u)$) by a contour joining the points $\sqrt{N'}, \sqrt{N'} - \epsilon\rho\sqrt{T}, x_m - \epsilon\rho\sqrt{T}, x_m + \epsilon\rho\sqrt{T}, \sqrt{N''} + \epsilon\rho\sqrt{T}, \sqrt{N''}$ by line segments, where $\epsilon > 0$ is small. We can use the functions $F(t) = Tu, G(t) = T^{\frac{1}{4}}, \mu(t) = b\sqrt{T}$ for a suitable positive constant b.

The first error term given by Theorem 20 is $G(x_m)\mu(x_m)F(x_m)^{-\frac{3}{2}} = bu^{-\frac{3}{2}}T^{-\frac{3}{4}}$. Since $\sqrt{N'} < x_m(u) < \sqrt{N''}$ we have $u \gg T^{-\frac{1}{2}}$ and $m \ll u\sqrt{T}$, then the contribution of such error terms to $S(u)$ is $\ll \sum_{m \ll u\sqrt{T}} u^{-\frac{3}{2}}T^{-\frac{3}{4}} \ll u^{-\frac{1}{2}}T^{-\frac{1}{4}}$ and their contribution to I_1 is $\ll \int_{c_1/\sqrt{T}}^{c} \frac{u^{-\frac{1}{2}}T^{-\frac{1}{4}}}{\sqrt{u}}\,\mathrm{d}u \ll T^{-\frac{1}{4}}\log T$.

Next we consider the contribution of the integral along the line segment going from $\sqrt{N'}$ to $\sqrt{N'} - \epsilon\rho\sqrt{T}$. Let $\gamma(t) = \sqrt{N'} - \rho t$ for $0 \leq t \leq \epsilon\sqrt{T}$. We write $g(t) = \sqrt{t}e^{-\frac{2\pi}{qT}t^2(1+u)}$ and $f(t) = (m - \frac{r}{2q})t - \frac{u}{2q}t^2$. Then the integral is

$$-\rho \int_0^{\epsilon\sqrt{T}} g(\gamma(t))e(f(\gamma(t)))\,\mathrm{d}t.$$

Then the contribution of the above integral to I_1 over the interval $0 \leq u \leq u(m)$

is

$$-\rho \sum_{0<m\ll\sqrt{T}} \int_0^{\epsilon\sqrt{T}} \gamma(t)^{\frac{1}{2}} e\big((m-\frac{r}{2q})\gamma(t)\big)\Big(\ulcorner \int_0^{u(m)} \frac{\tilde{w}(u)}{\sqrt{1+u}(a+\frac{i}{2}\log(1+u))} \times \tag{2.59}$$

$$\exp\big(\llcorner -\frac{2\pi}{qT}(1+u)\gamma(t)^2 - \frac{\pi i}{q}u\gamma(t)^2 + i\frac{T}{2}\log(1+u)\big)\lrcorner \,\mathrm{d}u\Big)^{\urcorner} \mathrm{d}t$$

The whole integrand is

$$\ll T^{\frac{1}{4}} \exp\big(-\frac{2\pi u}{q}t^2 - 2\pi t(m-\frac{r}{2q}-\frac{u\sqrt{N'}}{q})\big)\times$$
$$\frac{\exp\big(-\frac{2\pi}{qT}(1+u)(N'-2\sqrt{N'}t)\big)}{\sqrt{1+u}(4a^2+\log^2(1+u))^{\frac{1}{2}}}.$$

Since $\frac{2\pi(N'-2t\sqrt{N'})}{qT} \asymp 1$, we see that $\frac{\exp\left(-\frac{2\pi}{qT}(1+u)(N'-2\sqrt{N'}t)\right)}{\sqrt{1+u}(4a^2+\log^2(1+u))^{\frac{1}{2}}} \ll \frac{1}{1+u}$, hence the integrand is $\ll T^{\frac{1}{4}}(1+u)^{-1}\exp\big(-\frac{2\pi u}{q}t^2 - 2\pi t(m-\frac{r}{2q}-\frac{u\sqrt{N'}}{q})\big)$. Therefore, for $t_0 \gg \min\{u^{-\frac{1}{2}}, \left|m-\frac{r}{2q}-\frac{u\sqrt{N'}}{q}\right|^{-1}\}\log T$, we have

$$\int_0^{u(m)} \ulcorner\Big(\int_{t_0}^{\epsilon\sqrt{T}} T^{\frac{1}{4}}(1+u)^{-1}\exp\big(-\frac{2\pi u}{q}t^2 - 2\pi t(m-\frac{r}{2q}-\frac{u\sqrt{N'}}{q})\big)\,\mathrm{d}t\Big)^{\urcorner}\mathrm{d}u$$
$$\ll \int_0^{u(m)} T^{-\frac{3}{4}}(1+u)^{-1}\,\mathrm{d}u \ll T^{-\frac{3}{4}}$$

and hence we may truncate the integral to

$$t \ll \min\{u^{-\frac{1}{2}}, \left|m-\frac{r}{2q}-\frac{u\sqrt{N'}}{q}\right|^{-1}\}\log T. \tag{2.60}$$

We write the inner integral in (2.59) as $\int_0^{u(m)} \psi(u)e(\phi(u))\,\mathrm{d}u$ where $\phi(u) = -\frac{N'}{2q}u + \frac{\sqrt{N'}\rho t}{q}u + \frac{T}{4\pi}\log(1+u)$ and

$$\psi(u) = \frac{\tilde{w}(u)}{\sqrt{1+u}(a+\frac{i}{2}\log(1+u))}\exp\big(-\frac{2\pi}{qT}\gamma(t)^2(1+u) - \frac{2\pi t^2}{q}u\big) \tag{2.61}$$

in order to integrate by parts with respect to the factor $e(\phi(u))$. For this, we need a lower bound for $\phi'(u)$. Writing $u_1 = \frac{qT}{2\pi N'} - 1 = \sqrt{\frac{N}{N'}}$, we see that

$$\phi'(u) = \frac{\sqrt{N'}}{q}\rho t + \frac{T(u_1 - u)}{4\pi(1+u_1)(1+u)},$$

which shows that

$$|\phi'(u)| \gg T\,|u - u_1| \tag{2.62}$$

for $0 \leq u \leq u(m)$. Since $N' + \sqrt{NN'} = \frac{qT}{2\pi}$, we have

$$\phi'(u(m)) = \frac{\sqrt{N'}}{q}\rho t + \frac{T(\sqrt{N} - qm + \frac{r}{2})}{4\pi\sqrt{N'}(1+u_1)(1+u(m))}$$

so

$$\phi'(u(m)) \gg \sqrt{T}(\left|\sqrt{N} - qm\right| + 1). \tag{2.63}$$

By (2.62), the exponential $\phi(u)$ is stationary near u_1, so we estimate the corresponding integral in (2.59) by its absolute value around the point u_1, say for $|u - u_1| \leq T^{-\frac{1}{2}}$. The truncation (2.60) gives that the corresponding interval of integration for t is then of length $\ll \left|m - \frac{r}{2q} - \frac{\sqrt{N'}}{q}u_1\right|^{-1} \log T$, which is $\ll (\left|qm - \sqrt{N}\right| + 1)^{-1} \log T$. Hence the contribution of these terms to the sum (2.59) is

$$\ll \sum_{0<m\ll\sqrt{T}} \frac{T^{\frac{1}{4}} \log T}{\left|qm - \sqrt{N}\right| + 1} T^{-\frac{1}{2}} \ll T^{-\frac{1}{4}} \log^2 T.$$

It remains to estimate the contributions of the u-integrals over $|u - u_1| > T^{-\frac{1}{2}}$, that is, over intervals with end-points $u_1 \pm T^{-\frac{1}{2}}, u(m)$ and the end-points of the support of $\tilde{w}$. We will apply integration by parts to such intervals, thus obtaining integrated terms at $u_1 \pm \frac{1}{\sqrt{T}}$ and $u(m)$, the integrated terms corresponding to the end-points of the support of $\tilde{w}$ being 0. The integrated terms at $u_1 \pm \frac{1}{\sqrt{T}}$ are similar to the case $|u - u_1| \leq \frac{1}{\sqrt{T}}$ treated above, and they have a contribution of $O(T^{-\frac{1}{4}} \log^2 T)$.

The integrated term at $u(m)$ is

$$\frac{1}{2\pi i}\frac{\psi(u(m))}{\phi'(u(m))}e(\phi(u(m))) \ll \frac{T^{\frac{1}{4}}(m-\frac{r}{2q})^{-\frac{1}{2}}\exp(\frac{4\pi\sqrt{N'}}{qT}tu(m))}{\sqrt{T}(\left|\sqrt{N}-qm\right|+1)}e^{2\pi t(m-\frac{r}{2q})}$$

by (2.61) and (2.63), and the corresponding range of integration for t is of length $\ll u(m)^{-\frac{1}{2}}\log T \ll T^{\frac{1}{4}}(m-\frac{r}{2q})^{-\frac{1}{2}}\log T$, as $u(m)=\frac{2qm-r}{2\sqrt{N'}} \asymp T^{-\frac{1}{2}}(m-\frac{r}{2q})$. So the contribution of the integrated terms at $u(m)$ is

$$\ll T^{\frac{1}{4}} \sum_{0<m\ll\sqrt{T}} \frac{T^{\frac{3}{4}}(m-\frac{r}{2q})^{-\frac{3}{2}}}{\sqrt{T}(\left|\sqrt{N}-qm\right|+1)}\log T \ll \log T.$$

Continuing this way, the integrated terms and the integrand get smaller and smaller; their contribution is $O(\log T)$.

If $x_m(u) \le \sqrt{N'}$, then we use the contour joining $\sqrt{N'}, \sqrt{N'}+\epsilon\rho\sqrt{T}, \sqrt{N''}+\epsilon\rho\sqrt{T}$ and $\sqrt{N''}$ by line segments.

Finally, if $x_m(u) \ge \sqrt{N''}$, then we use the contour joining $\sqrt{N'}, \sqrt{N'}+\epsilon\overline{\rho}\sqrt{T}, \sqrt{N''}+\epsilon\overline{\rho}\sqrt{T}$ and $\sqrt{N''}$ by line segments.

This settles the error term on passing from (2.53) to (2.54).

Let us now estimate the error terms on applying Atkinson's saddle-point theorem to (2.55) to obtain (2.56). We use $G(y)=(n-\frac{r}{2q})^{\frac{1}{2}}, F(y)=y^{-1}T, \mu(y)=\beta y$ for a suitable constant $\beta>0$. Then the error term depending on y_0 is $G(y_0)\mu(y_0)F(y_0)^{-\frac{3}{2}}=(n-\frac{r}{2q})^{\frac{1}{2}}\beta y_0^{\frac{5}{2}}T^{-\frac{3}{2}}$, so the effect of these errors is

$$\begin{aligned}&\ll T^{-\frac{3}{2}}\sum_{0<n\ll\sqrt{T}}(n-\frac{r}{2q})y_0^{\frac{5}{2}}\\ &\ll T^{-\frac{3}{2}}T^{\frac{5}{4}}\sum_{0<n\ll\sqrt{T}}(n-\frac{r}{2q})^{-\frac{3}{2}} \ll T^{-\frac{1}{4}}.\end{aligned}$$

Next, the second error term is

$$\int_{A_1}^{B_1} G(t)\exp(-c\frac{q}{2}(n-\frac{r}{2q})^2\mu(t)-cF(t))(1+\mu'(t))\,\mathrm{d}t$$
$$\ll (n-\frac{r}{2q})^{\frac{1}{2}}\frac{\sqrt{T}}{n-\frac{r}{2q}}e^{-c\sqrt{T}(n-\frac{r}{2q})}$$

since $B_1 - A_1 \ll (n-\frac{r}{2q})^{-1}\sqrt{T}$ and $c\frac{q}{2}(n-\frac{r}{2q})^2\mu(t)+cF(t) \gg (n-\frac{r}{2q})^2\frac{2\sqrt{N'}}{2qn-r} + \frac{T}{\frac{2\sqrt{N''}}{2qn-r}} \gg \sqrt{T}(n-\frac{r}{2q})$. It follows that the effect of such terms is

$$\sqrt{T}\sum_{0<n\ll\sqrt{T}} e^{-c\sqrt{T}(n-\frac{r}{2q})} \ll 1.$$

Now, the third error term is

$$\frac{G(A_1)}{|\phi'(A_1)|+\sqrt{|\phi''(A_1)|}}.$$

We have

$$\phi'(A_1) = \big(-\frac{qT}{\pi\sqrt{N'}(2qn-r+2\sqrt{N'})}+1\big)\frac{q}{2}(n-\frac{r}{2q})^2 \tag{2.64}$$
$$= \frac{q}{2}\frac{(n-\frac{r}{2q})^2}{n-\frac{r}{2q}+\frac{\sqrt{N'}}{q}}(n-\frac{r}{2q}-\frac{\sqrt{N}}{q}). \tag{2.65}$$

and

$$\phi''(A_1) = \frac{T(1+2A_1)}{4\pi A_1^2(A_1+1)^2}. \tag{2.66}$$

From (2.65) and (2.66), it is clear that $\phi'(A_1) \gg \sqrt{T}\left|n-\frac{r}{2q}-\frac{\sqrt{N}}{q}\right|$ and $\phi''(A_1) \gg T$ for $n \asymp \sqrt{T}$ and (2.64) shows that $\phi'(A_1) \gg (n-\frac{r}{2q})^2$ if $n \ll \sqrt{T}$. Hence the contribution of these error terms is

$$\ll \sum_{n\asymp\sqrt{T}} \frac{(n-\frac{r}{2q})^{\frac{1}{2}}}{\max\{\sqrt{T}\left|n-\frac{r}{2q}-\frac{\sqrt{N}}{q}\right|, T\}} + \sum_{0<n\ll\sqrt{T}} \frac{(n-\frac{r}{2q})^{\frac{1}{2}}}{(n-\frac{r}{2q})^2} \ll 1.$$

Similarly, the fourth error terms have a contribution $\ll 1$ to I_1.

Thus, the contribution of all the error terms to I_1 is $\ll \log T$, and the proof is complete, that is, we have

$$F(T,\chi) = 2\Re S_1(T,\chi) + 2\Re S_2(T,\chi) + O(\log T)^{\frac{5}{4}},$$

where $S_1(T,\chi)$ and $S_2(T,\chi)$ are the main terms respectively of I_1 and I_2, as given by (2.57) and (2.50). We have used the abbreviations

$$\kappa^* = \frac{\sqrt{2}\kappa e^{\frac{5\pi i}{8}}}{q^{\frac{1}{4}}}, \tag{2.67}$$

$$\kappa_* = -\sqrt{2} i \kappa q^{\frac{1}{4}} e^{\frac{3\pi i}{8}} \tag{2.68}$$

in the statement of Theorem 21.

2.8 Proof of Corollary 22

Taking $f(u) = \frac{t}{2\pi}\log(u+\alpha)$ with $\alpha \geq 0, t \geq 2\pi$ in Lemma 19, we see that (1.30) is satisfied with $F = t$ and $\nu = 1$, so we can state

Lemma 35. *The following estimate for zeta sums holds:*

$$\sum_{N<n\leq M} (n+\alpha)^{it} \ll \sqrt{N} t^{\frac{1}{6}} \quad (t \geq 2\pi)$$

whenever $N < M \leq 2N$ *and* $N \leq t$.

An immediate corollary is:

Lemma 36. *The following estimate holds*

$$\sum_{n\leq x} (n+\alpha)^{it} \ll \sqrt{x} t^{\frac{1}{6}}$$

whenever $2 \leq x \leq 2t$ *and* $t \geq 2\pi$.

Proof. Let $2^m < x \le 2^{m+1}$. We write the sum $\sum_{n\le x}(n+\alpha)^{it}$ as

$$\sum_{n\le x}(n+\alpha)^{it} = (1+\alpha)^{it} + \sum_{k=0}^{m-1}\Big(\sum_{2^k<n\le 2^{k+1}}(n+\alpha)^{it}\Big) + \Big(\sum_{2^m<n\le x}(n+\alpha)^{it}\Big).$$

We can apply the previous lemma to each of the bracketed expressions to get the stated result. □

The following special case is what interests us most:

Lemma 37. *For $2 \le x \le 2t$ with $t \ge 3$, we have*

$$\sum_{\substack{n\le x\\ n\equiv r[q]}} n^{it} \ll_q \sqrt{x}t^{\frac{1}{6}}.$$

Here q is a positive integer and $1 \le r \le q-1$.

We can now prove Corollary 22:

Proof of Corollary 22. We note that

$$\sum_{n<\sqrt{N'}} \frac{\chi(n)}{\log(\frac{qT}{2\pi n^2})} n^{-\frac{1}{2}-iT} = \sum_{r=1}^{q}\chi(r)\sum_{\substack{n<\sqrt{N'}\\ n\equiv r[q]}} \frac{1}{\sqrt{n}\log(\frac{qT}{2\pi n^2})} n^{-iT}$$

so it suffices to show that the sum

$$\sum_{\substack{n<\sqrt{N'}\\ n\equiv r[q]}} \frac{1}{\sqrt{n}\log(\frac{qT}{2\pi n^2})} n^{-iT} \ll T^{\frac{1}{6}}\log T$$

for each $r = 1, ..., q-1$. We proceed by summation by parts. Let us write $m = \lfloor(\sqrt{N'}-r)/q\rfloor$, $h(u) = \frac{qT}{2\pi(qu+r)^2}$ and

$$f(u) = \begin{cases} \sum_{k\le u}(qk+r)^{-iT}, & \text{if } u \ge 1,\\ 0, & \text{if } u < 1;\end{cases}$$

Note that $\log(h(u)) \geq \log(h(m)) \asymp 1$. The sum gets rewritten as

$$\sum_{k=0}^{m} \frac{(qk+r)^{-iT}}{\sqrt{qk+r}\log h(k)} = \frac{r^{-iT}}{\sqrt{r}\log(\frac{qT}{2\pi r^2})} + \frac{f(m)}{\sqrt{qm+r}\log\big(\frac{qT}{2\pi(qm+r)^2}\big)} + q\int_1^m \frac{f(u)}{(qu+r)\log h(u)}\Big\{\frac{1}{2\sqrt{qu+r}} - \frac{2}{\log h(u)}\Big\}\,\mathrm{d}u.$$

We then use the estimate for $f(u)$ given in Lemma 37 to get that the sum is $\ll T^{\frac{1}{6}}\log T$. This completes the proof of the corollary. □

2.9 Description of the limit function $K_{q,r}(x)$

Let us first note that the inner sum in $S_1(T,\chi)$ may be rewritten as

$$\sum_{k=1}^{m} \frac{(-1)^{\frac{k(qk-r)}{2}}}{2qk-r}\,\mathsf{e}\big(T, q(k-\frac{r}{2q})^2\big)\exp\big(\frac{i}{2}\,\mathsf{f}(T, q(k-\tfrac{r}{2q})^2) - \frac{3\pi i}{8}\big) \tag{2.69}$$

where $m = \Big\lfloor (2\sqrt{N}+r)/(2q) \Big\rfloor \asymp \sqrt{T}$. Now, the Puiseux expansions of $\frac{1}{2}\,\mathsf{f}(T,\ell)$ and $\mathsf{e}(T,\ell)$ for big T are respectively

$$\frac{1}{2}\,\mathsf{f}(T,\ell) = -\frac{\pi}{8} + 2\pi\sqrt{\frac{T\ell}{2\pi}} + \psi(\sqrt{\ell/q}) \tag{2.70}$$

$$\mathsf{e}(T,\ell) = 1 - \frac{1}{12}\big(\frac{\pi\ell}{2T}\big) + \frac{97}{1440}\big(\frac{\pi\ell}{2T}\big)^2 + \cdots \tag{2.71}$$

$$= 1 + O\big(\frac{\ell}{T}\big)$$

with

$$\psi(\ell) = \frac{1}{6\sqrt{2}}\frac{(q\pi)^{3/2}}{\sqrt{T}}\ell^3 - \frac{1}{80\sqrt{2}}\frac{(q\pi)^{5/2}}{T^{3/2}}\ell^5 + \cdots \tag{2.72}$$

Writing $\sqrt{qT/2\pi} = qL + \vartheta$ with $L \in \mathbb{N}$ and $0 \leq \vartheta < q$, the sum gets rewritten as

$$(-1)^{rL+1} i \sum_{k=1}^{m} \frac{(-1)^{\frac{k(qk-r)}{2}}}{2qk - r} \mathrm{e}\left(T, q(k - \frac{r}{2q})^2\right) \exp\left\{2\pi(k - \frac{r}{2q})\vartheta i + i\psi(k - \frac{r}{2q})\right\} \tag{2.73}$$

Hence, ignoring the factor $\mathrm{e}(T, \ell) = 1 + O(\ell T^{-1})$ and the phase $\psi(\ell) = O(\ell^3 T^{-\frac{1}{2}})$, the sum in (2.73) leads us to consider the function

$$K_{q,r}(x) = \sum_{k=1}^{\infty} (-1)^{\frac{k(qk-r)}{2}} \frac{e\big((2qk - r)x\big)}{2qk - r} \tag{2.74}$$

which is of period 1. The behaviour of this function depends upon the parity of q.

To give a compact expression for $K_{q,r}(x)$ we have this short

Lemma 38. *The equation*

$$\sum_{\substack{n \geq 1 \\ n \equiv a[q]}} \frac{z^n}{n} = -\frac{1}{q} \sum_{k=1}^{q} e(-\frac{ak}{q}) \log\left(1 - ze^{\frac{2\pi ik}{q}}\right) \tag{2.75}$$

holds whenever both sides are defined.

Proof. We use the fact that

$$\log(1 - z) = -\sum_{n=1}^{\infty} \frac{z^n}{n} \quad (|z| \leq 1, z \neq 1)$$

and

$$\frac{1}{q} \sum_{k=1}^{q} e\Big(\frac{k(n-a)}{q}\Big) = \begin{cases} 1, \text{ if} & n \equiv a \bmod q, \\ 0, \text{ if} & n \not\equiv a \bmod q, \end{cases}$$

to obtain the formula (2.75) valid for $|z| \leq 1, z \neq e^{-\frac{2\pi ik}{q}}$ for any $k = 1, ..., q$. □

When q is even

Then we only have to consider even r; we see that $(-1)^{\frac{k(qk-r)}{2}} = (-1)^{k\epsilon_{q,r}}$ where $\epsilon_{q,r} = \frac{1-(-1)^{\frac{q-r}{2}}}{2}$. Thus the Fourier series (2.74) becomes

$$K_{q,r}(x) = \sum_{k=1}^{\infty} (-1)^{k\epsilon_{q,r}} \frac{e\big((2qk-r)x\big)}{2qk-r}.$$

Using (2.75), we immediately obtain

$$K_{q,r}(x) = -\frac{1}{2q} e\big(\frac{r\epsilon_{q,r}}{4q}\big) \sum_{k=1}^{2q} e\left(\frac{rk}{2q}\right) \log\big(1 - e(x + \frac{2k+\epsilon_{q,r}}{4q})\big). \tag{2.76}$$

When q is odd

In this case, we only need to consider odd r; it is easily seen that $(-1)^{\frac{k(qk-r)}{2}} = (-1)^{\frac{k(k-\delta_{q,r})}{2}}$ with $\delta_{q,r} = (-1)^{\frac{q-r}{2}}$. Using (2.75), we easily obtain

$$K_{q,r}(x) = -\frac{1}{4q} e\left(\tfrac{r}{8q}\right) \sum_{k=0}^{2q-1} (1 - \delta_{q,r}(-1)^k i) e\left(\frac{kr}{4q}\right) \times \tag{2.77}$$
$$\left\{ \log\left(1 - e(x + \frac{2k+1}{8q})\right) - \log\left(1 + e(x + \frac{2k+1}{8q})\right) \right\}.$$

Thus, we have given a compact expression for $K_{q,r}(x)$ in each case. It should be remarked that the right side of (2.76) is not defined when $x + \frac{2k+\epsilon_{q,r}}{4q}$ is an integer for some $1 \leq k \leq 2q$, and this happens exactly when x is of the form $n \pm \frac{2k+\epsilon_{q,r}}{4q}$ for some $1 \leq k \leq 2q$ and some $n \in \mathbb{Z}$. In particular, $K_{q,r}\left(\frac{\vartheta}{2q}\right)$ with ϑ as above is not defined when ϑ is of the form $\frac{2k+\epsilon_{q,r}}{2}$ for some $k = 1, 2, ..., 2q$. Hence we put $\vartheta_0 = \min\left\{ \left|\vartheta - \frac{n}{2}\right| : n \in \mathbb{Z}\right\}$ in case q and r are even.

Similarly, in view of (2.77), we put $\vartheta_0 = \min\left\{ \left|\vartheta - (q - \frac{2k+1}{4})\right| : 0 \leq k \leq 2q-1\right\}$ in case q and r are odd. Thus, in both cases, $K_{q,r}\left(\frac{\vartheta}{2q}\right)$ is not defined (the series does not converge) when $\vartheta_0 = 0$.

Let us call the sum in (2.73) $\Sigma_r(T)$ (without the factor $(-1)^{rL+1}i$). With the function $K_{q,r}$ in mind, we write, when $\vartheta_0 \neq 0$,

$$\Sigma_r(T) = K_{q,r}(\frac{\vartheta}{2q}) + A - B + C \tag{2.78}$$

where

$$A = \sum_{k \leq N_0} \frac{(-1)^{\frac{k(qk-r)}{2}}}{2qk - r} \Big\{ \mathsf{e}\left(T, q(k - \frac{r}{2q})^2\right) e\Big(\vartheta(k - \frac{r}{2q}) + \frac{1}{2\pi}\psi(k - \frac{r}{2q})\Big) - e\big(\vartheta(k - \frac{r}{2q})\big)\Big\},$$

$$B = \sum_{k > N_0} \frac{(-1)^{\frac{k(qk-r)}{2}}}{2qk - r} e\big(\vartheta(k - \frac{r}{2q})\big),$$

$$C = \sum_{N_0 < k \leq m} \frac{(-1)^{\frac{k(qk-r)}{2}}}{2qk - r} \mathsf{e}\left(T, q(k - \frac{r}{2q})^2\right) e\big(\vartheta(k - \frac{r}{2q}) + \frac{1}{2\pi}\psi(k - \frac{r}{2q})\big).$$

Here $N_0 \in]1, m[$ is to be chosen later. We estimate these sums one by one in later sections. Recall that the nature of these sums depends upon the parities of q and r, and that we only need to consider the cases where q and r are of the same parity, since the coefficient $\tau^*(r, \chi)$ in (2.6) vanishes when $q + r$ is odd. So we study the two cases separately.

We first note that since $\mathsf{e}(T, \ell) = 1 + O(\ell T^{-1})$, we trivially have

$$A = \sum_{k \leq N_0} (-1)^{\frac{k(qk-r)}{2}} \frac{e\big(\vartheta(k - \frac{r}{2q})\big)}{2qk - r} \Big\{ e\Big(\frac{1}{2\pi}\psi(k - \frac{r}{2q})\Big) - 1\Big\} + O\left(\frac{N_0^2}{T}\right). \tag{2.79}$$

Let us first study the behaviour of $\Sigma_r(T)$ when $\vartheta_0 = 0$; this is what gives the formula (2.10), as indicated in the statement itself. In fact, our proof of Theorem 23 itself is independent of that of (2.10).

2.10 Proof of (2.10) and (2.12)

The object of this section is to study the behaviour of the function $\Sigma_r(T)$ when $\vartheta_0 = 0$. We have to distinguish two cases, namely, q even and q odd.

When q is even, we have seen that $(-1)^{\frac{k(qk-r)}{2}} = (-1)^{k\epsilon_{q,r}}$ where $\epsilon_{q,r} = \frac{1-(-1)^{\frac{q-r}{2}}}{2}$. Moreover, $\vartheta_0 = 0$ means that $\vartheta = n/2$ for some $n \in \mathbb{Z}$. It follows that

$(-1)^{\frac{k(qk-r)}{2}} e^{2\pi k\vartheta i} = 1$ for all k (to see this, consider the cases $q \equiv r \pmod 4$ and $q \not\equiv r \pmod 4$ separately). Hence

$$\Sigma_r(T) = e\left(-\frac{r\vartheta}{2q}\right) \sum_{k=1}^{m} \frac{\exp\left\{i\psi(k-\frac{r}{2q})\right\}}{2qk-r} \, \mathrm{e}\left(T, q(k-\frac{r}{2q})^2\right).$$

To study this sum, let us first consider the sums

$$S_{N_1,N_2}(\varrho,\theta) = \sum_{N_1<k\leq N_2} \frac{\exp\left\{i\psi(\varrho k+\theta)\right\}}{\varrho k+\theta} \quad (0 \leq N_1 < N_2)$$

where $\varrho \geq 1$ and $-1 < \theta < 1$. Indeed, partial summation reduces the study of $\Sigma_r(T)$ to those of the sums $S_{N_1,N_2}(\varrho,\theta)$, since $\mathrm{e}(T,\ell) = 1 + O(\ell T^{-1})$. We first have the following lemma:

Lemma 39. *Let $n \asymp \sqrt{T}$. Then*

$$S_{0,n}(\varrho,\theta) = \frac{1}{6\varrho}\log T + \gamma(\varrho,\theta) + O(T^{-\frac{1}{12}}\log T) \tag{2.80}$$

where

$$\gamma(\varrho,\theta) = \frac{1}{2(\varrho+\theta)} - \varrho\int_1^\infty \frac{B_1(t)}{(\varrho t+\theta)^2}\,\mathrm{d}t - \frac{1}{3\varrho}\log\left(\frac{(q\pi)^{\frac{3}{2}}}{6\sqrt{2}}(\varrho+\theta)^3\right) - \frac{\gamma}{3\varrho} + \frac{\pi i}{6\varrho} \tag{2.81}$$

is a constant independent of T. Here, $B_1(t) = \{t\} - \frac{1}{2}$ is the usual Bernoulli function and γ is the Euler-Mascheroni constant.

Proof. Let $M = \lfloor \frac{1}{\varrho}(q\pi)^{-\frac{3}{4}} T^{\frac{1}{4}} \rfloor$. It is enough to prove that

$$S_{0,M}(\varrho,\theta) = \frac{1}{6\varrho}\log T + \gamma(\varrho,\theta) + O(T^{-\frac{1}{12}}\log T), \tag{2.82}$$

$$S_{M,n}(\varrho,\theta) \ll T^{-\frac{1}{12}}\log T. \tag{2.83}$$

From (2.72), we observe that

$$\exp\left(i\psi(\ell)\right) - \exp\left(\frac{i}{6\sqrt{2}}\frac{(q\pi)^{3/2}}{\sqrt{T}}\ell^3\right) \ll \frac{\ell^5}{T^{\frac{3}{2}}}$$

so we first simplify the sum $S_{0,M}(\varrho,\theta)$ to

$$S_{0,M}(\varrho,\theta) = \sum_{k\le M} \frac{e\left(c(\varrho k+\theta)^3\right)}{\varrho k+\theta} + O\left(T^{-\frac{1}{4}}\right)$$

where $c \asymp T^{-\frac{1}{2}}$. Take $M_1 \asymp T^{\frac{1}{3}}$. Using truncated Poisson summation [32, (10.2)], we have

$$S'_{0,M}(\varrho,\theta) = \sum_{|k|\le M_1} \int_1^M \frac{e\left(g_k(t)\right)}{\varrho t+\theta}\,\mathrm{d}t + O\left(T^{-\frac{1}{12}}\log T\right), \tag{2.84}$$

where $g_k(t) = c(\varrho t+\theta)^3 + kt$ and $S'_{0,M}(\varrho,\theta)$ is the sum obtained from $S_{0,M}(\varrho,\theta)$ by replacing the first and last terms by half of their respective values. Let us call the k^{th} integral in (2.84) $\mathcal{I}_k$. For $k=0$, we use Lemma 42 and get

$$\mathcal{I}_0 = -\frac{1}{3\varrho}\log\left(2\pi c(\varrho+\theta)^3\right) - \frac{\gamma}{3\varrho} + \frac{\pi i}{6\varrho} + O\left(T^{-\frac{1}{4}}\right).$$

Here, γ is the usual Euler-Mascheroni constant. For $k \neq 0$, our choice of M shows that $g'_k(t)$ has no zero and hence we can integrate by parts with respect to the oscillating factor. Thus

$$\mathcal{I}_k = -\frac{1}{2\pi i}\frac{e\left(c(\varrho+\theta)^3\right)}{(\varrho+\theta)(3c(\varrho+\theta)^2+k)} + \frac{\varrho}{2\pi i}\int_1^M \frac{e\left(g_k(t)\right)}{(\varrho t+\theta)^2 g'_k(t)}\,\mathrm{d}t + O\left(\frac{1}{M\,|k|}\right).$$

The quantity $g'_k(t)$ in the denominator of the integrand can be replaced by k with an error $\ll \frac{1}{k^2 M}$. We then divide the integral over $1 \le t \le M$ into two parts: $1 \le t \le M_2$ and $M_2 \le t \le M$ where $M_2 = \left\lfloor \sqrt{M} \right\rfloor$. The integral over the latter interval is obviously $\ll \frac{1}{|k|\sqrt{M}}$ so the contribution of this part to $S'_{0,M}(\varrho,\theta)$ is $\ll T^{-\frac{1}{8}}\log T$. On the first interval, we replace $g_k(t)$ by kt with an error of $O\left(\frac{1}{M}\right)$

contributing an overall error of $O\left(T^{-\frac{1}{4}}\log T\right)$ to $S'_{0,M}(\varrho,\theta)$. Thus, we have

$$S'_{0,M}(\varrho,\theta)=\mathcal{I}_0+\sum_{0<|k|\leq M_1}\frac{\varrho}{2\pi ik}\int_1^{M_2}\frac{e\left(kt\right)}{(\varrho t+\theta)^2}\,\mathrm{d}t+O\left(T^{-\frac{1}{12}}\log T\right). \tag{2.85}$$

Integrating by parts, we see that

$$\int_1^{M_2}\frac{e\left(kt\right)}{(\varrho t+\theta)^2}\,\mathrm{d}t=\frac{1}{\varrho(\varrho+\theta)}-\frac{1}{\varrho(\varrho M_2+\theta)}+\frac{2\pi ik}{\varrho}\int_1^{M_2}\frac{e\left(kt\right)}{\varrho t+\theta}\,\mathrm{d}t.$$

This simplifies (2.85) to

$$S'_{0,M}(\varrho,\theta)=\mathcal{I}_0+\sum_{0<|k|\leq M_1}\int_1^{M_2}\frac{e\left(kt\right)}{\varrho t+\theta}\,\mathrm{d}t+O\left(T^{-\frac{1}{12}}\log T\right).$$

But now, this sum is easily recognised to be a partial sum obtained from an application of the truncated Poisson formula to the sum $\sum_{k\leq M_2}\frac{1}{\varrho k+\theta}$, so we have

$$S'_{0,M}(\varrho,\theta)=\mathcal{I}_0-\varrho\int_1^{\infty}\frac{B_1(t)}{(\varrho t+\theta)^2}\,\mathrm{d}t+O\left(T^{-\frac{1}{12}}\log T\right)$$

where $B_1(t)=\{t\}-\frac{1}{2}$ is the first Bernoulli function. Plugging these into (2.84), we obtain (2.82).

To prove (2.83), we note that

$$S_{n_1,n_2}(\varrho,\theta)=\frac{s(n_1,n_2)}{\varrho n_2+\theta}+\varrho\int_{n_1}^{n_2}\frac{s(n_1,t)}{(\varrho t+\theta)^2}\,\mathrm{d}t$$

for $n_1<n_2$, where $s(t,u)=\sum_{t<k\leq u}\exp\left(i\psi(\varrho k+\theta)\right)$. Using the second and third derivative tests of the van der Corput method (see Theorems 17 & 18), we have

$$\begin{aligned}s(n_1,n_2)&\ll\min\left\{n_1^{\frac{3}{2}}T^{-\frac{1}{4}}+n_1^{-\frac{1}{2}}T^{\frac{1}{4}},n_1T^{-\frac{1}{12}}+n_1^{\frac{1}{2}}T^{\frac{1}{12}}\right\}\\&\ll n_1T^{-\frac{1}{12}}\end{aligned}$$

whenever $T^{\frac{1}{4}}\ll n_1<n_2\leq 2n_1\ll T^{\frac{1}{2}}$. It follows that $S_{n_1,n_2}(\varrho,\theta)\ll T^{-\frac{1}{12}}$ for each such pair n_1,n_2 and (2.83) follows. □

From (2.80), we obtain

$$\Sigma_r(T) = e\left(-\frac{r\vartheta}{2q}\right)\left\{\frac{1}{12q}\log T + \gamma(1, -\tfrac{r}{2q})\right\} + O\left(T^{-\frac{1}{12}}\log T\right). \tag{2.86}$$

Thus, the case of q even is complete.

We next consider the case of q odd. Take M as in our proof of Lemma 39. Assume $q \equiv r \pmod 4$ (the other case is similar). This time $\vartheta = \frac{2j+1}{4}$ for some j and we have $e(k\vartheta) = (-1)^{jk} i^k$ so

$$\begin{aligned}\Sigma_r(T) &= e\left(-\frac{r\vartheta}{2q}\right)\sum_{k\le M}\frac{(-1)^{\frac{k(k-1)}{2}+jk} i^k}{2qk-r}\exp\left\{i\psi(k-\frac{r}{2q})\right\} + O\left(T^{-\frac{1}{12}}\log T\right)\\ &= \frac{1}{2q}e\left(-\frac{r\vartheta}{2q}\right)\left(S_{0,\frac{M}{2}}(2,-\frac{r}{2q}) + (-1)^j i S_{0,\frac{M-1}{2}}(2, 1-\frac{r}{2q})\right) + O\left(T^{-\frac{1}{12}}\log T\right).\end{aligned}$$

Idem, *mutatis mutandis,* for the case $q \not\equiv r \pmod 4$. Using (2.82), we obtain

$$\Sigma_r(T) = \frac{1}{24q}e\left(-\frac{r\vartheta}{2q}\right)\left(1+\delta_{q,r}(-1)^j i\right)\log T + \gamma_j(2,r) + O\left(T^{-\frac{1}{12}}\log T\right) \tag{2.87}$$

where

$$\gamma_j(\varrho, r) = \frac{1}{2q}e\left(-\frac{r\vartheta}{2q}\right)\left\{\gamma(\varrho, -\frac{r}{2q}) + (-1)^j i\delta_{q,r}\gamma(\varrho, 1-\frac{r}{2q})\right\} \tag{2.88}$$

with $\delta_{q,r} = (-1)^{\frac{q-r}{2}}$. The proof of (2.10) and (2.12) is complete.

We now prove the two corollaries 23 and 25.

2.11 Proof of Theorem 23: Technical part I

Since we have just treated the case $\vartheta_0 = 0$, we assume henceforth that $\vartheta_0 \neq 0$. We thus have to estimate the three sums A, B and C of (2.78). This section concerns the case of q even and the cases $q \equiv r \pmod 4$ and $q \not\equiv r \pmod 4$ will be treated separately.

Case I: $q \equiv r \pmod 4$

We recall that in this case $\vartheta_0 = \|\vartheta\| \equiv \min\{\,|\vartheta - n| : n \in \mathbb{Z}\}$. We choose $N_0 \asymp T^{\frac{1}{6}}$.

2.11.1 Estimate of the sum A

Let us call A' the sum in (2.79), so it is simply

$$A' = \sum_{k \le N_0} \frac{e\big(\vartheta(k - \frac{r}{2q})\big)}{2qk - r} \left\{ e\left(\frac{1}{2\pi}\psi(k - \frac{r}{2q}) \right) - 1 \right\}.$$

By (2.72), we may replace (to simplify things) $\frac{1}{2\pi}\psi(k - \frac{r}{2q})$ by $c(k - \frac{r}{2q})^3$ (where $c = \frac{1}{12\sqrt{2}} \frac{\sqrt{q^3\pi}}{\sqrt{T}} \asymp T^{-\frac{1}{2}}$) with an error of $O(T^{-\frac{2}{3}})$, that is,

$$A' = \sum_{k \le N_0} \frac{e\big(\vartheta(k - \frac{r}{2q})\big)}{2qk - r} \left\{ e\left(c(k - \frac{r}{2q})^3 \right) - 1 \right\} + O\left(T^{-\frac{2}{3}}\right).$$

Choose an integer $M_0 \asymp T^{\frac{1}{8}}$. A trivial estimation (i.e., using absolute values) gives

$$\sum_{k \le M_0} \frac{e\big(\vartheta(k - \frac{r}{2q})\big)}{2qk - r} \left\{ e\left(c(k - \frac{r}{2q})^3 \right) - 1 \right\} \ll T^{-\frac{1}{8}};$$

thus,

$$A' = \sum_{M_0 \le k \le N_0} \frac{e\big(\vartheta(k - \frac{r}{2q})\big)}{2qk - r} \left\{ e\left(c(k - \frac{r}{2q})^3 \right) - 1 \right\} + O\left(T^{-\frac{1}{8}}\right).$$

Call this sum $A^\circ = A_1^\circ - A_2^\circ$, with obvious notations. By Poisson summation (1.8), we have

$$A_1^\circ = \sum_{n \in \mathbb{Z}} \int_{M_0}^{N_0} \frac{e\left(f_n(t)\right)}{2qt - r}\,\mathrm{d}t + O\left(T^{-\frac{1}{8}}\right)$$

where $f_n(t) = nt + \vartheta(t - \frac{r}{2q}) + c(t - \frac{r}{2q})^3$. Note that $f_n'(t)$ has no zero in $[M_0, N_0]$, so integrating by parts twice with respect to the oscillating factor, we see that $A_1^\circ \ll T^{-\frac{1}{8}} \max\{\vartheta^{-1}, 1\}$. We proceed similarly for A_2° and obtain the same result, namely $A_2^\circ \ll T^{-\frac{1}{8}} \max\{\vartheta^{-1}, 1\}$. Hence $A \ll T^{-\frac{1}{8}} \max\{\vartheta^{-1}, 1\}$. If we apply Kusmin-Landau to A_2°, we will obtain $A_2^\circ \ll T^{-\frac{1}{8}}\vartheta_0^{-1}$, which is *not* better than our bound.

Since it is trivially (i.e., by absolute values) true that $A \ll 1$, we have

$$A \ll \min\left\{T^{-\frac{1}{8}}\vartheta^{-1}, 1\right\}.$$

2.11.2 Estimate of the sum B

We have

$$B = \sum_{k > N_0} \frac{e\big(\vartheta(k - \frac{r}{2q})\big)}{2qk - r}.$$

By Kusmin-Landau theorem (see Theorem 15) and partial summation, we have $B \ll T^{-\frac{1}{6}}\vartheta_0^{-1}$. Since B is boundedly convergent, we also have $B \ll 1$, so

$$B \ll \min\left\{T^{-\frac{1}{6}}\vartheta_0^{-1}, 1\right\}.$$

2.11.3 Estimate of the sum C

Let us write $f(k) = \vartheta(k - \frac{r}{2q}) + \frac{1}{2\pi}\psi(k - \frac{r}{2q})$ and $g(k) = \frac{\mathrm{e}\left(T, q(k - \frac{r}{2q})^2\right)}{2qk - r}$ so $C = \sum_{N_0 < k \le m} g(k) e\left(f(k)\right)$. First, for any $N_1 < N_2$,

$$\sum_{N_1 < k \le N_2} g(k) e\left(f(k)\right) = g(m) S(N_1, N_2) - \int_{N_1}^{N_2} S(N_1, t) g'(t)\,\mathrm{d}t,$$

where $S(t, u) = \sum_{t < k \le u} e\left(f(k)\right)$. Since $g(k) \ll \frac{1}{k}$ and $g'(k) = -\frac{2q}{(2qk - r)^2} + O\left(\frac{1}{T}\right)$, we see that

$$\sum_{N_1 < k \le N_2} g(k) e\left(f(k)\right) \ll N_1^{-1} \max_{N_1 < t \le N_2} \left|S(N_1, t)\right|. \tag{2.89}$$

Thus, it suffices to estimate the sums $S(t, u)$.

Note that

$$\begin{aligned}\psi(\ell/q) &= T \operatorname{argsinh}\left(\sqrt{\frac{\pi}{2T}}\ell\right) + \frac{\ell}{2}\sqrt{(\pi\ell)^2 + 2\pi T} - 2\pi\sqrt{\frac{T}{2\pi}}\ell, \\ &= \int_0^\ell \left(\sqrt{\pi^2 x^2 + 2\pi T} - \sqrt{2\pi T}\right) \mathrm{d}x\end{aligned}$$

so $\psi'(\ell)$ and hence $f'(\ell)$ is clearly monotonic. We first suppose $\ell \ll T^{\frac{2}{9}}$. If $\vartheta_0 = \frac{1}{2}$, then $\|f'(\ell)\| = \frac{1}{2} - \frac{1}{2\pi}\psi'(\ell - \frac{r}{2q}) \gg 1 - cT^{-\frac{1}{18}}$. Kusmin-Landau then gives

$$S(t,u) \ll \frac{1}{1 - cT^{-\frac{1}{18}}} \ll 1$$

for $t < u \ll T^{\frac{2}{9}}$. If $\vartheta_0 < \frac{1}{2}$, then $\|f'(\ell)\| = \vartheta_0 + O(\ell^2 T^{-\frac{1}{2}}) \geq \vartheta_0 + cT^{-\frac{1}{6}}$ for some constant $c > 0$. By Kusmin-Landau again, we have

$$S(t,u) \ll \frac{1}{\vartheta_0 + cT^{-\frac{1}{6}}} \ll \min\{\vartheta_0^{-1}, T^{\frac{1}{6}}\}$$

for $t < u \ll T^{\frac{2}{9}}$. Hence, in any case,

$$\begin{aligned}\sum_{N_0 < k \leq M} g(k) e\left(f(k)\right) &\ll N_0^{-1} \min\{\vartheta_0^{-1}, T^{\frac{1}{6}}\} \\ &\ll \min\{T^{-\frac{1}{6}}\vartheta_0^{-1}, 1\}\end{aligned}$$

as long as $M \ll T^{\frac{2}{9}}$. Thus, it remains to estimate the contribution of the range $T^{\frac{2}{9}} \ll \ell \ll T^{\frac{1}{2}}$. So let $T^{\frac{2}{9}} \ll N_1 \leq N_2 \leq 2N_1 \ll T^{\frac{1}{2}}$. Applying the second and third derivative tests of the van der Corput method in (2.89), we have

$$\begin{aligned}\sum_{N_1 < k \leq N_2} g(k) e\left(f(k)\right) &\ll N_1^{-1} \min\left\{N_1^{\frac{3}{2}} T^{-\frac{1}{4}} + N_1^{-\frac{1}{2}} T^{\frac{1}{4}}, N_1 T^{-\frac{1}{12}} + N_1^{\frac{1}{2}} T^{\frac{1}{12}}\right\} \\ &\ll T^{-\frac{1}{12}}.\end{aligned}$$

Therefore, the contribution of the range $T^{\frac{2}{9}} \ll \ell \ll T^{\frac{1}{2}}$ to C is $O\left(T^{-\frac{1}{12}} \log T\right)$. This shows that

$$C \ll \min\left\{T^{-\frac{1}{6}}\vartheta_0^{-1}, 1\right\}.$$

Case II. $q \not\equiv r \pmod 4$

In this case, we have $(-1)^{\frac{k(qk-r)}{2}} = (-1)^k$, and we recall that $\vartheta_0 = \min\{\left|\vartheta - \frac{2j+1}{2}\right| : j = 0, 1, ...\}$.

We take $N_0 \asymp T^{\frac{1}{6}}$. Proceeding as in the above case, we obtain the same bounds as above for A, B and C.

2.12 Proof of Theorem 25: Technical part II

This section concerns the case of q odd. In this case, we have seen that $(-1)^{\frac{k(qk-r)}{2}} = (-1)^{\frac{k(k-1)}{2}}$ if $q \equiv r \pmod 4$ and $(-1)^{\frac{k(qk-r)}{2}} = (-1)^{\frac{k(k+1)}{2}}$ if $q \not\equiv r \pmod 4$. We deal with the two cases separately.

Case I: $q \equiv r \pmod 4$

In this case, we have $(-1)^{\frac{k(qk-r)}{2}} = (-1)^{\frac{k(k-1)}{2}}$ and we recall that $\vartheta_0 = \min\left\{\left|\vartheta - \frac{2j+1}{4}\right| : j \in \mathbb{Z}\right\}$; by assumption $\vartheta_0 \neq 0$.

We first have

$$A = \sum_{k \le N_0} (-1)^{\frac{k(k-1)}{2}} \frac{e\big(\vartheta(k - \frac{r}{2q})\big)}{2qk - r} \left\{ e\left(c(k - \frac{r}{2q})^3 \right) - 1 \right\} + O\left(T^{-\frac{2}{3}}\right)$$

with $c \asymp T^{-\frac{1}{2}}$. This sum may be written as $A_1 - A_2$ where

$$A_1 = \sum_{\substack{k \le N_0 \\ k \equiv 0,1 \pmod 4}} \frac{e\big(\vartheta(k - \frac{r}{2q})\big)}{2qk - r} \left\{ e\left(c(k - \frac{r}{2q})^3 \right) - 1 \right\},$$

$$A_2 = \sum_{\substack{k \le N_0 \\ k \equiv 2,3 \pmod 4}} \frac{e\big(\vartheta(k - \frac{r}{2q})\big)}{2qk - r} \left\{ e\left(c(k - \frac{r}{2q})^3 \right) - 1 \right\}$$

and now we can proceed similarly as in the previous cases to obtain a similar bound, namely,

$$A \ll \min\left\{T^{-\frac{1}{8}}\vartheta^{-1}, 1\right\}. \tag{2.90}$$

Similarly, B and C can be treated likewise to get the bounds

$$B \ll \min\left\{T^{-\frac{1}{6}}\vartheta_0^{-1}, 1\right\}, \quad C \ll \min\left\{T^{-\frac{1}{6}}\vartheta_0^{-1}, 1\right\}. \tag{2.91}$$

Case II: $q \not\equiv r \pmod 4$

In this case $(-1)^{\frac{k(qk-r)}{2}} = (-1)^{\frac{k(k+1)}{2}}$ and we have $\vartheta_0 = \min\left\{\left|\vartheta - \frac{2j+1}{4}\right| : j \in \mathbb{Z}\right\}$.

The treatment of this case is the same as the above, *mutatis mutandis.* We obtain the bounds (2.90) and (2.91).

Appendix: Lemmas on some sums and integrals

On more than one occasion, we had to estimate integrals of the form

$$I_k(v_0, b) = \int_{v_0}^{\infty} \frac{e^{-b(1+t)}}{(1+t)^{\alpha}\log^k(1+t)}\,\mathrm{d}t$$

where $0 < v_0 < 1$ and $\alpha \geq 0, b > 0, k \geq 1$. We want an upper bound in terms of v_0 and b. Such an upper bound is given by

Lemma 40. *We have*

$$I_k(v_0, b) \ll e^{-b}\max\left\{\frac{e^{-b}}{b}, \frac{\log\frac{1}{v_0}}{\log^{k-1}(1+v_0)}\right\}. \tag{2.92}$$

Proof. To estimate it, we separate the interval of integration into $[v_0, 1]$ and $[1, \infty[$. On $[1, \infty[$, we simply ignore the denominator and observe that

$$\int_1^{\infty} \frac{e^{-b(1+t)}}{(1+t)^{\alpha}\log^k(1+t)}\,\mathrm{d}t \ll e^{-b}\int_1^{\infty} e^{-bt}\,\mathrm{d}t = \frac{e^{-2b}}{b}.$$

On $[v_0, 1]$, we ignore the quantity $e^{-bt}/(1+t)^\alpha$ (since it changes very little in the interval) and observe that

$$\int_{v_0}^{1} \frac{e^{-b(1+t)}}{(1+t)^\alpha \log^k(1+t)}\,\mathrm{d}t \ll \frac{e^{-b}}{\log^{k-1}(1+v_0)} \int_{v_0}^{1} \frac{\mathrm{d}t}{\log(1+t)}$$
$$= \frac{e^{-b}}{\log^{k-1}(1+v_0)} \Big(-\log v_0 + \int_{v_0}^{1} \big(\frac{1}{\log(1+t)} - \frac{1}{t}\big)\,\mathrm{d}t \Big).$$

This last integral being bounded independently of v_0 (replace v_0 by 0), we find that the integral over $[v_0, 1]$ is $\ll e^{-b} \log^{1-k}(1+v_0) \log \frac{1}{v_0}$. This gives (2.92). □

Next, we estimate sums of the form

$$S_{\nu,\alpha}(x) = \sum_{k=1}^{\infty} \frac{e^{-2^k x}}{(1+2^k)^\alpha} k^\nu, \quad (0 < x < 1)$$

where $\nu > 0, \alpha \geq 0$ are real constants. We want an upper bound as a function of x.

Lemma 41. *For $\alpha > 0$, we have*

$$S_{\nu,\alpha}(x) \ll \max\big\{\frac{1}{2^\alpha \alpha}, \frac{x^\alpha}{(1+x)^\alpha}\big\} \log^\nu \frac{1}{x}. \tag{2.93}$$

For $\alpha = 0$,

$$S_{\nu,0} \ll \log^{\nu+1} \frac{1}{x}. \tag{2.94}$$

Proof. Write $X = (\log 2)^{-1} \log x^{-1}$. We may obviously assume that $x \leq e^{-\frac{2\nu}{\alpha}}$ in case $\alpha > 0$. Now,

$$\sum_{k>X} \frac{e^{-2^k x}}{(1+2^k)^\alpha} k^\nu \ll \int_X^\infty \frac{e^{-2^t x}}{(1+2^t)^\alpha} t^\nu \,\mathrm{d}t$$
$$\ll \frac{X^\nu}{(1+2^X)^\alpha} \int_X^\infty e^{-2^t x}\,\mathrm{d}t \ll \frac{x^\alpha}{(1+x)^\alpha} X^\nu.$$

On the other hand

$$\sum_{k\leq X}\frac{e^{-2^kx}}{(1+2^k)^\alpha}k^\nu \ll \int_1^X \frac{e^{-2^tx}}{(1+2^t)^\alpha}t^\nu\,\mathrm{d}t \leq \int_1^X t^\nu\exp(-t\alpha\log 2)\,\mathrm{d}t.$$

This is obviously $\ll \frac{X^\nu}{2^\alpha\alpha}$. Therefore, if $\alpha>0$,

$$S_{\nu,\alpha}(x)\ll \log^\nu\frac{1}{x}\max\left\{\frac{1}{2^\alpha\alpha},\frac{x^\alpha}{(1+x)^\alpha}\right\}.$$

For $\alpha=0$, one may similarly prove that $S_{\nu,0}(x)\ll\log^{\nu+1}\frac{1}{x}$. □

Next, consider the integral

$$J_n\equiv J_n(\varepsilon,\Delta,\varrho,\theta)=\int_1^\Delta\frac{e^{i\varepsilon(\varrho t+\theta)^n}}{\varrho t+\theta}\,\mathrm{d}t$$

where $n\geq 1,\varrho\geq 1,-1<\theta<1$ are fixed, and $\varepsilon\to 0^+,\Delta\to\infty$ in such a way that $\varepsilon(\varrho\Delta+\theta)^n\to\infty$.

Lemma 42. *We have*

$$J_n=-\frac{1}{n\varrho}\log\left(\varepsilon(\varrho+\theta)^n\right)-\frac{\gamma}{n\varrho}+\frac{\pi i}{2n\varrho}+O\left(\frac{\varepsilon(\varrho+\theta)^n}{n\varrho}\right)+O\left(\frac{1}{n\varrho\varepsilon(\varrho\Delta+\theta)^n}\right).$$

Proof. The change of variable $y(t)=\varepsilon(\varrho t+\theta)^n$ transforms the integral to

$$J_n=\frac{1}{n\varrho}\int_{y(1)}^{y(\Delta)}\frac{e^{iy}}{y}\,\mathrm{d}y.$$

Using the fact that

$$\int_0^x\frac{\sin t}{t}\,\mathrm{d}t=\begin{cases}O(x), & \text{near } x=0,\\ \frac{\pi}{2}+O(x^{-1}), & \text{near } x=\infty,\end{cases}$$

and

$$\int_0^x \frac{1-\cos t}{t}\,\mathrm{d}t = \begin{cases} O(x^2), & \text{near } x = 0, \\ \log x + \gamma + O(x^{-1}), & \text{near } x = \infty, \end{cases}$$

we immediately obtain the result. Here, γ is the Euler-Mascheroni constant. $\square$

Part III

Estimations explicites de certaines fonctions sommatoires des nombres premiers

Explicit estimates for some summatory functions of primes

Chapter 3

Historical remarks

It has been known since Euler's time that the sum of the reciprocals of prime numbers

$$\sum_p \frac{1}{p} = \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots \tag{3.1}$$

diverges. It was Mertens in 1874 who first proved the exact rate of divergence of this sum, proving, in modern notations, that the sum (3.1) upto a given real number x, usually denoted

$$\sum_{p \le x} \frac{1}{p}, \tag{3.2}$$

satisfies

$$\lim_{x\to\infty} \left(\sum_{p \le x} \frac{1}{p} - \log\log x - B \right) = 0, \tag{3.3}$$

where B is a constant we now call the Meissel-Mertens constant. In fact, he proves that the quantity on the left of (3.3) does not in absolute value exceed $\frac{4}{\log(x+1)} + \frac{2}{x\log x}$ for any $x \ge 2$. Mertens starts his paper with "two curious formulae" in Legendre's classic number theory textbook [52], one of the form

$$\sum_{p \le x} \frac{1}{p} = \log\big(\log x - 0.083\,66\big) + C$$

and another of the form

$$\prod_{p\le x}\big(1-\frac{1}{p}\big)=\frac{A}{\log x-0.083\,66} \tag{3.4}$$

for "certain positive constants" C and A. Mertens then goes on to prove (3.3) and the more exact formula for (3.4), namely

$$\lim_{x\to\infty}\log x\prod_{p\le x}\big(1-\frac{1}{p}\big)=e^{-\gamma},$$

thus freeing the equations of the number $0.083\,66$, "of doubtful value stemming from empirical observations" ([56]). Here γ is the usual Euler-Mascheroni constant. Earlier, Chebyshev had already proved that the sum (3.2) can be approximated by $\log\log x+B$ "once the value of B has been determined" ([77, § 7]), but his proof depends on the then unproved Prime Number Theorem, which we now know is equivalent to the fact that

$$\lim_{x\to\infty}\left(\sum_{p\le x}\frac{1}{p}-\log\log x-B\right)\log x=0,$$

a much stronger statement than the assertion (3.3). In any case, Legendre is without doubt the first eminent mathematician to have been interested in determining explicit approximate numerical values of the sum $\sum_{p\le x}{}^{1}/_{p}$ or of the product $\prod_{p\le x}\big(1-\frac{1}{p}\big)$ and to give concrete results (crude and asymptotically false though they are) about them.

Then came the problem of determining the error in approximating the sum (3.2) by $\log\log x+B$. The first major breakthrough in this direction came with the works [71] and [72] of Rosser and Schoenfeld who, exploiting the known zero-free region for the Riemann ζ-function, notably proved that the sum (3.2) is always less that $\log\log x+B+{}^{1}/_{\log^2 x}$ but more than $\log\log x+B-{}^{1}/_{2\log^2 x}$ ([72, Theorem 5 and its Corollary]). Thus, the error is of order at most ${}^{1}/_{\log^2 x}$. They also give approximations (among others) for the product (3.4) and for the sum $\sum_{p\le x}{}^{\log p}/_{p}$ with explicit errors of the order of ${}^{1}/_{\log^2 x}$ and ${}^{1}/_{\log x}$ respectively.

The main development since then has been directed towards the Chebyshev ψ-function: Verifying the Riemann hypothesis upto a large height ([81], [25], [62]), getting estimates for ψ ([73], [21], [22]) or getting better infinite zero-free region ([73], [40]). Related quantities like $\sum_{p\le x} 1/p$ or $\prod_{p\le x}(1-p^{-1})$ were considered only marginally. It may be surprising, but it is not automatic to derive quantitatively good estimates for such "derived quantities" from the estimates for the ψ-function, as is explained in [12]. Recently [65] dealt efficiently with $\sum_{p\le x}\Lambda(n)/n$ and this work may be seen as continuing this line of work.

Along this line, we also have the work of Bordellès [5] who studies the products (3.4) limited to those primes $\le x$ which belong to a particular residue class modulo a given positive integer, and of Diamond and Ford [10] who study the distribution of generalised Euler constants, and of Diamond and Pintz [11] who prove that the difference

$$\sqrt{x}\Big(\prod_{p\le x}\big(1-\frac{1}{p}\big)^{-1} - e^{\gamma}\log x\Big)$$

oscillates infinitely and unboundedly.

Concerning the sum, we prove that, given a zero-free region of the shape (called Hadamard-de la Vallée Poussin form of zero-free region for $\zeta(s)$)

$$\beta \ge 1 - \frac{1}{R\log|\gamma|},$$

the error in approximating the sum (3.2) by $\log\log x + B$ is less than

$$\frac{1.6}{R^{\frac14}\log^{\frac34} x}\exp\Big(-\sqrt{\frac{\log x}{R}}\Big) \tag{3.5}$$

for $\log x \ge 814R$. More simply and perhaps more interestingly, we also prove that the error is less than $1/\log^3 x$ as soon as $x \ge 50\,000$. Our computations also show that the error is smaller than $3/\log^3 x$ when $x \ge 1000$. The same remark holds *mutatis mutandis* for the products. In any case, an easy conclusion that we may draw from the error (3.5) is that, the error is asymptotically less than *any* power of $1/\log x$, although higher powers of $1/\log x$ are less interesting in that they need larger and larger constants. Moreover, approximations in terms of $1/\log^3 x$

are better than those given by (3.5) for small x.

This method in fact shows that the sum (3.2) differs from $\log\log x + B + \frac{\vartheta(x)-x}{x\log x}$ by less that $\frac{7\times 10^{-6}}{\log x}\exp\big(-\sqrt{\frac{2\log x}{R}}\big)$ as soon as $\log x \geq 814R$, thus devolving the burden to estimating the Chebyshev ϑ function, of which we now have good explicit approximations, thanks notably to the recent works of Dusart (cf. [20] and [19]), and of Ramaré and Rumely (see [66]) for primes in arithmetic progressions.

Now, one of the smallest values we know of R at the moment is $5.696\,93$, due to Kadiri [40], so that $814R$ is about 4637, thus requiring very big values of x for our approximations to hold. Luckily however, we have numerical tables for smaller numbers to fill this lacuna. For smaller values of x, we heavily rely on machine computations using Pari/GP.

Here is a specimen of what we prove regarding the sum (3.2) in the following pages:

$$
\begin{aligned}
|\sum_{p\leq x}\frac{1}{p} - \log\log x - B| &\leq \frac{4}{\log^3 x} \quad \text{for } x \geq 2,\\
|\sum_{p\leq x}\frac{1}{p} - \log\log x - B| &\leq \frac{2.3}{\log^3 x} \quad \text{for } x \geq 1000,\\
|\sum_{p\leq x}\frac{1}{p} - \log\log x - B| &\leq \frac{1}{\log^3 x} \quad \text{for } x \geq 24\,284;
\end{aligned}
$$

we also prove analogous results for the products (3.4), although for this case we content ourselves with approximations for very large values of x (for $x \geq \exp(22)$). But our method of proof shows that, if one wishes, one may as well give results for small values of x.

Chapter 4

Explicit estimates for some summatory functions of primes

4.1 Introduction and results

Explicit estimates in prime number theory have a long history, starting for the modern part with the two seminal papers [71] and [72]. The main development since then has been directed towards the Chebyshev ψ-function: Verifying the Riemann hypothesis until a large height ([81], [25], [62]), getting estimates for ψ ([73], [21], [22]) or getting better infinite zero-free region ([73], [40]). Related quantities like $\sum_{p\le x} 1/p$ or $\prod_{p\le x}(1-p^{-1})$ were considered only marginally. It may be surprising, but it is not automatic to derive quantitatively good estimates for such "derived quantities" from the estimates for the ψ-function, as is explained in [12]. Recently [65] dealt efficiently with $\sum_{p\le x} \Lambda(n)/n$ and this work may be seen as continuing this line of work. In passing we correct a mistake therein.

Here is one of our typical results:

Corollary 43. *We have*

$$\sum_{p\le x}\frac{1}{p} = \log\log x + B + O^*\left(\frac{4}{\log^3 x}\right) \quad (x\ge 2)$$
$$\sum_{p\le x}\frac{1}{p} = \log\log x + B + O^*\left(\frac{2.3}{\log^3 x}\right) \quad (x\ge 1000)$$

and for $x \ge 24\,284$*,*

$$\sum_{p\le x}\frac{1}{p} = \log\log x + B + O^*\left(\frac{1}{\log^3 x}\right).$$

Here and everywhere $f(x) = O^*\big(g(x)\big)$ means $|f(x)| \le g(x)$.

This is to be compared with [72, (3.17),(3.18)] where the authors have the error term $1/(2\log^2 x)$. Our method proposes a better handling of the error term, which is why we can reach the size $c/\log^3 x$. However, we also give results with error terms of shape $c/\log x$ in the last section. We heavily rely on Pari/GP (see [61]) computations for small values of the variable x. It is also of interest to get better error terms, even if the variable x is asked to be large and in this direction, we prove the following:

Corollary 44. *When* $\log x \ge 4635$*,*

$$\sum_{p\le x}\frac{1}{p} = \log\log x + B + O^*\left(1.1\frac{\exp(-\sqrt{0.175\log x})}{(\log x)^{3/4}}\right) \tag{4.1}$$

Such results are dependent on the size of the zero-free region for the Riemann zeta function and may change if the latter is modified, so we provide a result to reflect the size of the known zero-free region. Let us assume that $\zeta(s)$ does not vanish in the region

$$\Re s \ge 1 - \frac{1}{R\log|\Im s|}, \quad (|\Im s| \ge t_0)$$

where $R > 0$. For instance, thanks to [40], we can choose $R = 5.696\,93$ and

$t_0 = 10$. We use the notation

$$\lambda(x) = \sum_{p \le x} \frac{1}{p}. \tag{4.2}$$

Under such a hypothesis, we have the following.

Corollary 45. *For* $\log x \ge 814R$*,*

$$\lambda(x) = \log\log x + B + O^*\left(\frac{1.6}{R^{\frac{1}{4}} \log^{\frac{3}{4}} x} \exp\big(-\sqrt{\frac{\log x}{R}}\big)\right).$$

This is derived from our main Theorem.

Theorem 46. *For* $x \ge \exp(814R)$*, we have*

$$\lambda(x) = \log\log x + B + \frac{\vartheta(x) - x}{x \log x} + O^*\left(\frac{7 \times 10^{-6}}{\log x} \exp\big(-\sqrt{\frac{2\log x}{R}}\big)\right). \tag{4.3}$$

For $x \ge 2$*,*

$$\lambda(x) = \log\log x + B + \frac{\vartheta(x) - x}{x \log x} + O^*\left(\frac{1 + \log x}{\log^2 x} \alpha^*(x)\right)$$

with

$$\alpha^*(x) = \frac{2.1}{\sqrt{x}} + \frac{4.5}{x^{\frac{2}{3}}} + \frac{2.84}{x} + 1.751 \times 10^{-12}.$$

Proof of Corollary 45. This comes directly from (4.3) and a recent result of Dusart (see [20, Theorem 1.1]) which says that, if $\log x \ge 70R$, then

$$\left|\frac{\vartheta(x) - x}{x}\right| < \sqrt{\frac{8}{\pi}} \left(\frac{\log x}{R}\right)^{\frac{1}{4}} \exp\big(-\sqrt{\frac{\log x}{R}}\big). \tag{4.4}$$

□

Using the value $R = 5.696\,93$ mentioned earlier then gives (4.1).

Eulerian products

As an application of our estimates on $\lambda(x)$, we give estimates for the (finite Euler) products $\prod_{p\le x}(1+{}^{z}/_{p})$.

Theorem 47. *Let z be a complex number with $0<|z|<2$. Then for $x\ge\exp(22)$,*

$$\prod_{p\le x}\left(1+\frac{z}{p}\right)=e^{\gamma(z)}\big(\log x\big)^{z}\left\{1+O^{*}\left(\frac{0.841}{\log^{3}x}\right)\right\} \tag{4.5}$$

where

$$\gamma(z)=zB+\sum_{p}\sum_{n=2}^{\infty}(-1)^{n+1}\frac{z^{n}}{np^{n}}.$$

The cases $z=\pm1$ are most commonly studied, with Mertens himself treating the case $z=-1$ in [56], without giving explicit error terms. A preliminary form for this result is found in [52]. One may compare this with the error term ${}^{1}/_{(2\log^{2}x)}$ given in [72, Theorem 7] for the case $z=-1$. In [11] it is proved that the difference

$$\prod_{p\le x}\left(1-\frac{1}{p}\right)^{-1}-e^{\gamma}\log x$$

changes sign infinitely often. Similar products are studied in [5, 10], the latter dealing with aspects other than explicit bounds. We remark that $\gamma(-1)=-\gamma$, $\gamma(1)=\gamma+\log(6/\pi^{2})$. We give in the last section results with error terms of shape $c/\log x$.

Mertens sums

We next study two closely related sums:

$$\Upsilon(x)=\sum_{p\le x}\frac{\log p}{p},\tilde{\psi}(x)=\sum_{n\le x}\frac{\Lambda(n)}{n}.$$

We will content ourselves with giving explicit approximations for very large values of x.

Theorem 48. *For* $\log x \geq 814R$*,*

$$\Upsilon(x) = \log x + E + \frac{\vartheta(x) - x}{x} + O^*\Bigg(6.5 \times 10^{-6} \exp\big(-\sqrt{\frac{2\log x}{R}}\big)\Bigg)$$

where $E = -\gamma - \sum_{n=2}^{\infty} \sum_p {}^{\log p}/_{p^n} = -1.332\,582\,275\,733\,221\ldots$ [1]
For all $x \geq 2$*, we have*

$$\Upsilon(x) = \log x + E + \frac{\vartheta(x) - x}{x} + O^*\Bigg(\frac{2.0494}{x^{\frac{1}{2}}} + \frac{4.5}{x^{\frac{2}{3}}} + \frac{1.838}{x} + 1.75 \times 10^{-12}\Bigg)$$

Corollary 49. *For* $x \geq \exp(814R)$*,*

$$\Upsilon(x) = \log x + E + O^*\Bigg(1.6\left(\frac{\log x}{R}\right)^{\frac{1}{4}} \exp\big(-\sqrt{\frac{\log x}{R}}\big)\Bigg).$$

Proof. One uses (4.4) in Theorem 48. □

The value $R = 5.696\,93$ then gives

Corollary 50. *For* $x \geq \exp(2319)$*,*

$$\Upsilon(x) = \log x + E + O^*\Bigg(1.036(\log x)^{1/4} \exp\big(-\sqrt{0.175\log x}\big)\Bigg).$$

In [72, Theorem 6], we find an error term of ${}^1/_{(2\log x)}$ for this sum. Landau [49, § 55] gives error terms of $\exp\big(-(\log x)^{\frac{1}{14}}\big)$ for both $\Upsilon(x)$ and $\lambda(x)$.

Finally, we rectify the estimate for $\tilde{\psi}(x)$ given in [65, Theorem 1.1]:

Theorem 51. *When* $\log x \geq 407R$*,*

$$\tilde{\psi}(x) = \log x - \gamma + \frac{\psi(x) - x}{x} + O^*\left(\frac{0.05}{\sqrt{x}}\right) + O^*\Bigg(6.4 \times 10^{-6} \exp\big(-\sqrt{\frac{2\log x}{R}}\big)\Bigg).$$

1. See [72, (2.11)] for the numerical value.

Here γ is the usual Euler-Mascheroni constant. For all $x \geq 2$, we have

$$\tilde{\psi}(x) = \log x - \gamma + \frac{\psi(x) - x}{x} + O^*\left(\frac{0.047}{\sqrt{x}} + \frac{1.884}{x} + 1.75 \times 10^{-12}\right).$$

We find this sum for example in [2, Theorem 4.9] or [30, Theorem 424] in the more rudimentary form $\tilde{\psi}(x) = \log x + O(1)$.

Corollary 52. *For $\log x \geq 407R$,*

$$\tilde{\psi}(x) = \log x - \gamma + O^*\left(1.6\left(\frac{\log x}{R}\right)^{\frac{1}{4}} \exp\big(-\sqrt{\frac{\log x}{R}}\big)\right).$$

Proof. We use the estimate

$$0 \leq \psi(x) - \vartheta(x) \leq 1.0012\sqrt{x} + 3x^{\frac{1}{3}}, \quad (x > 0) \tag{4.6}$$

(see [73, Theorem 6]) together with (4.4) and round off appropriately. □

One may also give an analogue of Corollary 50 for this function.

Notation

We introduce some more notations, also recalling those we have already introduced. For $x \geq 0$ we put

$$\lambda(x) = \sum_{p \leq x} \frac{1}{p}, \quad \Upsilon(x) = \sum_{p \leq x} \frac{\log p}{p},$$
$$\tilde{\psi}(x) = \sum_{n \leq x} \frac{\Lambda(n)}{n}.$$

Following [65], by $f(x) = O^*(g(x))$ we mean $|f(x)| \leq g(x)$. We define the following functions:

$$J(x) = \sum_{\rho} \frac{x^{\rho-1}}{\rho(\rho-1)}, \tag{4.7}$$

$$S_m(x) = \sum_{\rho} \frac{x^{\beta-1}}{|\gamma|^{m+1}} \tag{4.8}$$

for $x > 0$ and $m \geq 1$. In both cases, the sum runs through all the nontrivial zeros ρ of the the Riemann zeta function. It has been verified (see [25]) that at least the first 10^{13} zeros of $\zeta(s)$ lie on the critical line $\Re s = \frac{1}{2}$. Hence we may consider the Riemann Hypothesis verified upto height $T_0 = 2.44 \times 10^{12}$. As mentioned earlier, we also suppose that there is no nontrivial zero $\rho = \beta + i\gamma$ for $\zeta(s)$ satisfying

$$\beta \geq 1 - \varphi(\gamma) = 1 - \frac{1}{R \log |\gamma|}, \quad (|\gamma| \geq t_0) \tag{4.9}$$

where R is a positive constant. For explicit computations, we will however assume throughout that $R \geq 1$. We unhesitatingly use γ as the imaginary part of a nontrivial zero of $\zeta(s)$ and as the Euler-Mascheroni constant. This is unlikely to cause any confusion. The symbols ψ and ϑ always denote the Chebyshev functions, whereas we have defined φ in (4.9), abrogating its traditional use as the Euler totient. We follow other usual number-theoretic conventions, such as writing s for a complex variable, etc. Further notations will be introduced as necessary.

Organisation of the paper

The results stated in the INTRODUCTION AND RESULTS section above are not restated. The § 4.2 is independent of other sections, and so may be read separately. The theorems stated in § 4.1 are proved in § 4.3; only the proof of Theorem 47 there depends on § 4.4 which comes after it. This last section consists mainly of numerical computations (using Pari/GP) to bridge the gap between extremely big values of x and bounded intervals. It uses the results stated in § 4.1.

4.2 Lemmas on the zeros of ζ

We recall that $N(T)$ denotes the number of zeros $\rho = \beta + i\gamma$ of the Riemann zeta function $\zeta(s)$ with $0 < \gamma \leq T$ and $0 < \beta < 1$.

We quote [67, Lemma 2]:

Lemma 53. *For $m \geq 1$ and $T \geq 1\,000$,*

$$\sum_{\substack{\rho \\ |\gamma|>T}} \frac{1}{|\gamma|^{m+1}} = \frac{1}{m\pi T^m}(\log \tfrac{T}{2\pi} + \frac{1}{m}) + O^*\big(\frac{1.34}{T^{m+1}}(2\log \tfrac{T}{2\pi} + 1)\big). \tag{4.10}$$

We next quote [65] for an estimate of $J(x)$:

Lemma 54. *We have*

$$|J(x)| \leq \frac{0.047}{\sqrt{x}} + 1.75 \times 10^{-12} \quad (x \geq 2). \tag{4.11}$$

Further, when $\log x \geq 407R$,

$$|J(x)| \leq \frac{0.047}{\sqrt{x}} + 6.4 \times 10^{-6} \exp\big(-\sqrt{\frac{2\log x}{R}}\big) \tag{4.12}$$

To give an estimate for $S_m(x)$, we need this

Lemma 55. *Let $n \geq 1$ and $T \geq 1$. If $\log x \geq \frac{1}{2}nR\log^2 T$ then we have*

$$I_n(T,x) \stackrel{\text{def}}{=} \int_T^\infty \frac{x^{-\varphi(t)}}{t^{n+1}} \log t \,\mathrm{d}t \leq \frac{4+2n\log T}{n^2 T^{\frac{n}{2}}} \exp\big(-\sqrt{\frac{2n\log x}{R}}\big). \tag{4.13}$$

If $\log x \leq \frac{1}{2}nR\log^2 T$ then

$$\int_T^\infty \frac{x^{-\varphi(t)}}{t^{n+1}} \log t \,\mathrm{d}t \leq \big(\frac{4+2n\log T}{n^2 T^{\frac{n}{2}}}\big) x^{-\frac{1}{R\log T}}$$

Proof. We transform the integral by writing $u = \log t$ to

$$I_n(T,x) = \int_{\log T}^\infty \exp\big(-\frac{\log x}{Ru} - nu\big) u \,\mathrm{d}u.$$

Now, this may be rewritten as

$$I_n(T,x) = \int_{\log T}^{\infty} \exp\big(-\frac{\log x}{Ru} - \frac{nu}{2}\big) e^{-\frac{nu}{2}} u \,\mathrm{d}u.$$

The function

$$\exp\big(-\frac{\log x}{Ru} - \frac{1}{2}nu\big) \tag{4.14}$$

has a maximum at $u = \sqrt{\frac{2\log x}{nR}}$ (which is $\geq \log T$ by assumption) so we have

$$\begin{aligned} I_n(T,x) &\leq \exp\big(-\sqrt{\frac{2n\log x}{R}}\big) \int_{\log T}^{\infty} u e^{-\frac{nu}{2}} \,\mathrm{d}u \\ &= \frac{4 + 2n\log T}{n^2 T^{\frac{n}{2}}} \exp\big(-\sqrt{\frac{2n\log x}{R}}\big). \end{aligned}$$

The second assertion is obvious since then the function (4.14) is decreasing in the interval of integration. □

Of course, the factor $\frac{1}{2}$ in (4.14) may be replaced by any positive $\epsilon < 1$; in that case, (4.13) will become

$$I_n(T,x) \leq \frac{1 + (1-\epsilon)n\log T}{(1-\epsilon)^2 n^2 T^{(1-\epsilon)n}} \exp\big(-2\sqrt{\frac{\epsilon n\log x}{R}}\big)$$

for $\log x \geq \epsilon nR\log^2 T$. This is interesting if we want to gain in powers of T in the denominator to the detriment of the factor of $\log x$ inside the exponential, and vice versa. For example, by choosing $\epsilon = \frac{1}{4}$, we obtain

$$I_n(T,x) \leq \frac{16 + 12n\log T}{9n^2 T^{\frac{3}{4}n}} \exp\big(-\sqrt{\frac{n\log x}{R}}\big)$$

for $\log x \geq \frac{1}{4}nR\log^2 T$.

Lemma 56. *For $x \geq 1$,*

$$S_m(x) \leq \frac{S_m(1)}{\sqrt{x}} + \Big(\frac{0.67}{T_0}\log\frac{T_0}{2\pi} - \frac{\log 2\pi}{2m\pi}\Big)\frac{x^{-\varphi(T_0)}}{T_0^m} + \frac{1}{2\pi}I_m(T_0,x) \qquad (4.15)$$
$$+ 0.67\Big(m+1+\frac{\log x}{R\log^2 T_0}\Big)I_{m+1}(T_0,x).$$

Moreover, if $m \leq 3.6\times 10^{10} \leq \big(T_0\log(2\pi)\big)/\big(1.34\pi\log(\frac{T_0}{2\pi})\big)$, then we can ignore the second term in (4.15)*, that is,*

$$S_m(x) \leq \frac{S_m(1)}{\sqrt{x}} + \frac{1}{2\pi}I_m(T_0,x) + 0.67\Big(m+1+\frac{\log x}{R\log^2 T_0}\Big)I_{m+1}(T_0,x).$$

Proof. Since $1-\rho$ is a nontrivial zero whenever ρ is, we have

$$S_m(x) = \sum_{\gamma>0}\frac{x^{-\beta}+x^{\beta-1}}{\gamma^{m+1}}$$
$$= \frac{2}{\sqrt{x}}\sum_{0<\gamma\leq T_0}\frac{1}{\gamma^{m+1}} + \sum_{\gamma>T_0}\frac{x^{-\beta}+x^{\beta-1}}{\gamma^{m+1}}.$$

Using (4.9), we easily see that $x^{-\beta}+x^{\beta-1} \leq x^{-\frac{1}{2}} + x^{-\varphi(\gamma)}$, so that

$$\sum_{\gamma>T_0}\frac{x^{-\beta}+x^{\beta-1}}{\gamma^{m+1}} \leq \frac{1}{\sqrt{x}}\sum_{\gamma>T_0}\frac{1}{\gamma^{m+1}} + \sum_{\gamma>T_0}\frac{x^{-\varphi(\gamma)}}{\gamma^{m+1}};$$

we can apply (4.10) to the first sum, and we evaluate the second sum as follows. Set $\varphi_m(t) = \frac{x^{-\varphi(t)}}{t^{m+1}}$. We write

$$\sum_{\gamma>T_0}\frac{x^{-\varphi(\gamma)}}{\gamma^{m+1}} = -\int_{T_0}^{\infty} N(t)\varphi_m'(t)\,\mathrm{d}t - N(T_0)\varphi_m(T_0)$$
$$= \int_{T_0}^{\infty}(N^*(t)-N(t))\varphi_m'(t)\,\mathrm{d}t - \int_{T_0}^{\infty}N^*(t)\varphi_m'(t)\,\mathrm{d}t - N(T_0)\varphi_m(T_0).$$

Integration by parts (of the middle term) and an appeal to the asymptotic (1.25) yields

$$\sum_{\gamma>T_0} \frac{x^{-\varphi(\gamma)}}{\gamma^{m+1}} = (N^*(T_0) - N(T_0))\varphi_m(T_0) + \int_{T_0}^{\infty} (N^*(t) - N(t))\varphi_m'(t)\,\mathrm{d}t$$
$$+ \frac{1}{2\pi}\int_{T_0}^{\infty} \frac{x^{-\varphi(t)}}{t^{m+1}} \log\frac{t}{2\pi}\,\mathrm{d}t$$
$$= (N^*(T_0) - N(T_0))\frac{x^{-\varphi(T_0)}}{T_0^{m+1}} - \frac{\log 2\pi}{2\pi}\int_{T_0}^{\infty} \frac{x^{-\varphi(t)}}{t^{m+1}}\,\mathrm{d}t$$
$$+ O^*\big(0.67(m+1+\frac{\log x}{R\log^2 T_0})I_{m+1}(T_0,x)\big) + O^*\big(\frac{1}{2\pi}I_m(T_0,x)\big).$$

The first assertion follows readily; the second is obvious in view of the first. □

Corollary 57. *For* $\log x \geq R\log^2 T_0$,

$$S_2(x) \leq \frac{0.001\,460}{\sqrt{x}} + 2\times 10^{-12}\exp\big(-2\sqrt{\frac{\log x}{R}}\big).$$

For $\log x \leq R\log^2 T_0$*,*

$$S_2(x) \leq \frac{0.001\,460}{\sqrt{x}} + 2\times 10^{-12} x^{-\frac{1}{29R}}$$
$$+ 3.5\times 10^{-18}\big(3 + \frac{\log x}{813R}\big)x^{-\frac{1}{29R}}.$$

Proof. The number $0.001\,460$ comes from a Pari/GP (see [61]) computation making use of the file of the first $100\,000$ zeros of $\zeta(s)$ provided by [60] (see also [59]); we only need to use the first few thousand zeros to get a 5-digit precision (in fact, we have used the first $20\,000$ zeros and get $0.001\,459\,09...$ Since the $20\,001^{\text{st}}$ zero has $\gamma = 18\,047.134\,530\,33...$ and $1/\gamma^3$ is then about 1.7×10^{-13}, we do not run the risk of committing an error, given the 28-digit precision in Pari/GP). □

Note that the equation (4.10) and our evaluation of $S_m(x)$ prove that for small x, one may profitably use the bound given in the following

Lemma 58. *For $x \geq 1$,*

$$S_m(x) \leq \frac{S_m(1)}{\sqrt{x}} + \frac{1}{2\pi m T_0^m}\big(\log\frac{T_0}{2\pi} + \frac{1}{m}\big) + \frac{0.67}{T_0^{m+1}}\big(2\log\frac{T_0}{2\pi} + 1\big).$$

This gives, in particular, that

$$S_2(x) \leq \frac{0.001\,460}{\sqrt{x}} + 3.7 \times 10^{-25} \quad (x \geq 2) \tag{4.16}$$

4.3 Proof of the Theorems

Proof of Theorem 1

We easily see, using (Stieltjes) integration by parts, that

$$\begin{aligned}\lambda(x) &= \int_{2^-}^{x} \frac{\mathrm{d}\vartheta(t)}{t\log t} = \frac{\vartheta(x)}{x\log x} + \int_2^x \frac{1+\log t}{t^2\log^2 t}\vartheta(t)\,\mathrm{d}t \\ &= \log\log x + B + \frac{\vartheta(x)-x}{x\log x} - \int_x^\infty \frac{1+\log t}{t^2\log^2 t}(\vartheta(t)-t)\,\mathrm{d}t \end{aligned} \tag{4.17}$$

where

$$\begin{aligned}B &= \frac{1}{\log 2} - \log\log 2 + \int_2^\infty \frac{1+\log t}{t^2\log^2 t}(\vartheta(t)-t)\,\mathrm{d}t \\ &= \gamma + \sum_p \big\{\log(1-\frac{1}{p}) + \frac{1}{p}\big\} \\ &= 0.26149\,72128\,47643\ldots\end{aligned}$$

is called the Meissel-Mertens constant; see [72, (2.10)] for the numerical value and [33, p. 23] for the second line. Here γ is Euler's constant. The integral in (4.17) is

$$\begin{aligned}&\int_x^\infty \frac{1+\log t}{t^2\log^2 t}(\vartheta(t)-t)\,\mathrm{d}t \\ &= \int_x^\infty \frac{1+\log t}{t^2\log^2 t}(\vartheta(t)-\psi(t))\,\mathrm{d}t + \int_x^\infty \frac{1+\log t}{t^2\log^2 t}(\psi(t)-t)\,\mathrm{d}t.\end{aligned} \tag{4.18}$$

Since we know that (see [73, Theorem 6])

$$0 \le \psi(t) - \vartheta(t) \le 1.0012\sqrt{t} + 3t^{\frac{1}{3}} \quad (t > 0)$$

we have

$$\int_x^\infty \frac{1+\log t}{t^2 \log^2 t}(\psi(t) - \vartheta(t))\,\mathrm{d}t \le \Big(\frac{2.0024}{\sqrt{x}} + \frac{9}{2x^{\frac{2}{3}}}\Big)\frac{1+\log x}{\log^2 x}, \tag{4.19}$$

which gives an explicit estimate for the first integral in (4.18). In order to estimate the second integral, we use Lemma 13 with $g(t) = \frac{1+\log t}{t^2 \log^2 t}$ to get

$$\begin{aligned}\int_x^Y \frac{1+\log t}{t^2 \log^2 t}(\psi(t) - t)\,\mathrm{d}t = &-\sum_\rho \int_x^Y \frac{t^{\rho-2}}{\rho}\frac{1+\log t}{\log^2 t}\,\mathrm{d}t \\ &+ \int_x^Y (\log 2\pi - \tfrac{1}{2}\log(1-t^{-2}))\frac{1+\log t}{t^2\log^2 t}\,\mathrm{d}t.\end{aligned} \tag{4.20}$$

We easily see that

$$\int_x^Y \frac{1+\log t}{t^2 \log^2 t}\,\mathrm{d}t \le \frac{1+\log x}{x\log^2 x}$$

and

$$\begin{aligned}-\int_x^Y \log(1-t^{-2}))\frac{1+\log t}{t^2\log^2 t}\,\mathrm{d}t &\le -\frac{1+\log x}{\log^2 x}\int_x^Y \frac{\log(1-t^{-2})}{t^2}\,\mathrm{d}t \\ &= \frac{1+\log x}{\log^2 x}\bigg(\log\frac{Y+1}{Y-1} + \log\frac{x-1}{x+1} + \frac{1}{Y}\log\frac{Y^2-1}{Y^2} \\ &\qquad - \frac{1}{x}\log\frac{x^2-1}{x^2} + \frac{2}{x} - \frac{2}{Y}\bigg).\end{aligned}$$

We would like to send Y to infinity; for this, it suffices to prove the absolute convergence of all the sums and integrals in (4.20). First of all, integration by parts gives

$$\int_x^Y \frac{t^{\rho-2}}{\rho}\frac{1+\log t}{\log^2 t}\,\mathrm{d}t = \frac{1+\log Y}{\log^2 Y}\frac{Y^{\rho-1}}{\rho(\rho-1)} - \frac{1+\log x}{\log^2 x}\frac{x^{\rho-1}}{\rho(\rho-1)} + \int_x^Y \frac{t^{\rho-2}}{\rho(\rho-1)}\frac{2+\log t}{\log^3 t}\,\mathrm{d}t.$$

This last integral is clearly absolutely convergent and since we know that the sum $\sum_\rho \frac{1}{\rho(\rho-1)}$ converges absolutely, we can let Y tend to infinity on the right of (4.20) and on the left as well. We thus obtain

$$\int_x^\infty \frac{1+\log t}{t^2\log^2 t}(\psi(t)-t)\,\mathrm{d}t = \frac{1+\log x}{\log^2 x}\sum_\rho \frac{x^{\rho-1}}{\rho(\rho-1)} - \sum_\rho \int_x^\infty \frac{t^{\rho-2}}{\rho(\rho-1)}\frac{2+\log t}{\log^3 t}\,\mathrm{d}t + O^*\big(\frac{1+\log x}{x\log^2 x}\log(2\pi e)\big).$$

Now,

$$\begin{aligned}\int_x^\infty \frac{t^{\rho-2}}{\rho(\rho-1)}\frac{2+\log t}{\log^3 t}\,\mathrm{d}t &= -2\int_x^\infty \frac{t^{\rho-2}}{\rho(\rho-1)}\Big(\int_t^\infty \frac{3+\log u}{u\log^4 u}\,\mathrm{d}u\Big)\,\mathrm{d}t\\ &= -2\int_x^\infty \frac{3+\log u}{u\log^4 u}\Big(\int_x^u \frac{t^{\rho-2}}{\rho(\rho-1)}\,\mathrm{d}t\Big)\,\mathrm{d}u\\ &= -2\int_x^\infty \frac{3+\log u}{u\log^4 u}\Big(\frac{u^{\rho-1}-x^{\rho-1}}{\rho(\rho-1)^2}\Big)\,\mathrm{d}u.\end{aligned}$$

The absolute value of the left member is therefore

$$\begin{aligned}&\le \frac{4x^{\beta-1}}{|\rho(\rho-1)^2|}\int_x^\infty \frac{3+\log u}{u\log^4 u}\,\mathrm{d}u = \frac{2x^{\beta-1}}{|\rho(\rho-1)^2|}\frac{2+\log x}{\log^3 x}\\ &\le \frac{4+2\log x}{\log^3 x}\frac{x^{\beta-1}}{|\gamma|^3}.\end{aligned}$$

Using this and (4.18) in (4.17) we get

$$\lambda(x) = \log\log x + B + \frac{\vartheta(x) - x}{x\log x} + \frac{1+\log x}{\log^2 x} J(x) + O^*\left(\frac{4+2\log x}{\log^3 x} S_2(x)\right) + O^*\left(\left(\frac{2.0024}{\sqrt{x}} + \frac{9}{2x^{\frac{2}{3}}} + \frac{\log(2\pi e)}{x}\right)\frac{1+\log x}{\log^2 x}\right).$$

This is valid for any $x \geq 2$.

Lemma 54 and the estimate (4.16) give

$$\lambda(x) = \log\log x + B + \frac{\vartheta(x) - x}{x\log x} + O^*\left(\frac{1+\log x}{\log^2 x}\alpha(x)\right) + O^*\left(\frac{4+2\log x}{\log^3 x}\left(\frac{0.001\,460}{\sqrt{x}} + 3.7\times 10^{-25}\right)\right) \tag{4.21}$$

for all $x \geq 2$, where

$$\alpha(x) = \frac{2.0494}{\sqrt{x}} + \frac{4.5}{x^{\frac{2}{3}}} + \frac{2.84}{x} + 1.75\times 10^{-12}.$$

Retaining only the biggest terms and rounding off appropriately gives the first statement of the theorem. For bigger values of x, we use Lemmas 54–56 and the fact that

$$\left(\frac{2.0494}{\sqrt{x}} + \frac{9}{2x^{\frac{2}{3}}} + \frac{\log(2\pi e)}{x}\right)\frac{1+\log x}{\log^2 x} \leq \frac{3}{\sqrt{x}\log x}, \quad \frac{1+\log x}{\log^2 x} \leq \frac{1.09}{\log x}$$
$$\frac{4+2\log x}{\log^3 x} \leq \frac{2.36}{\log^2 x}$$

as soon as $x \geq 74\,000$. The proof is complete.

Proof of Theorem 47

We first prove the following lemma:

Lemma 59. *Let z be a complex number with $0 < |z| < 2$ and let $g(x)$ be a positive* *function which tends to* 0 *as* $x \to \infty$*. Suppose we have*

$$\lambda(x) = \log\log x + B + O^*\big(g(x)\big)$$

and

$$\frac{|z|}{2}\log\frac{x-1}{x-1-|z|} + |z|\,g(x) \le \frac{1}{2}\log 3 \tag{4.22}$$

for some values of x*, say* $x \in A \subset \mathbb{R}$*. Then we also have*

$$\left|e^{-zB-\gamma(z)}(\log x)^{-z}\prod_{p\le x}\big(1+\frac{z}{p}\big)-1\right| \le |z|\left(\log\frac{x-1}{x-1-|z|}+2g(x)\right), \quad (x\in A).$$

Proof. Let us write $\psi(z,x) = \sum_{p\le x}\log(1+\frac{z}{p})$ for $0<|z|<2$. We have

$$\log(1+\frac{z}{p}) - \frac{z}{p} = \sum_{n=2}^{\infty}(-1)^{n+1}\frac{z^n}{np^n}$$

so that

$$\psi(z,x) - z\lambda(x) = \eta(z) + f(z,x) \tag{4.23}$$

with

$$\eta(z) = \sum_p\sum_{n=2}^{\infty}(-1)^{n+1}\frac{z^n}{np^n},$$
$$f(z,x) = \sum_{p>x}\sum_{n=2}^{\infty}(-1)^{n}\frac{z^n}{np^n}.$$

It is easily seen that

$$|f(z,x)| < \frac{|z|}{2}\log\frac{x-1}{x-1-|z|}. \tag{4.24}$$

Let us write $h(x) = \lambda(x) - \log\log x - B$ so $|h(x)| \le g(x)$. Equation (4.23) then gives

$$\left| e^{-zB-\eta(z)} (\log x)^{-z} \prod_{p \le x} \big(1 + \frac{z}{p}\big) - 1 \right| = \left| e^{f(z,x)+zh(x)} - 1 \right| .$$

Using the fact that $|e^z - 1| \le 2\,|z|$ if $|\Re z| \le \frac{1}{2}\log 3$ the assertion follows. (The number $\frac{1}{2}\log 3$ is chosen for convenience; it may be replaced by any other value with corresponding change in the resulting inequality). □

Using our estimates on $\lambda(x)$ in Corollary 61 (below), we obtain Theorem 47. The theorem is stated with an error term of $O({}^1\!/\!{\log^3 x})$ for convenience, although one may state it for error terms based on other functions, such as the one in Corollary 45. Also, for smaller values of x one may apply the results in Corollary 61.

The result shows that we have

$$\lim_{x\to\infty} (\log x)^{-z} \prod_{p \le x} \big(1 + \frac{z}{p}\big) = e^{\gamma(z)}$$

uniformly for complex z in $0 < |z| < 2$.

Proof of Theorem 48

We use the same procedure as in the proof of Theorem 46; here the functions involved are even simpler. In effect,

$$\begin{aligned} \Upsilon(x) &= \int_{2^-}^{x} \frac{\mathrm{d}\vartheta(t)}{t} = \frac{\vartheta(x)}{x} + \int_{2}^{x} \frac{\vartheta(t)}{t^2}\,\mathrm{d}t \\ &= \log x + E + \frac{\vartheta(x) - x}{x} + \int_{x}^{\infty} \frac{\psi(t) - \vartheta(t)}{t^2}\,\mathrm{d}t - \int_{x}^{\infty} \frac{\psi(t) - t}{t^2}\,\mathrm{d}t \end{aligned}$$

and this last integral is the same as the one occurring in [65, Proof of Lemma 2.2]; the result follows immediately. See also Proof of Theorem 51 below.

Proof of Theorem 51

We proceed again by integration by parts:

$$\begin{aligned}\tilde{\psi}(x) &= \int_{2^-}^{x} \frac{\mathrm{d}\psi(t)}{t} = \frac{\psi(x)}{x} + \int_{2}^{x} \frac{\psi(t)}{t^2}\,\mathrm{d}t \\ &= \log x - \gamma + \frac{\psi(x)-x}{x} - \int_{x}^{\infty} \frac{\psi(t)-t}{t^2}\,\mathrm{d}t\end{aligned}$$

where

$$\gamma = \log 2 - 1 - \int_{2}^{\infty} \frac{\psi(t)-t}{t^2}\,\mathrm{d}t$$

is Euler's constant (see [49, § 55]). Proceeding as in [65, Proof of Lemma 2.2], we get

$$\int_{x}^{\infty} \frac{\psi(t)-t}{t^2}\,\mathrm{d}t = J(x) - \frac{B(x)}{x}$$

with

$$B(x) = \frac{x}{2}\log\left(\frac{x+1}{x-1}\right) + \log\left(1-\frac{1}{x^2}\right) - \log(2\pi) - 1. \tag{4.25}$$

Thus

$$\tilde{\psi}(x) = \log x - \gamma + \frac{\psi(x)-x}{x} - J(x) + \frac{B(x)}{x}.$$

Using our estimates (4.11) and (4.12) for $J(x)$ and the fact that

$$\begin{aligned}|B(x)| &\le \log(2\pi) + 1 + \log 2 - 1.5\log 3 = 1.883\,105\,81... \quad (x \ge 2) \\ &< 1.884\end{aligned}$$

we obtain the result.

Remark. There is a minor error in [65] in the evaluation of the integral I in the proof of Theorem 1.1. The estimates for $J(x)$ given in [65, Proof of Theorem 1.1] should be replaced by our Lemma 54. The expression for $B(x)$ in [65, Lemma 2.2] should also be replaced by (4.25).

4.4 Results for bounded intervals

Corollary 45 gives good results for very large values of x. For example, when $x = \exp(20\,000)$, Corollary 45 says that the error in approximating $\sum_{p\leq x} 1/p$ by $\log\log x + B$ is less than 1.33×10^{-29}, which is very interesting to know, since we cannot easily compute all primes $\leq \exp(20\,000)$. In this section, we give bounds for moderately big values of x. We first state a corollary of Theorem 46 for big x:

Corollary 60. *For* $x \geq \exp(4635)$*,*

$$\lambda(x) = \log\log x + B + O^*\Big(\frac{0.21}{\log^3 x}\Big).$$

Proof. This comes immediately from (4.3) and the estimate on $|\vartheta(x) - x|$ given in [19, Théorème 1.4]. □

We now determine the constants required for smaller values of x in order to get an error term of $O\left(1/\log^3 x\right)$. For this, we use the second assertion of Theorem 46 together with [19, Table 1.1] and [20, Table 2]. Using the inequality (4.6) and (4.21), we obtain

$$|\lambda(x) - \log\log x - B| \leq \frac{1}{\log x}\Big(\frac{1.0012}{\sqrt{x}} + \frac{3}{x^{\frac{2}{3}}} + \epsilon\Big) + \frac{1+\log x}{\log^2 x}\alpha(x) + \frac{4+2\log x}{\log^3 x}\Big(\frac{0.001\,460}{\sqrt{x}} + 3.7\times 10^{-25}\Big) \tag{4.26}$$

for $x \geq e^b$, where the ϵ are as in the aforementioned tables. A Pari/GP ([61]) computation then gives the inequalities

$$|\lambda(x) - \log\log x - B| \leq \frac{\eta_n}{\log^3 x} \quad \big(\exp(b_n) \leq x \leq \exp(b_{n+1})\big)$$

where b_n, ϵ_n and the corresponding η_n are tabulated in Table 4.1. Note that b_n and ϵ_n are correlated by the inequality

$$|\psi(x) - x| \leq \epsilon_n x \quad \big(x \geq \exp(b_n)\big).$$

Also, we observe that η_n need not decrease with increasing b_n, as is clear from (4.26).

We also give the following short-interval result as a curiosity and to complement Table 4.1 (this table starts from $x = 10^8$):

Corollary 61. *We have the following bounds in the indicated intervals:*

$$\lambda(x) = \log\log x + B + O^*\left(\frac{1.835}{\log^3 x}\right), \quad (2 \le x \le 10)$$
$$\lambda(x) = \log\log x + B + O^*\left(\frac{3.690}{\log^3 x}\right), \quad (x \ge 10)$$
$$\lambda(x) = \log\log x + B + O^*\left(\frac{0.820}{\log^3 x}\right), \quad (x \ge 50\,000)$$
$$\lambda(x) = \log\log x + B + O^*\left(\frac{0.210}{\log^3 x}\right), \quad (x \ge 2 \times 10^6, x \notin [10^8, \exp(22)])$$

Proof. Write $f(x) = (\lambda(x) - \log\log x - B)\log^3 x$ for $x \ge 2$. We make a Pari/GP computation of all $f(k)$ for integers k in the range $2 \le k \le 10^8$. Table 4.2 gives the minima m_n and maxima M_n attained by $f(k)$ for k in the interval $x_n \le k \le x_{n+1}$. The columns y_n, Y_n are the unique integers $x_n \le y_n, Y_n \le x_{n+1}$ for which the quantities $m_n = f(y_n)$ and $M_n = f(Y_n)$ are the smallest and biggest, respectively. The quantities m_n and M_n given are truncated after the sixth decimal digit without rounding off. The value given for the last row corresponds to the number $f(10^8)$. Also, our calculations show that f does not change sign in $[2, 10^{18}]$.

We remark that to find the maxima of $f(x)$ for $4 \le x \le 10^8$, it is enough to evaluate $f(x)$ at integral and prime x, since $f(x)$ decreases between two consecutive primes, attaining its local maxima at primes (because the derivative $f'(x)$ of $f(x)$ is negative as soon as $x \ge 4$). □

We also read from the table that

$$|\lambda(x) - \log\log x - B| \ge \frac{0.009}{\log^3 x}$$

for $2 \le x \le 10^8$, although such a lower bound cannot hold for all x, in view of Corollary 45.

n	b_n	ϵ_n	η_n
1	18.42	$1.186\,414\,000 \times 10^{-3}$	0.522 463 178
2	19	$9.416\,472\,060 \times 10^{-4}$	0.438 928 475
3	20	$6.302\,000\,000 \times 10^{-4}$	0.314 349 592
4	21	$4.197\,685\,060 \times 10^{-4}$	0.230 174 445
5	22	$2.786\,520\,000 \times 10^{-4}$	0.165 125 235
6	23	$1.843\,645\,000 \times 10^{-4}$	0.117 791 272
7	24	$1.216\,119\,620 \times 10^{-4}$	0.083 581 394
8	25	$7.998\,895\,869 \times 10^{-5}$	0.072 862 959
9	30	$9.778\,040\,657 \times 10^{-6}$	0.024 445 214
10	50	$9.049\,928\,595 \times 10^{-8}$	0.000 905 011
11	100	$8.842\,626\,429 \times 10^{-8}$	0.003 537 121
12	200	$8.561\,316\,979 \times 10^{-8}$	0.013 698 388
13	400	$8.000\,089\,705 \times 10^{-8}$	0.028 800 954
14	600	$7.442\,047\,763 \times 10^{-8}$	0.074 422 230
15	1000	$6.337\,118\,668 \times 10^{-8}$	0.107 100 266
16	1300	$5.518\,819\,789 \times 10^{-8}$	0.124 177 386
17	1500	$4.980\,115\,883 \times 10^{-8}$	0.161 361 428
18	1800	$4.191\,337\,100 \times 10^{-8}$	0.167 660 488
19	2000	$3.674\,711\,889 \times 10^{-8}$	0.194 401 521
20	2300	$2.917\,036\,000 \times 10^{-8}$	0.182 325 692
21	2500	$2.439\,460\,000 \times 10^{-8}$	0.184 497 402
22	2750	$1.876\,943\,507 \times 10^{-8}$	0.168 940 671
23	3000	$1.376\,020\,000 \times 10^{-8}$	0.168 583 894
24	3500	$6.165\,300\,000 \times 10^{-9}$	0.098 672 807
25	4000	$2.405\,714\,403 \times 10^{-9}$	0.053 180 897
26	4700	$1.734\,200\,000 \times 10^{-12}$	0.000 087 114
27	10 000	$6.228\,800\,000 \times 10^{-18}$	0.000 338 143
28	20 000	$2.229\,400\,000 \times 10^{-25}$	–
n	b_n	ϵ_n	η_n

Table 4.1 – $|\lambda(x) - \log\log x - B| \le \frac{\eta_n}{\log^3 x}$ for $e^{b_n} \le x \le e^{b_{n+1}}$ and $|\psi(x) - x| \le \epsilon_n x$ for $x \ge e^{b_n}$.

n	x_n	β_n	y_n	m_n	Y_n	M_n
1	2	1.835	2	0.201 485	7	1.834 441
2	10	3.055	58	1.186 615	73	3.054 472
3	100	3.690	556	0.715 234	113	3.689 944
4	1000	2.247	1422	0.312 136	1327	2.246 529
5	5000	1.425	7450	0.356 194	5881	1.424 019
6	10 000	1.270	19 372	0.159 575	10 343	1.269 310
7	20 000	1.107	32 050	0.187 937	24 137	1.106 448
8	50 000	0.820	69 990	0.165 231	59 797	0.819 324
9	100 000	0.596	302 830	0.067 158	102 679	0.595 960
10	500 000	0.343	643 846	0.103 429	617 819	0.342 335
11	700 000	0.288	993 820	0.085 181	910 229	0.287 257
12	1 000 000	0.275	1 090 696	0.053 584	1 195 247	0.274 719
13	2 000 000	0.209	4 409 886	0.036 799	2 275 771	0.208 742
14	5 000 000	0.151	9 993 078	0.036 926	5 001 779	0.150 128
15	10 000 000	0.120	10 219 590	0.026 636	12 871 811	0.119 603
16	30 000 000	0.089	36 917 098	0.009 107	30 909 673	0.088 092
17	50 000 000	0.057	65 404 318	0.016 282	51 841 303	0.056 192
18	70 000 000	0.055	89 823 540	0.015 339	76 020 569	0.054 421
19	90 000 000	0.041	93 798 766	0.015 401	97 931 143	0.040 071
20	100 000 000	–	–	0.025 190	–	–

Table 4.2 – $|\lambda(x) - \log\log x - B| \leq \frac{\beta_n}{\log^3 x}$ for $x_n \leq x \leq x_{n+1}$.

Finally, in view of our computations and theoretical results, the following result is clear:

Theorem 62. *For $x \geq 24\,284$, we have*

$$\sum_{p\leq x} \frac{1}{p} = \log\log x + B + O^*\left(\frac{1}{\log^3 x}\right). \qquad (4.27)$$

Indeed, our computation shows that (4.27) does not hold for $x = 24\,283$ but holds for $24\,284 \leq x \leq 10^8$, hence for all $x \geq 24\,284$ in view of our theoretical results. Corollary 43 can be read off immediately from our tables and other results of this section.

4.4.1 Error terms of shape $c/\log x$

We collect here some computation results and corollaries of the theorems we have proved to give error terms of the form $c/\log x$ for the sum $\sum_{p\le x} {}^1\!/_p$ and the products $\prod_{p\le x}\left(1+{}^z\!/_p\right)$.

Theorem 63. *The following bounds hold in the indicated intervals:*

$$\sum_{p\le x}\frac{1}{p} = \log\log x + B + O^*\left(\frac{c_n}{\log x}\right), \quad (x \ge x_n),$$
$$\sum_{p\le x}\frac{1}{p} = \log\log x + B + O^*\left(\frac{0.001\,863}{\log x}\right), \quad (x \ge 678\,407),$$

where x_n, c_n are tabulated in Table 4.3.

Proof. We use the bound [73, (5.7)]

$$|\vartheta(x) - x| < \frac{x}{40\log x} \quad (x \ge 678\,407)$$

in (4.3) to obtain

$$\lambda(x) = \log\log x + B + O^*\left(\frac{0.001\,863}{\log x}\right), \quad (x \ge 678\,407).$$

For lower values of x, we use Pari/GP, employing the same strategy as Corollary 61. The table is analogous to the table for Corollary 61. □

An immediate corollary is the following:

Corollary 64. *The following bounds hold in the intervals indicated:*

$$\prod_{p\le x}\left(1+\frac{z}{p}\right) = e^{\gamma(z)}(\log x)^z\left\{1 + O^*\left(\frac{4c_n}{\log x} + \frac{4}{x-3}\right)\right\}, \quad (x_n \le x \le 678\,407)$$
$$\prod_{p\le x}\left(1+\frac{z}{p}\right) = e^{\gamma(z)}(\log x)^z\left\{1 + O^*\left(\frac{0.008}{\log x}\right)\right\}, \quad (x \ge 678\,407)$$

uniformly for complex $0 < |z| < 2$, where c_n, x_n are as in Theorem 63.

n	x_n	c_n	y_n	m_n	Y_n	M_n
1	2	0.525	10	0.185 728	3	0.524 904
2	10	0.361	96	0.058 126	13	0.360 597
3	100	0.166	556	0.017 902	113	0.165 111
4	1000	0.044	1422	0.005 922	1129	0.043 803
5	5000	0.019	7450	0.004 480	5881	0.018 902
6	10 000	0.015	19 372	0.001 637	10 343	0.014 853
7	20 000	0.011	32 050	0.001 745	24 137	0.010 864
8	50 000	0.007	88 788	0.001 288	59 797	0.006 772
9	100 000	0.005	302 830	0.000 421	102 679	0.004 475
10	500 000	0.002	643 846	0.000 578	617 819	0.001 925
11	678 407	–	–			

Table 4.3 – $|\lambda(x) - \log\log x - B| \leq \frac{c_n}{\log x}$ for $x \geq x_n$.

Proof. It is a direct application of Lemma 59; we may take $x \geq 14$ so as to satisfy (4.22). We only remark that if we want an error term dependent on z, then we may easily obtain it in the following form:

$$h_n(z,x) = \frac{2\,|z|\,c_n}{\log x} + \frac{|z|^2}{(x-1-|z|)}$$

within the appropriate intervals. Thus, we in fact have

$$\left|e^{-\gamma(z)}(\log x)^{-z}\prod_{p\leq x}\big(1+\frac{z}{p}\big)-1\right| \leq h_n(z,x) \tag{4.28}$$

within the appropriate intervals. See also our proof of Theorem 47. □

In particular, taking $z = \pm 1$ in (4.28), we obtain the following:

Corollary 65. *The following asymptotics hold:*

$$\prod_{p\leq x}\big(1-\frac{1}{p}\big) = \frac{e^{-\gamma}}{\log x}\left\{1+O^*\left(\frac{2c_n}{\log x}+\frac{1}{x-2}\right)\right\}, \quad (x\geq x_n)$$
$$\prod_{p\leq x}\big(1+\frac{1}{p}\big) = \frac{6}{\pi^2}e^{\gamma}(\log x)\left\{1+O^*\left(\frac{2c_n}{\log x}+\frac{1}{x-2}\right)\right\}, \quad (x\geq x_n).$$

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Annexe A

Scripts GP

Listing A.1 – Nullité de λ_o pour puissances de nombres premiers ≤ 100

```
{GetLambdaOddVec(q, Verbose = 1)=
   /* The modulus q has to be a power of an odd
   prime as we need a primitive root modulo q */
   local(gsum, gsumvec, gsumquadloc, gsumquadvecloc,
        gsumquadvec,  kappa, kappastar, kappastarvec,
         lambdaoddvec, rootnum, rootnumvec, rootnumabsvec,
         signevec, signe, g, count, indexOFquad);
   gsumvec = []; gsumquadvec = [];
   kappa = 0; kappastar = 0; kappavec = []; kappastarvec = [];
   rootnum = 0; rootnumvec = []; rootnumabsvec = [];
   signe = 0; signevec = [];
   g = znprimroot(q);

   count = 0;

   for(j = 1, eulerphi(q)-1,
      if(gcd(j, q) == 1,
         gsum = 0;
         kappa = 0;
         count += 1; \*counts the number of
                primitive characters modulo q*/
         /* Index of the quadratic character : */
         if(2*j == eulerphi(q), indexOFquad = count,);
         /* Compute Gauss sum of character
         tagged by j according to g: */
```

```
         for(i = 1, q, gsum += Dchipp(i, q, g, j)*exp(2*Pi*I*i/q));
         gsumvec = concat(gsumvec, gsum);
         if(abs((1 - Dchipp(-1, q, g, j))/2)< 0.01,
            signe = 0, signe = 1); /*signe = 0 if the
                                         character is even, 1 else*/
         signevec = concat(signevec, signe);
         rootnum = gsum/((I^signe)*sqrt(q));
         rootnumvec = concat(rootnumvec, rootnum);
         rootnumabsvec = concat(rootnumabsvec, abs(rootnum));
         kappa = sqrt(2/rootnum)*exp(Pi*I*(signe-2)/4)/q^(0.25);
         kappavec = concat(kappavec,kappa);

         kappastar = sqrt(2)*kappa*exp(5*Pi*I/8)/q^(1/4);
         kappastarvec = concat(kappastarvec, kappastar);

         gsumquadvecloc = [];
         for(r = 0, 2*q-1,
            gsumquadloc = 0;
            if((q+r)%2 == 0,
               for(i = 1, 2*q,
                  gsumquadloc += Dchipp(i,q,g,j)
                                *exp(Pi*I*(i*r-i*i)/q);),
               gsumquadloc = 0);
            gsumquadvecloc = concat(gsumquadvecloc,
                             gsumquadloc*exp(-Pi*I*r^2/(8*q)));
         );
         gsumquadvec = concat(gsumquadvec, [gsumquadvecloc]);
        ); //Else gcd(j,q) != 1 and we do nothing
   );

   if(Verbose,
        print("Signs of the successives characters:",
        signevec),
   );

   lambdaoddvec = vector(count);
   for(j = 1, count, // Careful! This is not the j of above!
      lambdaodd = vector(2*q-1, ell, 0); /*lambdaodd[ell]
                                         --> vartheta = (2ell+1)/4*/
      if(signevec[j] == 0,
```

```
         //Even characters:
         for(ell = 1, 2*q-1,
            for(r = 0, 2*q-1,
               lambdaodd[ell] += -I*kappastarvec[j]
                                *gsumquadvec[j][r+1]
                                *exp(-Pi*I*r*(2*ell+1)/(4*q))
                                *(1+I*((-1)^((q-r)/2))*((-1)^ell));
            );
            lambdaodd[ell] = real(lambdaodd[ell]));
            lambdaoddvec[j] = lambdaodd;  /*we should
                        also multiply by (-1)^L/(12q)*/
         ,
         //Odd characters:
         lambdaoddvec[j] = "odd")
   );
   if(Verbose == 1,
      print("Here are the functions lambdaodd((2 ell+1)/4)");
      print("for the successive characters
                and 1 <= ell <= 2q-1 ");
      print("(the entry \"odd\" means
                the relevant character is odd):"),
     if(Verbose == 2,
        print("Here is the function
                lambdaodd((2 ell+1)/4)");
        print(" for the quadratic character: ");
        print(lambdaoddvec[indexOFquad]);
        print("with quadratic Gauss sum: ");
        print(gsumquadvec[indexOFquad]),)
     );
   return(lambdaoddvec);
}

{Dchipp(n, q, g, tag)=
        /*Returns the value at n of character number tag
        according to the primitive root g modulo q */
   if(omega(q)>1,
        print("On veut une puissance d'un nombre premier!");
    return;
    );
   if(gcd(n,q)>1, return(0););
```

```

   return(exp(2*Pi*I*tag*znlog(n, g)/eulerphi(q)));
}

{Run(borneinf, bornesup) =
   local(lambdaoddvec, ok, nbbadchi);
   for(q = max(2, borneinf), bornesup,
      if((q%2==1) && (omega(q)==1),
         lambdaoddvec = GetLambdaOddVec(q, 0);
         nbbadchi = 0;
         for(j = 1, length(lambdaoddvec),
            if(lambdaoddvec[j] != "odd",
               ok = 0;
               for(ell = 1, 2*q-1,
                  if(abs(lambdaoddvec[j][ell])>0.00001,
                     ok = 1; break, ));
               if(ok == 0,
                  nbbadchi++;
                 ,)
               ,nbbadchi++ //add the odd characters
               );
            );
         if(nbbadchi == length(lambdaoddvec),
            print("Every even charater modulo ",q,
            " has zero lambda_o"),
            print("There is a character modulo ", q,
            " that has non zero lamba_o"));
   ,));
}
```

One runs the script like this :

```
< Run(a,b)
```

and this produces the moduli q with $a \leqslant q \leqslant b$ for which there is a character χ for which one of the coefficients λ_o is nonzero like this :

```
< There is a character modulo q  that
has a nonzero $ \lambda_o $.
```

In case there is no such character, it prints

```
< Every even character modulo $ q $ has zero lambda_o .
```

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Table des matières

FONCTIONS DE HARDY DES SÉRIES L ET SOMMES DE MERTENS EXPLICITES

Résumé

Cette thèse comporte deux parties principales.

Dans la première partie, nous étudions les fonctions de Hardy des fonctions L de Dirichlet. La fonction de Hardy $Z(t,\chi)$ liée à la fonction $L(s,\chi)$ est une fonction à valeurs réelles de la variable réelle t dont les zéros correspondent exactement aux zéros de $L(s,\chi)$ sur la droite critique $\Re s = \frac{1}{2}$; en effet, on a $|Z(t,\chi)| = |L(\frac{1}{2}+it,\chi)|$ pour tout $t \in \mathbb{R}$. Nous étudions sa primitive $F(T,\chi) \stackrel{\text{def}}{=} \int_0^T Z(t,\chi)\,\mathrm{d}t$. L'étude asymptotique de la fonction de Hardy liée à la fonction zêta de Riemann a connu récemment un regain d'activité grâce à Ivić [34] qui a montré entre autres la majoration

$$F(T) = O(T^{\frac{1}{4}+\epsilon})$$

et a conjecturé le comportement $F(T) = \Omega_{\pm}(T^{\frac{1}{4}})$. Cette dernière conjecture a été démontrée par Korolëv [48], qui a également exhibé un comportement surprenant de $F(T)$; en effet, il montre que $F(T)$ peut être approchée asymptotiquement par une fonction étagée périodique qui prend des valeurs positives et négatives. Ensuite, Jutila [39, 38] a donné une démonstration indépendante de ces résultats, avec un traitement plus uniforme de l'approximation et des termes d'erreur. Il montre d'abord une formule de type Atkinson pour $F(T)$ par le biais de la transformée de Laplace et en déduit les résultats de Korolëv. En suivant Jutila, nous étendons ces résultats aux fonctions L de Dirichlet, et nous montrons également que le comportement de $F(T,\chi)$ dépend notamment de la parité de χ et celle du conducteur.

Sauf en un point majeur où Jutila utilise très spécifiquement les coefficients de la série de Dirichlet (ici 1 !), ce travail suit assez fidèlement celui de Jutila, bien que de nombreuses modifications soient nécessaires. Au niveau de ce point majeur, notre travail s'écarte notoirement de celui de Jutila, et nous ne savons conclure que dans le cas des caractères pairs et pour les deux caractères impairs modulo 4 et 8. Ensuite, l'approximation par une fonction simple pose elle aussi plusieurs problèmes que nous traitons de façon satisfaisante aux points génériques, mais le comportement en certains points spéciaux pose encore de nombreuses questions.

Dans la seconde partie, nous étudions certaines fonctions sommatoires des nombres premiers en vue d'estimations explicites dans la lignée de Rosser et Schoenfeld [72]. Nous donnons principalement des estimations explicites pour les termes d'erreur des sommes de Mertens $\sum_{p\le x}\frac{1}{p}, \sum_{p\le x}\frac{\log p}{p}, \sum_{n\le x}\frac{\Lambda(n)}{n}$ et des produits d'Euler $\prod_{p\le x}\left(1+\frac{z}{p}\right)$. La méthode utilisée est celle suggérée par un article récent de Ramaré [65] ; des estimations explicites très précises sont données au moyen d'une région explicite sans zéros de type de la Vallée Poussin pour la fonction zêta de Riemann.

Mots clefs : série de Dirichlet, fonction L de Dirichlet, fonction de Hardy, formule d'Atkinson, somme de Gauss, somme d'exponentielles, fonction sommatoire des nombres premiers, estimation explicite.

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HARDY'S FUNCTIONS OF L-SERIES AND EXPLICIT MERTENS SUMS

Abstract

The thesis consists of two main parts.

In the first part of this thesis, we study Hardy's functions of Dirichlet L-functions. The *Hardy's function* $Z(t,\chi)$ corresponding to a Dirichlet L-function $L(s,\chi)$ is a real-valued function of the real variable t whose zeros coincide exactly with the zeros of $L(s,\chi)$ on the critical line $\Re s = \frac{1}{2}$; in fact, we have $|Z(t,\chi)| = |L(\frac{1}{2}+it,\chi)|$ for all $t \in \mathbb{R}$. We study its primitive $F(T,\chi) \overset{\text{def}}{=} \int_0^T Z(t,\chi)\,\mathrm{d}t$. The asymptotic study of Hardy's function corresponding to the Riemann zeta function was recently revived by Ivić [34] who proved among others the asymptotic

$$F(T) = O(T^{\frac{1}{4}+\epsilon})$$

and conjectured the behaviour $F(T) = \Omega_{\pm}(T^{\frac{1}{4}})$. This conjecture was proved by Korolëv [48], who further exhibited a rather surprising property of $F(T)$; in fact, he proved that $F(T)$ can be asymptotically approximated by a periodic step function which takes both positive and negative values. Later, Jutila [39] gave another proof of these results, giving more uniform treatment of the approximation and the error terms. He first proves an Atkinson-like formula for $F(T)$ via the Laplace transform and therefrom derives the results of Korolëv. Following Jutila, we extend these results to Dirichlet L-functions, and show that the behaviour of $F(T,\chi)$ depends on the parity of the character as well as that of the conductor.

Except at one major point where Jutila very specifically uses the coefficients of the Dirichlet series (here equal to 1!), this adaptation quite faithfully follows the work of Jutila, although a number of modifications are necessary. Regarding this major point, our work deviates manifestly from that of Jutila, and we can conclude only for even characters and the two odd characters modulo 4 and 8. The approximation by a function also poses several problems which we treat in a satisfying manner at the generic points, but the behaviour at certain points still poses several questions.

In the second part, we study some summatory functions of primes in view of explicit estimates in the line of Rosser and Schoenfeld [72]. Principally, we give explicit estimates for the error terms in the Mertens sums $\sum_{p\le x}\frac{1}{p}, \sum_{p\le x}\frac{\log p}{p}, \sum_{n\le x}\frac{\Lambda(n)}{n}$ and the Euler products $\prod_{p\le x}\left(1+\frac{z}{p}\right)$. The method used is the one suggested by a recent paper of Ramaré [65]; precise explicit estimates are obtained using explicit de la Vallée Poussin zero-free regions for the Riemann zeta function.

Keywords: Dirichlet series, Dirichlet L-function, Hardy's function, Atkinson's formula, Gauss sum, exponential sum, summatory function of primes, explicit estimate.